FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 22, No. 1 (2007), pp. 91–103

ON THE CONVERGENCE OF THE THIRD ORDER ROOT-SOLVER*

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Abstract. The construction of computationally verifiable initial conditions that provide both the guaranteed and fast convergence of a numerical method for solving nonlinear equations is one of the most important tasks in the field of iterative processes. A suitable convergence procedure, based partially on Smale's "point estimation theory" from 1981, is applied in this paper to a new cubically convergent derivative free iterative method for the simultaneous approximation of simple zeros of polynomials. We have stated initial conditions which guarantee the convergence of this method. These conditions are of significant practical importance since they depend only on available data: the coefficients of a given polynomial, its degree n and initial approximations to polynomial zeros.

1. Introduction

Last years a great attention is paid to state computationally verifiable initial conditions which enable both the guaranteed and fast convergence of the applied iterative method for solving a nonlinear equation $f(z) = 0$. This challenging problem of the theory and practice of iterative processes is often considered in the literature during the last fifty years, but the results were rather of theoretical importance; namely, the established initial conditions depend on unattainable data such as suitable (but unknown) constants, "reasonable good initial approximations" (without a proper estimate of their accuracy), or even the sought zeros of an equation to be solved.

In general, the construction of computationally verifiable initial conditions is a very difficult problem, even in the case of simple functions such

Received February 2, 2007.
2000 *Mathematics Subject Classification.* 65H05.
*This work was supported by the Serbian Ministry of Science under grant 144024.

as algebraic polynomials. A great progress in this topic was made in 1981 when Stiven Smale [14] developed a concept known as *point estimation theory* which treats convergence condition using only the information of f at the initial point z_0. Considering a monic polynomial of the form

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \quad (a_i \in \mathbb{C})$$

in the spirit of Smale's approach, initial conditions should depend only on the coefficients of P, its degree n and initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ to the zeros $\zeta_1, \ldots, \zeta_n$ of P.

For $m = 0, 1, \ldots$ and $i \in I_n := \{1, \ldots, n\}$ let

$$d^{(m)} = \min_{\substack{1 \le i,j \le n \\ i \ne j}} |z_i^{(m)} - z_j^{(m)}|$$

be the minimal distance between approximations obtained in the mth iteration by an iterative root-solvers, and let

$$W_i^{(m)} = \frac{P(z_i^{(m)})}{\prod_{\substack{j=1 \\ j \ne i}}^{n} (z_i^{(m)} - z_j^{(m)})}, \quad w^{(m)} = \max_{1 \le i \le n} |W_i^{(m)}|.$$

Let us note that $W_i^{(m)}$ is often called the Weierstrass correction since it appeared in Weierstrass' work [19]. In [17] D. Wang and Zhao refined Smale's result for Newton's method and applied it to the Weierstrass (or Durand-Kerner) method for the simultaneous approximation of simple zeros of polynomials. Their procedure led to an initial condition of the form

$$w^{(0)} \le c_n \cdot d^{(0)}. \tag{1}$$

The constant c_n, which depends only on the polynomial degree n, should be chosen to be as large as possible, see the discussion in [6], [8] and [13]. A quite different approach given in [5] for the same method, also resulted in the condition of the form (1). Results which are related to the point estimation theory and iterative processes for the simultaneous determination of polynomial zeros were presented in [1, 4, 5, 6, 8, 9, 13, 16, 17, 18] and the book [7]. Convergence analysis presented in these papers and [7] showed that the condition (1) is quite suitable for a wide class of simultaneous methods.

In this paper we state initial conditions that guarantee the convergence of the new cubically convergent iterative method for the simultaneous determination of simple zeros of the polynomial P,

$$(2)\qquad z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 - \dfrac{P\big(z_i^{(m)} - W_i^{(m)}\big)}{P\big(z_i^{(m)}\big)}} \qquad (i \in I_n;\ m = 0, 1, 2, \ldots).$$

This method was proposed in [12] and the following theorem was proved therein:

Theorem 1. *If $z_1, \ldots, z_n$ are sufficiently close approximations to the zeros $\zeta_1, \ldots, \zeta_n$ of P, the order of convergence of the iterative method* (4) *is three.*

Remark 1. The simultaneous method (2) can be derived from the Newton-secant method (see Traub [15, p. 184])

$$\phi = z + \frac{u(z)f(z)}{f\big(z - u(z)\big) - f(z)}, \quad u(z) = \frac{f(z)}{f'(z)}$$

taking $z = z_i,\ f \equiv P$ and substituting Weierstrass' correction W_i in the above formula instead of Newton's correction $u(z)$.

Remark 2. The efficient *a posteriori* error bound method based on the method (2), which gives automatically the upper error bound of the obtained approximations, was proposed in [11].

Remark 3. The method (2) requires less numerical operations than another derivative free method of the third order, the so-called the Börsch-Supan method [2],

$$z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 - \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_j^{(m)}}{z_j^{(m)} - z_i^{(m)}}} \qquad (i \in I_n;\ m = 0, 1, \ldots).$$

Considering Theorem 1 we observe that the cubic convergence of the method (2) is stated under initial conditions that assume "sufficiently close approximations to the exact zeros," without any quantitative (and computationally verifiable) characterization of the closeness of these approximations to the zeros. To overcome this difficult that appear in the traditional treating the convergence conditions based on the asymptotical convergence analysis, in Section 3 we present computationally verifiable initial conditions, which is of significant practical importance.

2. Preliminary Results

Most of the iterative methods for the simultaneous approximation of polynomial zeros can be represented in the form

$$z_i^{(m+1)} = z_i^{(m)} - C_i(z_1^{(m)}, \ldots, z_n^{(m)}) \quad (i \in I_n,\ m = 0, 1, \ldots), \tag{3}$$

where $z_1^{(m)}, \ldots, z_n^{(m)}$ are some distinct approximations to the zeros $\zeta_i, \ldots, \zeta_n$ respectively, obtained in the m-th iterative step by the method (3). The term

$$C_i^{(m)} = C_i(z_1^{(m)}, \ldots, z_n^{(m)}) \quad (i \in I_n)$$

is often called the *iterative correction* or simply the *correction.*

Let $\Lambda(\zeta_i)$ be a reasonably close neighborhood of the zero ζ_i $(i \in I_n)$ of P. The convergence theorem which will be used in our study of the method (2) assumes that corrections C_i can be represented in the form

$$C_i(z_1, \ldots, z_n) = \frac{P(z_i)}{F_i(z_1, \ldots, z_n)} \quad (i \in I_n), \tag{4}$$

where the function $(z_1, \ldots, z_n) \mapsto F_i(z_1, \ldots, z_n)$ satisfies the following conditions for each $i \in I_n$:

1° $F_i(\zeta_1, \ldots, \zeta_n) \neq 0$,

2° $F_i(z_1, \ldots, z_n) \neq 0$ for distinct approximations $z_i \in \Lambda(\zeta_i)$,

3° $F_i(z_1, \ldots, z_n)$ is continuous in $\mathbb{C}^n$.

In the convergence analysis we will use a real function $t \mapsto g(t)$ defined on the open interval $(0, 1)$ by

$$g(t) = \begin{cases} 1 + 2t, & 0 < t \leq \dfrac{1}{2}, \\ \dfrac{1}{1-t}, & \dfrac{1}{2} < t < 1. \end{cases}$$

The following theorem [7, Theorem 5.1] (see, also, [6]) has the main role in our convergence analysis of the simultaneous method (2).

Theorem 2. *Let the iterative method* (3) *have the correction term of the form* (4) *for which the conditions* $1^\circ - 3^\circ$ *hold, and let* $z_1^{(0)}, \ldots, z_n^{(0)}$ *be distinct initial approximations to the zeros of* P. *If there exists a real number* $\beta \in (0, 1)$ *such that the following two inequalities*

(i) $|C_i^{(m+1)}| \leq \beta |C_i^{(m)}|$ $(m = 0, 1, \ldots)$,

(ii) $|z_i^{(0)} - z_j^{(0)}| > g(\beta)\Big(|C_i^{(0)}| + |C_j^{(0)}|\Big) \quad (i \neq j, \quad i, j \in I_n),$

are valid, then the iterative method (3) *is convergent.*

Remark 4. The assertions (i) and (ii) are concerned with the monotonicity of the sequences of corrections $\{C_i^{(m)}\}$ and the disjunctivity of disks

$$\{z_1^{(0)}; g(\beta)|C_1^{(0)}|\}, \dots, \{z_n^{(0)}; g(\beta)|C_n^{(0)}|\}$$

(see (5)), respectively.

The proof of Theorem 2 is similar to that proved in [8], where the function g is defined in a slightly different manner. For this reason, we omit the proof.

To provide estimates of some complex quantities, in this paper we use some properties of circular complex interval arithmetic. For more details see [10, Ch. 1].

A disk Z with center $c = \operatorname{mid} Z$ and radius $r = \operatorname{rad} Z$ will be denoted with $Z = \{c; r\} = \{z : |z - c| \leq r\}$. If $Z_k = \{c_k; r_k\}$ $(k = 1, 2)$, then

$$\begin{aligned} &Z_1 \pm Z_2 := \{c_1 \pm c_2; r_1 + r_2\} = \{z_1 \pm z_2 : z \in Z_1, z_2 \in Z_2\}, \\ (5) \qquad &\{c_1; r_1\} \cap \{c_2; r_2\} = \varnothing \quad \Leftrightarrow \quad |c_1 - c_2| > r_1 + r_2, \\ &w\{c\,;\, r\} = \{wc\,;\, |w|r\} \quad (w \in \mathbb{C}), \\ (6) \qquad &\max\{0, |\operatorname{mid} Z| - \operatorname{rad} Z\} \leq |z| \leq |\operatorname{mid} Z| + \operatorname{rad} Z \quad \text{for all } z \in Z. \end{aligned}$$

The product $Z_1 \cdot Z_2$ is defined as in [3]:

$$Z_1 \cdot Z_2 := \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\} \supseteq \{z_1 z_2 : z_1 \in Z_1, z_2 \in Z_2\}.$$

In particular, from the last formula one obtains

$$\{c; r\}^n = \{c^n; (|c| + r)^n - |c|^n\}. \tag{7}$$

If $F(Z) \supseteq \{f(z) : \ z \in Z\}$ is a *circular interval extension* of a given closed complex function f over a disk Z, then

$$z \in Z \ \Rightarrow \ f(z) \in F(Z). \tag{8}$$

We note that all considerations in this paper are given for $n \geq 3$ regarding that algebraic equations of the order ≤ 2 are trivial.

3. Initial Conditions and Guaranteed Convergence

In this section we apply Theorem 2 and an initial condition of the form (1) to state the convergence theorem for the simultaneous method (2). To simplify the denotation, we will omit sometimes the iteration index m and denote quantities in the subsequent $(m+1)$-st iteration by $\widehat{\ }$ (*"hat"*). For example, the iterative formula (2) is written as follows,

$$\hat{z}_i = z_i - \frac{W_i}{1-q_i} \quad (i \in I_n), \tag{9}$$

where $q_i = P(z_i - W_i)/P(z_i)$.

In our analysis we will use Lagrange's interpolation formula

$$\begin{aligned} P(z) &= \left(\sum_{j=1}^{n} \frac{W_j}{z-z_j} + 1\right) \prod_{j=1}^{n} (z - z_j) \\ &= W_i \prod_{j\neq i} (z-z_j) + \prod_{j=1}^{n} (z-z_j) \left(\sum_{j\neq i} \frac{W_j}{z-z_j} + 1\right). \end{aligned} \tag{10}$$

For $z = z_i - W_i$ from (10) one obtains

$$P(z_i - W_i) = -W_i \prod_{j\neq i} (z_i - W_i - z_j) \sum_{j\neq i} \frac{W_j}{z_i - W_i - z_j}.$$

Taking into account that $P(z_i) = W_i \prod_{j\neq i}(z_i - z_j)$, we find

$$q_i = \frac{P(z_i - W_i)}{P(z_i)} = -\prod_{j\neq i}\Big(1 - \frac{W_i}{z_i - z_j}\Big) \sum_{j\neq i} \frac{W_j}{z_i - W_i - z_j}. \tag{11}$$

Before establishing the main results, we give some necessary inequalities in the following lemma.

Lemma 1. *Let $z_1, \dots, z_n$ be distinct approximations to the zeros $\zeta_1, \dots, \zeta_n$ of a polynomial P of degree n, and let $\hat{z}_1, \dots, \hat{z}_n$ be new respective approximations obtained by the iterative method* (9). *If $\hat{d} = \min_{j\neq i} |\hat{z}_i - \hat{z}_j|$ and the inequality*

$$w := \max_{1\le i\le n} |W_i| < \frac{2}{9(n-1)} d = c_n d \tag{12}$$

holds, then for $i, j \in I_n$ we have

(i) $|q_i| < 0.32$; (ii) $|\widehat{W}_i| < \frac{1}{5}|W_i|$;

(iii) $|\widehat{W}_i| < \dfrac{2}{9(n-1)}\hat{d} = c_n \hat{d}$.

Proof. (i) We will often use the inequality (12) written in the form

$$\frac{w}{d} < \frac{2}{9(n-1)} = c_n \le \frac{1}{9} \tag{13}$$

without being cited. Using the definition of w and the minimal distance d, we find

$$|z_i - z_j| \ge d, \quad |z_i - W_i - z_j| \ge |z_i - z_j| - |W_i| \ge d - w. \tag{14}$$

Applying (13) and (14), from (11) we estimate

$$\begin{aligned} |q_i| &\le \prod_{j\ne i}\Big|1 - \frac{W_i}{z_i - z_j}\Big| \sum_{j\ne i} \frac{|W_j|}{|z_i - W_i - z_j|} \le \prod_{j\ne i}\Big(1 + \frac{w}{d}\Big) \cdot \frac{(n-1)w}{d-w} \\ &< (1+c_n)^{n-1}\frac{(n-1)c_n}{1-c_n} < \frac{e^{2/9}}{\frac{9}{2} - \frac{1}{n-1}} \le \frac{e^{2/9}}{4} \approx 0.3122 < 0.32. \end{aligned}$$

(ii) By (i) of Lemma 1 we find

$$|\hat{z}_i - z_i| = \frac{|W_i|}{|1-q_i|} \le \frac{|W_i|}{1-|q_i|} < 1.5|W_i| < \frac{d}{3(n-1)}, \tag{15}$$

$$|\hat{z}_i - z_j| \ge |z_i - z_j| - |\hat{z}_i - z_i| > d - \frac{d}{3(n-1)} = \frac{n-4/3}{n-1}d, \tag{16}$$

$$|\hat{z}_i - \hat{z}_j| \ge |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| > d - \frac{2d}{3(n-1)} = \frac{n-5/3}{n-1}d. \tag{17}$$

From the iterative formula (9) we have

$$\frac{W_i}{\hat{z}_i - z_i} = q_i - 1$$

so that, using (11), we obtain

$$\begin{aligned} \sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1 &= \frac{W_i}{\hat{z}_i - z_i} + \sum_{j\ne i}\frac{W_j}{\hat{z}_i - z_j} + 1 = q_i + \sum_{j\ne i}\frac{W_j}{\hat{z}_i - z_j} \\ &= -\prod_{j\ne i}\Big(1 - \frac{W_i}{z_i - z_j}\Big)\sum_{j\ne i}\frac{W_j}{z_i - W_i - z_j} + \sum_{j\ne i}\frac{W_j}{\hat{z}_i - z_j}. \end{aligned} \tag{18}$$

Since

$$0 < \frac{|W_i|}{|z_i - z_j|} \le \frac{w}{d} < c_n,$$

it follows

$$\frac{W_i}{z_i - z_j} \in \{0; c_n\}.$$

Applying circular arithmetic operations (Section 2), (7) and (8), we find

$$\begin{aligned} -\prod_{j\neq i}\Big(1 - \frac{W_i}{z_i - z_j}\Big) &\in -\prod_{j\neq i}(1 - \{0; c_n\}) = -\{1; c_n\}^{n-1} \\ &\subset -\Big\{1; (1 + c_n)^{n-1} - 1\Big\} = \Big\{-1; \Big(1 + \frac{2}{9(n-1)}\Big)^{n-1} - 1\Big\} \\ &\subset \Big\{-1; e^{2/9} - 1\Big\} \subset \{-1; 0.25\}. \end{aligned}$$

Using this inclusion we return into (18) and obtain

$$\begin{aligned} &\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1 \in \{-1; 0.25\} \sum_{j\neq i} \frac{W_j}{z_i - W_i - z_j} + \sum_{j\neq i} \frac{W_j}{\hat{z}_i - z_j} \\ = \quad & \Big\{\sum_{j\neq i} W_j \Big(\frac{1}{\hat{z}_i - z_j} - \frac{1}{z_i - W_i - z_j}\Big); 0.25 \sum_{j\neq i} \frac{|W_j|}{|z_i - W_i - z_j|}\Big\} \\ \subset \quad & \Big\{\eta_i; \frac{0.25(n-1)w}{d - w}\Big\} \subset \Big\{\eta_i; \frac{1}{16}\Big\} =: Z_i, \end{aligned}$$

where

$$\operatorname{mid} Z_i = \eta_i = (z_i - W_i - \hat{z}_i) \sum_{j\neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - W_i - z_j)}, \quad \operatorname{rad} Z_i = \frac{1}{16}.$$

By (i) of Lemma 1 we estimate

$$(19)\ |z_i - W_i - \hat{z}_i| = \Big|\frac{W_i}{1 - q_i} - W_i\Big| = |W_i| \frac{|q_i|}{1 - |q_i|} < |W_i| \frac{0.32}{1 - 0.32} < 0.5w.$$

According to (13), (16) and (19) we find the upper bound of $|\eta_i|$,

$$\begin{aligned} |\eta_i| \quad &\le \quad |z_i - W_i - \hat{z}_i| \sum_{j\neq i} \frac{|W_j|}{|\hat{z}_i - z_j||z_i - W_i - z_j|} < 0.5w \frac{(n-1)w}{\frac{n - 4/3}{n-1} d \cdot (d - w)} \\ &< \quad \frac{0.5(n-1)^2 c_n^2}{(n - 4/3)(1 - c_n)} < \frac{1}{60}. \end{aligned}$$

Using (6) and the above bounds we estimate

$$\left|\sum_{j=1}^{n}\frac{W_j}{\hat{z}_i - z_j}+1\right| < |\mathrm{mid}\,Z_i| + \mathrm{rad}\,Z_i < \frac{1}{60}+\frac{1}{16} < \frac{2}{25}. \tag{20}$$

Furthermore, by the inequalities (15) and (17) we obtain

$$\prod_{j\neq i}\left|\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| \le \prod_{j\neq i}\Big(1+\frac{|\hat{z}_j - z_j|}{|\hat{z}_i - \hat{z}_j|}\Big) < \prod_{j\neq i}\left(1+\frac{\frac{1}{3(n-1)}d}{\frac{n-5/3}{n-1}d}\right) = \Big(1+\frac{1}{3(n-5/3)}\Big)^{n-1}.$$

The sequence defined by $u_n = \left(1+\frac{1}{3(n-5/3)}\right)^{n-1}$ is monotonically decreasing so that $u_n \le u_3 = 1.5625$, and whence

$$\prod_{j\neq i}\left|\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| < 1.5625. \tag{21}$$

From (10) we obtain for $z = \hat{z}_i$

$$P(\hat{z}_i) = (\hat{z}_i - z_i)\Big(\sum_{j=1}^{n}\frac{W_i}{\hat{z}_i - z_j}+1\Big)\prod_{j\neq i}(\hat{z}_i - z_j)$$

so that, after dividing by $\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)$, we find

$$\widehat{W}_i = (\hat{z}_i - z_i)\Big(\sum_{j=1}^{n}\frac{W_i}{\hat{z}_i - z_j}+1\Big)\prod_{j\neq i}\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}.$$

Starting from the last relation and using the estimates (15), (20) and (21), we obtain

$$|\widehat{W}_i| = |\hat{z}_i - z_i|\Big|\sum_{j=1}^{n}\frac{W_i}{\hat{z}_i - z_j}+1\Big|\prod_{j\neq i}\left|\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| < 1.5|W_i|\cdot\frac{2}{25}\cdot 1.5625 < \frac{1}{5}|W_i|,$$

which proves (ii) of Lemma 1.

We make use of (ii) of Lemma 1, (13) and (17) to get

$$|\widehat{W}_i| < \frac{1}{5}|W_i| < \frac{1}{5}\cdot\frac{2d}{9(n-1)} < \frac{2}{45(n-1)}\cdot\frac{n-1}{n-5/3}\hat{d} < \frac{2}{9(n-1)}\hat{d}.$$

Therefore, we have proved the implication

$$\frac{w}{d} < c_n \quad\Rightarrow\quad \frac{\hat{w}}{\hat{d}} < c_n,$$

which completes the proof of the assertion (iii) of Lemma 1. □

Using Lemma 1 and Theorem 2 we establish the convergence theorem for the iterative method (2).

Theorem 3. *The iterative method* (2) *is convergent under the condition*

$$w^{(0)} < \frac{2}{9(n-1)} d^{(0)}. \tag{22}$$

Proof. From (2) we observe that corrections $C_i^{(m)}$ are given by

$$C_i^{(m)} = \frac{W_i^{(m)}}{1 - \dfrac{P(z_i^{(m)} - W_i^{(m)})}{P(z_i^{(m)})}} = \frac{W_i^{(m)}}{1 - q_i^{(m)}}. \tag{23}$$

Let us note that these corrections have the required form $C_i = \dfrac{P(z_i)}{F(z_1, \dots, z_n)}$ with

$$F(z_1, \dots, z_n) = (1 - q_i) \prod_{j \neq i} (z_i - z_j). \tag{24}$$

To prove the convergence of the method (2), we will show that the assertion (i) of Theorem 2 holds in this particular case. Therefore, we should prove that the sequences $\{C_i^{(m)}\}$ $(i \in I_n)$, given by (23), are monotonically decreasing. Starting from (23) and omitting iteration indices we obtain by (15) (which holds under the condition (22))

$$|C_i| = |\hat{z}_i - z_i| < 1.5|W_i|. \tag{25}$$

In Lemma 1 (assertion (iii)) we have proved the implication

$$\frac{w}{d} < c_n \quad \Rightarrow \quad \frac{\hat{w}}{\hat{d}} < c_n.$$

In a similar way we prove by induction that the initial condition (22) implies $w^{(m)} < c_n d^{(m)}$ for each $m = 1, 2, \dots$. Hence, by (ii) of Lemma 1 we get

$$|W_i^{(m+1)}| < \frac{1}{5}|W_i^{(m)}| \quad (i \in I_n,\ m = 0, 1, \dots).$$

Taking into consideration this inequalities and the inequalities (12) and (24), we find

$$|\widehat{C}_i| < 1.5|\widehat{W}_i| < \frac{3}{10}|W_i| = \frac{3}{10}\left|\frac{W_i}{1 - q_i}\right||1 - q_i| = \frac{3}{10}|C_i||1 - q_i| < \frac{3}{10}|C_i|(1 + 0.32),$$

whence

$$|\widehat{C}_i| < 0.4|C_i|. \tag{26}$$

Using the same argumentation we prove by induction that $|C_i^{(m+1)}| < 0.4|C_i^{(m)}|$ is valid for each $i \in I_n$ and $m = 0, 1, \ldots$. Therefore, the sequences $\{C_i^{(m)}\}$ $(i \in I_n)$ given by (23) are monotonically decreasing, which completes the first part of the theorem (see the assertion (i) of Theorem 2.).

Let us prove now the assertion (ii) of Theorem 2. From (26) we see that $\beta = 0.4 \in (0, 0.5)$ and calculate $g(\beta) = g(0.4) = 1.8$ using formula $g(\beta) = 1 + 2\beta$. By (22) and (25) we find

$$|z_i^{(0)} - z_j^{(0)}| \geq d^{(0)} > \frac{9}{2}(n-1)w^{(0)} > \frac{9(n-1)}{2} \cdot \frac{|C_i^{(0)}|}{1.5} = 3(n-1)|C_i^{(0)}|.$$

Hence, it follows

$$|z_i^{(0)} - z_j^{(0)}| > \frac{3(n-1)}{2}\Big(|C_i^{(0)}| + |C_j^{(0)}|\Big) > g(0.4)\Big(|C_i^{(0)}| + |C_j^{(0)}|\Big)$$

since $3(n-1)/2 > g(0.4) = 1.8$ for every $n \geq 3$.

Finally, since

$$\prod_{j \neq i}(z_i - z_j) \neq 0 \quad \text{and} \quad |q_i| < 0.32 < 1,$$

it follows from (24) that F never takes the value 0. Therefore, the iterative method (2) is well defined. $\square$

REFERENCES

1. P. Batra: *Improvement of a convergence condition for the Durand-Kerner iteration.* J. Comput. Appl. Math. **96** (1998), 117–125.
2. W. Börsch-Supan: *Residuenabschätzung für Polynom-Nullstellen mittels Lagrange-Interpolation.* Numer. Math. **14** (1970), 287–296.
3. I. Gargantini and P. Henrici: *Circular arithmetic and the determination of polynomial zeros.* Numer. Math. **18** (1972), 305–320.
4. M. S. Petković: *On initial conditions for the convergence of simultaneous root finding methods.* Computing **57** (1996), 163–177.
5. M. S. Petković, C. Carstensen and M. Trajković: *Weierstrass' formula and zero-finding methods.* Numer. Math. **69** (1995), 353–372.

6. M. S. Petković and Đ. Herceg: *Point estimation of simultaneous methods for solving polynomial equations: a survey.* J. Comput. Appl. Math. **91** (2001), 283–307.

7. M. S. Petković, Đ. Herceg and S. Ilić: *Point Estimation Theory and its Applications.* Institute of Mathematics, Novi Sad 1997.

8. M. S. Petković, Đ. Herceg and S. Ilić: *Safe convergence of simultaneous methods for polynomial zeros. Numer. Algorithms* **17** (1998), 313–331.

9. M. S. Petković, Đ. Herceg and S. Ilić: *Point estimation and some applications to iterative methods.* BIT **38** (1998), 112–126.

10. M. S. Petković and Lj. D. Petković: *Complex Interval Arithmetic and its Applications.* Wiley-VCH, Berlin-Weinheim-New York 1998.

11. M. S. Petković and Lj. D. Petković: *On the convergence of the sequences of Gerschgorin-like disks.* Numer. Algorithms 42 (2006), 363–377.

12. M. S. Petković and Lj. D. Petković: *On a cubically convergent derivative free root finding method.* Intern. J. Comput. Math. (to appear).

13. M. S. Petković, L. Rančić, Lj. D. Petković: *Point estimation of simultaneous methods for solving polynomial equations: a survey (II).* J. Comput. Appl. Math. 205 (2007), 32–52.

14. S. Smale: *The fundamental theorem of algebra and complexity theory.* Bull. Amer. Math. Soc. **4**, 1–35, (1981).

15. J. F. Traub: *Iterative Methods for the Solution of Equations.* Prentice Hall, Englewood Cliffs, New Jersey 1964.

16. D. Wang and F. Zhao: *On the determination of the safe initial approximation for the Durand-Kerner algorithm.* J. Comput. Appl. Math. **38** (1991), 447–456.

17. D. Wang and F. Zhao: *The theory of Smale's point estimation and its application.* J. Comput. Appl. Math. **60** (1995), 253–269.

18. X. Wang: *Convergence of Newton's method and inverse function theorem.* Math. Comp. **68** (1999), 169–186.

19. K. Weierstrass: *Neuer Beweis des Satzes, dass jede ganze rationale Funktion einer Veränderlichen dargestellt werden kann als ein Product aus linearen Funktionen derstelben Verändelichen.* Ges. Werke **3** (1903), 251–269 (Johnson Reprint Corp., New York 1967).

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