

**ON THE NUMBERS RELATED TO THE STIRLING NUMBERS
 OF THE SECOND KIND**

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Abstract. In this paper we give the new explicit expression for the numbers c_{nk} treated by Mijajlović and Marković [Facta Univ. Ser. Math. Inform. **13** (1998), 7–17]. The new combinatorial identities are also given.

1. Introduction

In this paper, we consider the numbers c_{nk} , which may be defined by the operational relation

$$(x^{\alpha+1}D)^n = \sum_{k=1}^n c_{nk} x^{n\alpha+k} D^k,$$

and by the recurrence relation (triangular or Pascal-type relation)

$$c_{n+1k} = c_{nk-1} + (n\alpha + k)c_{nk}, \quad 1 \leq k \leq n, \quad c_{11} = 1, \quad c_{nk} = 0 \text{ for } k > n,$$

where $\alpha \in \mathbb{R}$ (see [5]). It is clear that for $\alpha = 0$, c_{nk} are the usual Stirling numbers of the second kind. The aim of this paper is to prove the new explicit expression for the numbers c_{nk} . Also, we give some additional remarks and comments.

2. Main Result

In [5] Mijajlović and Marković studied the numbers c_{nk} and proved that

$$(x| - \alpha)_n = \sum_{k=0}^n c_{nk} (x|1)_k,$$

where $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $(x|\alpha)_n$ denotes the generalized factorial of the form

$$(x|\alpha)_n = \prod_{j=0}^{n-1} (x - j\alpha), \quad n \geq 1, \quad (x|\alpha)_0 = 1.$$

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and in particular $(x|1)_n = (x)^{(n)} = x(x-1)\cdots(x-n+1)$, $(x)^{(0)} = 1$.

Theorem. *The explicit expression for the numbers c_{nk} defined in [5] is given by:*

$$(2.1) \quad c_{nk} = \sum_{\substack{i_1 + \cdots + i_{n-1} = n-k \\ i_j \in \{0, 1\}}} \binom{1+\alpha}{i_1} \cdots \binom{(n-1)(1+\alpha) - i_1 - \cdots - i_{n-2}}{i_{n-1}}.$$

Proof. Starting from (2.1), for $i_n \in \{0, 1\}$ we get

$$\begin{aligned} c_{n+1k} &= \sum_{\substack{i_1 + \cdots + i_n = n+1-k \\ i_j \in \{0, 1\}}} \binom{1+\alpha}{i_1} \cdots \binom{n(1+\alpha) - i_1 - \cdots - i_{n-1}}{i_n} \\ &= \sum_{\substack{i_1 + \cdots + i_n = n+1-k \\ i_j \in \{0, 1\}}} \binom{1+\alpha}{i_1} \cdots \binom{(n-1)(1+\alpha) - i_1 - \cdots - i_{n-2}}{i_{n-1}} + \\ &\quad (n\alpha + k) \sum_{\substack{i_1 + \cdots + i_{n-1} = n-k \\ i_j \in \{0, 1\}}} \binom{1+\alpha}{i_1} \cdots \binom{(n-1)(1+\alpha) - i_1 - \cdots - i_{n-2}}{i_{n-1}} \end{aligned}$$

i.e., $c_{n+1k} = c_{nk-1} + (n\alpha + k)c_{nk}$, because

$$n(1+\alpha) - i_1 - \cdots - i(n-1) = n(1+\alpha) - n + k = n\alpha + k.$$

This completes the proof. \square

Corollary. Unsigned Lah numbers, $L(n, k)$ (see, for example [2] and [3]) defined by triangular recurrence relation

$$L(n, k) = L(n-1, k-1) + (n+k-1)L(n-1, k),$$

$$L(n, n) = L(n, 1) = 1, \quad L(n, k) = 0 \text{ for } n < k,$$

have the explicit representation

$$L(n, k) = \sum_{\substack{i_1 + i_2 + \cdots + i_{n-1} = n-k \\ i_j \in \{0, 1\}}} \binom{2}{i_1} \binom{4-i_1}{i_2} \binom{6-i_1-i_2}{i_3} \cdots \binom{2(n-1)-i_1-\cdots-i_{n-2}}{i_{n-1}}.$$

This expression follows from (2.1) for $\alpha = 1$.

Naturally, it is well-known that (see [3])

$$L(n, k) = \frac{n!}{k!} \binom{n-1}{k-1}.$$

So, the combinatorial identity

$$\begin{aligned} \frac{n!}{k!} \binom{n-1}{k-1} &= \\ \sum_{\substack{i_1 + \dots + i_{n-1} = n-k \\ i_j \in \{0, 1\}}} \binom{2}{i_1} \binom{4-i_1}{i_2} \binom{6-i_1-i_2}{i_3} \dots \binom{2(n-1)-i_1-\dots-i_{n-2}}{i_{n-1}} \end{aligned}$$

is valid.

Remark 2.1. Explicit expression for the numbers c_{nk} from [5] is

$$(2.2) \quad c_{nk} = \frac{1}{k!} \sum_{k=1}^n (-1)^{k-i} \binom{k}{i} i(i+\alpha) \dots (i+(n-1)\alpha),$$

and therefore, if we identify the right sides of (2.1) and (2.2), we get the following combinatorial identity

$$\begin{aligned} \sum_{\substack{i_1 + \dots + i_{n-1} = n-k \\ i_j \in \{0, 1\}}} \binom{1+\alpha}{i_1} \binom{2(1+\alpha)-i_1}{i_2} \dots \binom{(n-1)(1+\alpha)-i_1-\dots-i_{n-2}}{i_{n-1}} \\ = \frac{1}{k!} \sum_{k=1}^n (-1)^{k-i} \binom{k}{i} i(i+\alpha) \dots (i+(n-1)\alpha). \end{aligned}$$

Remark 2.2. For $\alpha = 0$ we get the explicit formula for the Stirling numbers of the second kind (see [1] and comment by L. Carlitz therein)

$$S(n, k) = \sum_{\substack{i_1 + \dots + i_{n-1} = n-k \\ i_j \in \{0, 1\}}} \binom{1}{i_1} \binom{2-i_1}{i_2} \dots \binom{n-1-i_1-\dots-i_{n-2}}{i_{n-1}}.$$

Remark 2.3. The numbers c_{nk} are the special case of a generalized Stirling numbers $S(n, k; \alpha, \beta, \gamma)$, introduced by Hsu and Shiue ([4]). Namely, $c_{n,k} = S(n, k; -\alpha, 1, \gamma)$, where the following relations holds

$$(x|\alpha)_n = \sum_{k=0}^n S(n, k; \alpha, \beta, \gamma) (x - \gamma|\beta)_k,$$

$$S(n+1, k; \alpha, \beta, \gamma) = S(n, k-1) + (k\beta - n\alpha + \gamma) S(n, k; \alpha, \beta, \gamma).$$

Evidently, the classical Stirling numbers $s(n, k)$ and $S(n, k)$ are in fact $S(n, k; 1, 0, 0)$ and $S(n, k; 0, 1, 0)$ respectively. Also, the binomial coefficients are given by $S(n, k; 0, 0, 1)$.

In a forthcoming paper we give an explicit formula for generalized Stirling numbers $S(n, k; \alpha, \beta, \gamma)$.

R E F E R E N C E S

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