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NORM AND NUMERICAL RADIUS INEQUALITIES FOR SUMS OF BOUNDED LINEAR OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities for the operator norm and numerical radius of sums of bounded linear operators in Hilbert spaces are given. Applications for the Cartesian decomposition of an operator are also provided.

1. Introduction

Let $B(H)$ denote the $C^{*}-$algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. For $A \in B(H)$, let $\|A\|$ denote the the usual operator norm of A.

It is well known that, the following generalized triangle inequality holds true

$$\left\| \sum_{j=1}^{n} A_j \right\| \leq \sum_{j=1}^{n} \|A_j\|$$

for any $A_1, \dots, A_n \in B(H)$ and n a natural number.

In [1] we obtained several inequalities that provide alternative upper bounds for the norm of the sum $\sum_{j=1}^{n} A_j$:

$$(1.1) \quad \left\| \sum_{j=1}^{n} A_j \right\|^2$$

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$$\leq \begin{cases} \sum_{j=1}^{n} \|A_j\|^2 ; \\ n^{1/p} \left(\sum_{j=1}^{n} \|A_j\|^{2q} \right)^{1/q} \\ \text{if } p > 1, \ 1/p + 1/q = 1; \\ n \max_{1 \leq j \leq n} \|A_j\|^2 ; \end{cases} + \begin{cases} \sum_{1 \leq j \neq k \leq n} \|A_j A_k^*\| ; \\ [n(n-1)]^{1/r} \left(\sum_{1 \leq j \neq k \leq n} \|A_j A_k^*\|^s \right)^{1/s} \\ \text{if } r > 1, \ 1/r + 1/s = 1; \\ n(n-1) \max_{1 \leq j \neq k \leq n} \|A_j A_k^*\| ; \end{cases}$$

where (1.1) should be seen as all 9 possible configurations.

Out of these inequalities, one can remark the following results of interest

$$\left\| \sum_{j=1}^{n} A_j \right\| \leq \left(\sum_{j,k=1}^{n} \|A_j A_k^*\| \right)^{1/2} = \left(\sum_{j=1}^{n} \|A_j\|^2 + \sum_{1 \leq j \neq k \leq n} \|A_j A_k^*\| \right)^{1/2} \leq \sum_{j=1}^{n} \|A_j\| , \text{ a refinement of the triangle inequality;} \tag{1.2}$$

$$\left\| \sum_{j=1}^{n} A_j \right\| \leq \sqrt{n} \left[\max_{1 \leq j \leq n} \|A_j\|^2 + (n-1) \max_{1 \leq j \neq k \leq n} \|A_j A_k^*\| \right]^{1/2} ; \tag{1.3}$$

$$\left\| \sum_{j=1}^{n} A_j \right\| \leq \sqrt{n} \left[\max_{1 \leq j \leq n} \|A_j\|^2 + \left(\sum_{1 \leq j \neq k \leq n} \|A_j A_k^*\|^2 \right)^{1/2} \right]^{1/2} \tag{1.4}$$

and

$$\left\| \sum_{j=1}^{n} A_j \right\| \leq n^{1/2p} \left[\left(\sum_{j=1}^{n} \|A_j\|^{2q} \right)^{1/q} + (n-1)^{1/p} \left(\sum_{1 \leq j \neq k \leq n} \|A_j A_k^*\|^q \right)^{1/q} \right]^{1/2} \tag{1.5}$$

for $p, q > 1$ with $1/p + 1/q = 1$, which, for $p = q = 2$ provides

$$\left\| \sum_{j=1}^{n} A_j \right\| \leq n^{1/4} \left[\left(\sum_{j=1}^{n} \|A_j\|^4 \right)^{1/2} + (n-1)^{1/2} \left(\sum_{1 \leq j \neq k \leq n} \|A_j A_k^*\|^2 \right)^{1/2} \right]^{1/2} . \tag{1.6}$$

A different approach employed in [2] has lead to the following inequalities for iterated sums:

$$\left\|\sum_{j=1}^{n} A_j\right\|^2 \leq \begin{cases} \alpha \\ \beta \\ \gamma \end{cases} \tag{1.7}$$

where

$$\alpha := \begin{cases} \sum\limits_{j,k=1}^{n} \|A_j A_k^*\|\,; \\ n^{1/r}\left[\sum\limits_{j=1}^{n}\left(\sum\limits_{k=1}^{n}\|A_j A_k^*\|\right)^s\right]^{1/s} \\ \text{if } r>1,\ 1/r+1/s=1; \\ n\max\limits_{1\leq j\leq n}\left(\sum\limits_{k=1}^{n}\|A_j A_k^*\|\right); \end{cases}$$

$$\beta := \begin{cases} n^{1/p}\sum\limits_{j=1}^{n}\left(\sum\limits_{k=1}^{n}\|A_j A_k^*\|^q\right)^{1/q}; \\ n^{1/t+1/p}\left[\sum\limits_{j=1}^{n}\left(\sum\limits_{k=1}^{n}\|A_j A_k^*\|^q\right)^{u/q}\right]^{1/u} \\ \text{if } u>1,\ 1/u+1/t=1; \\ n^{1+1/p}\max\limits_{1\leq j\leq n}\left(\sum\limits_{k=1}^{n}\|A_j A_k^*\|^q\right)^{1/q}; \end{cases}$$

where $p > 1$, $1/p + 1/q = 1$, and

$$\gamma := \begin{cases} n\sum\limits_{j=1}^{n}\left(\max\limits_{1\leq k\leq n}\|A_j A_k^*\|\right); \\ n^{1/m+1}\left[\sum\limits_{j=1}^{n}\left(\max\limits_{1\leq k\leq n}\|A_j A_k^*\|^{\ell}\right)\right]^{1/\ell} \\ \text{if } m>1,\ 1/m+1/\ell=1; \\ n^2\max\limits_{1\leq j,k\leq n}\|A_j A_k^*\|\,. \end{cases}$$

Note that the choice $p = t = 2$ (therefore $u = q = 2$) will produce from the $\beta-$branch the inequality

$$\left\|\sum_{j=1}^{n} A_j\right\|^2 \leq n\left[\sum_{j=1}^{n}\|A_j\|^4 + \sum_{1\leq j\neq k\leq n}\|A_j A_k^*\|^2\right]^{1/2} \leq n\sum_{j=1}^{n}\|A_j\|^2\,, \tag{1.8}$$

which is a refinement of the Cauchy-Bunyakovsky-Schwarz inequality.

The aim of this paper is to establish various new inequalities for the operator norm and numerical radius of sums of bounded linear operators in Hilbert spaces. In particular, two refinements of the generalized triangle inequality for operator norm are obtained. Particular cases of interest for two bounded linear operators and their applications for the Cartesian decomposition of an operator are also considered.

2. Some General Results

Theorem 2.1. *For any sequence of operators $A_1, \dots, A_n \in B(H)$ we have:*

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \leq \left\|\sum_{k=1}^{n} A_k^* A_k\right\| + \frac{1}{2}\left\|\sum_{1\leq k\neq j\leq n} A_j^* A_k\right\|^2 + \frac{1}{2} \tag{2.1}$$
$$\left(= \left\|\sum_{k=1}^{n} A_k^* A_k\right\| + \frac{1}{2}\left\|\sum_{j=1}^{n} A_j^* \sum_{k=1}^{n} A_k - \sum_{k=1}^{n} A_k^* A_k\right\|^2 + \frac{1}{2}\right).$$

Proof. For any $x \in H$, observe that

$$\begin{aligned} \left\|\sum_{k=1}^{n} A_k x\right\|^2 &= \operatorname{Re}\left[\sum_{j=1}^{n}\sum_{k=1}^{n} \langle A_k x, A_j x\rangle\right] = \sum_{j=1}^{n}\sum_{k=1}^{n} \operatorname{Re}\langle A_k x, A_j x\rangle \\ &= \sum_{k=1}^{n} \|A_k x\|^2 + \sum_{1\leq k\neq j\leq n} \operatorname{Re}\langle A_k x, A_j x\rangle \\ &= \left\langle \left(\sum_{k=1}^{n} A_k^* A_k\right) x, x\right\rangle + \operatorname{Re}\left\langle \sum_{1\leq k\neq j\leq n} \left(A_j^* A_k\right) x, x\right\rangle. \end{aligned} \tag{2.2}$$

Then

$$\left\|\sum_{k=1}^{n} A_k x\right\|^2 \leq \left\langle \left(\sum_{k=1}^{n} A_k^* A_k\right) x, x\right\rangle + \frac{1}{2}\left[\left\|\sum_{1\leq k\neq j\leq n} \left(A_j^* A_k\right) x\right\|^2 + \|x\|^2\right], \tag{2.3}$$

where, for the last inequality we have used the elementary inequality in $(H; \langle\cdot,\cdot\rangle)$:

$$\operatorname{Re}\langle z, u\rangle \leq \frac{1}{2}\left[\|z\|^2 + \|u\|^2\right], \quad z, u \in H.$$

Taking the supremum in (2.3) over $x \in H$, $\|x\| = 1$, we get

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \leq \left\|\sum_{k=1}^{n} A_k^* A_k\right\| + \frac{1}{2}\left\|\sum_{1\leq k\neq j\leq n} A_j^* A_k\right\|^2 + \frac{1}{2}. \tag{2.4}$$

Since

$$\sum_{1\le k\ne j\le n} A_j^* A_k = \sum_{j=1}^{n} A_j^* \sum_{k=1}^{n} A_k - \sum_{k=1}^{n} A_k^* A_k,$$

then the last part of (2.1) is also proved. $\square$

Remark 2.1. For the case of two operators, we can state that

$$\|B+C\|^2 \le \|B^*B + C^*C\| + \frac{1}{2}\|B^*C + C^*B\|^2 + \frac{1}{2} \tag{2.5}$$

for any $B, C \in B(H)$. If in this inequality we choose $B = A$, $C = A^*$, then we get

$$\|A + A^*\|^2 \le \|A^*A + AA^*\| + \frac{1}{2}\left\|A^2 + (A^*)^2\right\|^2 + \frac{1}{2} \tag{2.6}$$

for any $A \in B(H)$.

Now, if $A = B + C$ with $B = (A + A^*)/2$, $C = (A - A^*)/2$, i.e., B and C are the Cartesian decomposition of A, then applying (2.5) for B and C as above will give the inequality

$$\|A\|^2 \le \frac{1}{2}\|A^*A + AA^*\| + \frac{1}{4}\|A^*A - AA^*\|^2 + \frac{1}{2} \tag{2.7}$$

for any $A \in B(H)$.

The following result may be stated as well.

Theorem 2.2. *For any $A_1, \dots, A_n \in B(H)$ we have:*

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \le \left\|\sum_{k=1}^{n} A_k^* A_k\right\| + \frac{1}{2}\left\|(n-2)\sum_{k=1}^{n} A_k^* A_k + \sum_{k=1}^{n} A_k^* \sum_{k=1}^{n} A_k\right\|. \tag{2.8}$$

Proof. Utilising the elementary inequality

$$\operatorname{Re}\langle z, u\rangle \le \frac{1}{4}\|z+u\|^2, \quad z, u \in H,$$

we then have:

$$\begin{aligned}
\sum_{1\le k\ne j\le n} \operatorname{Re}\langle A_k x, A_j x\rangle &\le \frac{1}{4}\sum_{1\le k\ne j\le n} \|(A_k + A_j)x\|^2 \\
&= \frac{1}{4}\sum_{1\le k\ne j\le n} \langle (A_k + A_j)^*(A_k + A_j)x, x\rangle \\
&= \frac{1}{4}\left\langle \sum_{1\le k\ne j\le n} \left(A_k^* + A_j^*\right)(A_k + A_j)x, x\right\rangle.
\end{aligned}$$

Now, on making use of the identity (2.2), we can state that:

$$\left\|\sum_{k=1}^{n} A_k x\right\|^2 \le \left\langle \left(\sum_{k=1}^{n} A_k^* A_k\right) x, x \right\rangle + \frac{1}{4}\left\langle \sum_{1\le k\ne j\le n} \left(A_k^* + A_j^*\right)\left(A_k + A_j\right) x, x \right\rangle \tag{2.9}$$

for any $x \in H$.

Taking the supremum in (2.9), we get

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \le \left\|\sum_{k=1}^{n} A_k^* A_k\right\| + \frac{1}{4}\left\|\sum_{1\le k\ne j\le n} \left(A_k^* + A_j^*\right)\left(A_k + A_j\right)\right\|. \tag{2.10}$$

Since

$$\begin{aligned}
&\sum_{1\le k\ne j\le n} \left(A_k^* + A_j^*\right)\left(A_k + A_j\right) \\
&\quad = \sum_{k,j=1}^{n} \left(A_k^* A_k + A_j^* A_k + A_k^* A_j + A_j^* A_j\right) - 4\sum_{k=1}^{n} A_k^* A_k \\
&\quad = 2n\sum_{k=1}^{n} A_k^* A_k + 2\sum_{k=1}^{n} A_k^* \sum_{k=1}^{n} A_k - 4\sum_{k=1}^{n} A_k^* A_k \\
&\quad = 2\left[(n-2)\sum_{k=1}^{n} A_k^* A_k + \sum_{k=1}^{n} A_k^* \sum_{k=1}^{n} A_k\right],
\end{aligned}$$

hence by (2.10) we deduce the desired inequality (2.8). □

Remark 2.2. Since, by the triangle inequality we have that

$$\left\|(n-2)\sum_{k=1}^{n} A_k^* A_k + \sum_{k=1}^{n} A_k^* \sum_{k=1}^{n} A_k\right\| \le (n-2)\left\|\sum_{k=1}^{n} A_k^* A_k\right\| + \left\|\sum_{k=1}^{n} A_k\right\|^2$$

hence by (2.8) we deduce that

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \le n\left\|\sum_{k=1}^{n} A_k^* A_k\right\|$$

for $n \ge 2$ and $A_1, \ldots, A_n \in B(H)$.

Remark 2.3. The case $n = 2$ provides the following interesting inequality

$$\left\| \frac{B+C}{2} \right\|^2 \leq \left\| \frac{B^*B + C^*C}{2} \right\| \tag{2.11}$$

for any $B, C \in B(H)$. If in this inequality we choose $B = A$, $C = A^*$, then we get

$$\left\| \frac{A+A^*}{2} \right\|^2 \leq \left\| \frac{A^*A + AA^*}{2} \right\| \tag{2.12}$$

for each $A \in B(H)$.

Moreover, if in (2.11) we chose the Cartesian decomposition of an operator A, then we get

$$\|A\|^2 \leq \|A^*A + AA^*\| \tag{2.13}$$

for any $A \in B(H)$.

Remark 2.4. Note that, as pointed out in [5], (2.13) follows by the inequality (33) from [6]. We have shown above that it can be also easily deduced from (2.11).

A similar approach which provides another inequality for the operator norm is incorporated in:

Theorem 2.3. *If $A_1, \dots, A_n \in B(H)$, then*

$$\left\| \sum_{k=1}^{n} A_k \right\|^2 + \sum_{k=1}^{n} \|A_k\|^2 \leq \left\| \sum_{k=1}^{n} A_k^* A_k \right\| + \sum_{k,j=1}^{n} \left\| \frac{A_k + A_j}{2} \right\|^2. \tag{2.14}$$

Proof. Since, by the proof of Theorem 2.2, we have

$$\sum_{1 \leq k \neq j \leq n} \operatorname{Re} \langle A_k x, A_j x \rangle \leq \frac{1}{4} \sum_{1 \leq k \neq j \leq n} \|(A_k + A_j)\, x\|^2,$$

then, by (2.2), we can state that:

$$\left\| \sum_{k=1}^{n} A_k x \right\|^2 \leq \left\langle \left(\sum_{k=1}^{n} A_k^* A_k \right) x, x \right\rangle + \frac{1}{4} \sum_{1 \leq k \neq j \leq n} \|(A_k + A_j)\, x\|^2 \tag{2.15}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over x, $\|x\| = 1$, we deduce that

$$\begin{aligned} \left\| \sum_{k=1}^{n} A_k \right\|^2 &\leq \left\| \sum_{k=1}^{n} A_k^* A_k \right\| + \frac{1}{4} \sum_{1 \leq k \neq j \leq n} \|A_k + A_j\|^2 \\ &= \left\| \sum_{k=1}^{n} A_k^* A_k \right\| + \frac{1}{4} \sum_{k,j=1}^{n} \|A_k + A_j\|^2 - \sum_{k=1}^{n} \|A_k\|^2, \end{aligned}$$

which is exactly the desired result (2.14). $\square$

Remark 2.5. The case $n = 2$ will also provide the inequality (2.11) obtained above from a different inequality.

Finally, we have:

Theorem 2.4. *If $A_1, \dots, A_n \in B(H)$, then*

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \le \left\|\sum_{k=1}^{n} A_k^* A_k\right\| + \frac{1}{4}\left\|\sum_{1\le k\neq j\le n} A_j^* A_k + I\right\|^2. \tag{2.16}$$

Proof. It follows by the identity (2.2) on noticing that

$$\operatorname{Re}\left\langle \sum_{1\le k\neq j\le n} \left(A_j^* A_k\right) x, x\right\rangle \le \frac{1}{4}\left\|\sum_{1\le k\neq j\le n} A_j^* A_k + x\right\|^2$$

for any $x \in H$. The details are omitted. □

Remark 2.6. The case $n = 2$ provides the following inequality

$$\|B + C\|^2 \le \|B^*B + C^*C\| + \frac{1}{4}\|B^*C + C^*B + I\|^2 \tag{2.17}$$

for any $B, C \in B(H)$. If in this inequality we choose $B = A$ and $C = A^*$, then we get

$$\|A + A^*\|^2 \le \|A^*A + AA^*\| + \frac{1}{4}\|A^2 + (A^*)^2 + I\|^2 \tag{2.18}$$

for any $A \in B(H)$.

Finally, the inequality (2.17) provides for the Cartesian decomposition of an operator $A \in B(H)$ the following result as well:

$$\|A\|^2 \le \frac{1}{2}\left[\|AA^* + A^*A\| + \frac{1}{2}\left\|\frac{A^*A - AA^*}{2} + I\right\|^2\right] \tag{2.19}$$

for any $A \in B(H)$.

3. Some Results for Commuting Operators

For a bounded linear operator A, let $w(A)$ denotes its numerical radius [3], [4].

We recall the following results concerning the numerical radius of a product of two operators [3, pp. 37-40] that will be used in deriving various inequalities for the sums of operators on Hilbert spaces:

Theorem 3.1. *For any two bounded linear operators A, B we have*

$$w(AB) \leq 4w(A)\,w(B)\,. \tag{3.1}$$

If A and B commute, i.e., $AB = BA$, then

$$w(AB) \leq 2w(A)\,w(B)\,. \tag{3.2}$$

To get closer to the elusive inequality

$$w(AB) \leq w(A)\,w(B)\,. \tag{3.3}$$

we recall the following result [3, p. 38]:

Theorem 3.2. *If the operators A and B double commute, i.e., $AB = BA$ and $AB^* = B^*A$, then*

$$w(AB) \leq w(B)\,\|A\|\,. \tag{3.4}$$

As a particular case of interest that provides an affirmative answer for the validity of the inequality (3.3) for some pair of operators (A, B), we can state [3, p. 39]:

Corollary 3.2. *Let A be a normal operator, i.e., $AA^* = A^*A$. If A commutes with B, then* (3.3) *holds true.*

Another sufficient condition for (A, B) to satisfy (3.4) is incorporated in [3, p. 39].

Theorem 3.3. *Let A commutes with B and $A^2 = \alpha I$ for some $\alpha \in \mathbb{C}$. Then* (3.4) *holds true.*

Given that A and B are commutative, the question of whether

$$w(AB) \leq w(A)\,\|B\|$$

was open for about twenty years. The issue was finally resolved by V. Miller in 1988, see for instance [3, p. 40]. Miller's approach was computational, and a counterexample was found in a 12-dimensional Hilbert space.

The related question of the best constant $C > 0$ for the inequality

$$w(AB) \leq Cw(A)\,\|B\| \tag{3.5}$$

for commuting A and B has also been considered by K. Okubo and T. Ando in 1976 and by K. Davidson and J. Holbrook in 1988, see for instance [3, p. 42].

For other results on numerical radius, see the classical problem book [4].

The following result may be stated.

Theorem 3.4. *If $A_1, \ldots, A_n \in B(H)$, then*

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \leq 4w\left(\sum_{k=1}^{n} A_k\right)\sum_{j=1}^{n} w(A_j). \tag{3.6}$$

Moreover, if $\sum_{k=1}^{n} A_k$ commutes with each A_j^* for $j \in \{1, \ldots, n\}$, then

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \leq 2w\left(\sum_{k=1}^{n} A_k\right)\sum_{j=1}^{n} w(A_j). \tag{3.7}$$

Proof. For any $x \in H$ we have

$$\begin{aligned} \left\|\sum_{k=1}^{n} A_k x\right\|^2 &= \left|\sum_{j=1}^{n}\left\langle\left(\sum_{k=1}^{n} A_k\right)x, A_j x\right\rangle\right| \\ &\leq \sum_{j=1}^{n}\left|\left\langle\left(\sum_{k=1}^{n} A_k\right)x, A_j x\right\rangle\right| = \sum_{j=1}^{n}\left|\left\langle A_j^*\left(\sum_{k=1}^{n} A_k\right)x, x\right\rangle\right|. \end{aligned} \tag{3.8}$$

Taking the supremum over $x \in H$, $\|x\| = 1$ in (3.8), we get

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \leq \sum_{j=1}^{n} w\left[A_j^*\left(\sum_{k=1}^{n} A_k\right)\right], \tag{3.9}$$

which is an inequality of interest in itself.

Now, since, by (3.1) applied for A_j^* $(j = 1, \ldots, n)$ and $\sum_{k=1}^{n} A_k$ we can state that

$$w\left[A_j^*\left(\sum_{k=1}^{n} A_k\right)\right] \leq 4w(A_j)\, w\left(\sum_{k=1}^{n} A_k\right) \tag{3.10}$$

for any $j \in \{1, \ldots, n\}$, then by (3.9) and (3.10) we deduce (3.6).

Now, if A_j^* $(j = 1, \ldots, n)$ commutes with $\sum_{k=1}^{n} A_k$, then on utilising the second part of Theorem 3.1 we can state that:

$$w\left[A_j^*\left(\sum_{k=1}^{n} A_k\right)\right] \leq 2w(A_j)\, w\left(\sum_{k=1}^{n} A_k\right), \quad j \in \{1, \ldots, n\} \tag{3.11}$$

which, by (3.9) will imply the desired inequality (3.7). $\square$

The following particular case may be of interest.

Corollary 3.4. *If $A_1, \dots, A_n \in B(H)$ are normal operators and $*-$commute with each other, i.e., $A_k A_j^* = A_j^* A_k$ for $k, j \in \{1, \dots, n\}$, $k \neq j$, then* (3.7) *holds true.*

Remark 3.1. It is useful to observe that the case $n = 2$ in the inequality (3.9) provides the result

$$\|B + C\|^2 \le w(B^*B + B^*C) + w(C^*B + C^*C) \tag{3.12}$$

for any $B, C \in B(H)$. In particular, for $B = A$ and $C = A^*$, we get from (3.12) the inequality

$$\|A + A^*\|^2 \le w\left(A^*A + (A^*)^2\right) + w\left(A^2 + AA^*\right) \tag{3.13}$$

for any $A \in B(H)$.

Now, if we assume in (3.12) that B and C are the Cartesian decomposition of $A \in B(H)$, then

$$\|A\|^2 \le \frac{1}{2}\left[w\left(A^2 + A^*A\right) + w\left(A^2 - A^*A\right)\right].$$

The constant $\frac{1}{2}$ here is best possible in the sense that it cannot be replaced by a smaller constant. The equality case is, for instance, realised when A is a self adjoint operator.

A more interesting case which provides refinements of the generalized triangle inequality for operator norm is incorporated in the following:

Theorem 3.5. *Let $A_1, \dots, A_n \in B(H)$. If A_j^* $(j = 1, \dots, n)$ double commutes with $\sum_{k=1}^n A_k$, then we have the following refinement of the generalized triangle inequality:*

$$\left\|\sum_{k=1}^n A_k\right\| \le \sum_{k=1}^n w(A_k) \left(\le \sum_{k=1}^n \|A_k\|\right). \tag{3.14}$$

Moreover, if $\sum_{k=1}^n A_k \neq 0$, then also

$$(1 \le) \frac{\left\|\sum_{k=1}^n A_k\right\|}{w\left(\sum_{k=1}^n A_k\right)} \le \frac{\sum_{k=1}^n \|A_k\|}{\left\|\sum_{k=1}^n A_k\right\|}. \tag{3.15}$$

Proof. If $\sum_{k=1}^{n} A_k = 0$, then (3.14) is obvious.

Now, suppose $\sum_{k=1}^{n} A_k \neq 0$ and A_j^* $(j = 1, \dots, n)$ double commutes with $\sum_{k=1}^{n} A_k$. By Theorem 3.2, we then have

$$w\left[A_j^*\left(\sum_{k=1}^{n} A_k\right)\right] \leq w\left(A_j\right)\left\|\sum_{k=1}^{n} A_k\right\|, \quad j \in \{1, \dots, n\}$$

which, by (3.9), implies that

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \leq \left\|\sum_{k=1}^{n} A_k\right\| \sum_{j=1}^{n} w\left(A_j\right),$$

which is clearly equivalent with (3.14).

By the same Theorem 3.2, we can also state that

$$w\left[A_j^*\left(\sum_{k=1}^{n} A_k\right)\right] \leq w\left(\sum_{k=1}^{n} A_k\right)\left\|A_j\right\|, \quad j \in \{1, \dots, n\},$$

which, together with (3.9) imply (3.15). □

The following corollary may be stated.

Corollary 3.5. *If $A_1, \dots, A_n$ are normal operators which double commute with each other, i.e., $A_k A_j = A_j A_k$ and $A_k^* A_j = A_j A_k^*$ for each $k, j \in \{1, \dots, n\}$; then both* (3.14) *and* (3.15) *hold true.*

Utilising Theorem 3.3 we are able to state the following result as well.

Corollary 3.5. *Let $A_1, \dots, A_n \in B(H)$ such that A_j^* commutes with the sum $\sum_{k=1}^{n} A_k$. If either $\left(\sum_{k=1}^{n} A_k\right)^2 = \beta I$ for some $\beta \in \mathbb{C}$ or $(A_j^*)^2 = \beta_j I$ for some $\beta_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$; are valid, then both* (3.14) *and* (3.15) *hold true.*

From a different perspective, we can state the following result.

Theorem 3.6. *Let $A_1, \dots, A_n \in B(H)$. Then*

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 + 4\sum_{k=1}^{n} w^2\left(A_k\right) \leq \left\|\sum_{k=1}^{n} A_k^* A_k\right\| + 4\left(\sum_{k=1}^{n} w\left(A_k\right)\right)^2. \tag{3.16}$$

If A_j^* commutes with A_k for any $k, j \in \{1, \dots, n\}$, $k \neq j$, then

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 + 2\sum_{k=1}^{n} w^2\left(A_k\right) \leq \left\|\sum_{k=1}^{n} A_k^* A_k\right\| + 2\left(\sum_{k=1}^{n} w\left(A_k\right)\right)^2. \tag{3.17}$$

Proof. Let $x \in H$. Then

$$\left\|\sum_{k=1}^{n} A_k x\right\|^2 = \left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle A_k x, A_j x\rangle\right| \tag{3.18}$$
$$= \left|\sum_{k=1}^{n}\langle A_k x, A_k x\rangle + \sum_{1\le k\neq j\le n}\langle A_k x, A_j x\rangle\right|$$
$$\le \left|\sum_{k=1}^{n}\langle A_k^* A_k x, x\rangle\right| + \sum_{1\le k\neq j\le n}\left|\left\langle A_j^* A_k x, x\right\rangle\right|$$
$$= \left|\left\langle\left(\sum_{k=1}^{n} A_k^* A_k\right)x, x\right\rangle\right| + \sum_{1\le k\neq j\le n}\left|\left\langle A_j^* A_k x, x\right\rangle\right|.$$

Taking the supremum in (3.18) over $x \in H$, $\|x\| = 1$, we deduce the following inequality

$$\left\|\sum_{k=1}^{n} A_k\right\|^2 \le \left\|\sum_{k=1}^{n} A_k^* A_k\right\| + \sum_{1\le k\neq j\le n} w(A_j^* A_k), \tag{3.19}$$

that is of interest in itself.

Since, by Theorem 3.1, in general we have

$$w\left(A_j^* A_k\right) \le 4w\left(A_j\right)w\left(A_k\right), \quad k, j \in \{1, \ldots, n\}, k \neq j$$

then

$$\sum_{1\le k\neq j\le n} w(A_j^* A_k) \le 4\sum_{1\le k\neq j\le n} w(A_j)w(A_k)$$
$$= 4\left[\sum_{i,j=1} w(A_j)w(A_k) - \sum_{k=1}^{n} w^2(A_k)\right]$$
$$= 4\left[\left(\sum_{k=1}^{n} w(A_k)\right)^2 - \sum_{k=1}^{n} w^2(A_k)\right],$$

which, together with (3.19) provides the desired result (3.16).

The second inequality follows by making use of the second part of Theorem 3.1. The details are omitted. □

Remark 3.2. The case $n = 2$ in (3.19) will produce the following sharp inequality

$$\left\|\frac{B+C}{2}\right\|^2 \le \frac{1}{2}\left[\left\|\frac{B^*B + C^*C}{2}\right\| + w\left(B^*C\right)\right], \tag{3.20}$$

for any $B, C \in B(H)$. Here, the multiplicative constant $1/2$ is best possible. We get equality in (3.20) if $B = C$.

If we choose $B = A^*$ and $C = A$ in (3.20), then we get

$$\left\| \frac{A + A^*}{2} \right\|^2 \leq \frac{1}{2} \left[\left\| \frac{A^* A + AA^*}{2} \right\| + w\left(A^2\right) \right] \tag{3.21}$$

for any $A \in B(H)$. Here the constant $1/2$ is also best possible. The equality case holds if A is self adjoint.

Finally, if B and C are chosen in (3.20) to be the Cartesian decomposition of an operator A, then we have

$$\|A\|^2 \leq \frac{1}{2} \left[\|A^* A + AA^*\| + w\left[(A + A^*)(A - A^*)\right] \right].$$

The constant $1/2$ is sharp.

If one were to place more conditions on the operators involved, the following result could be stated as well:

Theorem 3.7. *If $A_1, \ldots, A_n \in B(H)$ are such that A_j^* double commutes with A_k for $k, j \in \{1, \ldots, n\}$, $k \neq j$, then*

$$\left\| \sum_{k=1}^{n} A_k \right\|^2 + \sum_{k=1}^{n} w(A_k) \|A_k\| \leq \left\| \sum_{k=1}^{n} A_k^* A_k \right\| + \sum_{k=1}^{n} w(A_k) \cdot \sum_{k=1}^{n} \|A_k\|. \tag{3.22}$$

Proof. Since A_j^* double commutes with A_k for $k, j \in \{1, \ldots, n\}$, $k \neq j$, then, by Theorem 3.2 we have

$$w(A_j^* A_k) \leq w(A_j)\|A_k\|,$$

for $k, j \in \{1, \ldots, n\}$, $k \neq j$. This implies that

$$\begin{aligned} \sum_{1 \leq k \neq j \leq n} w\left(A_j^* A_k\right) &\leq \sum_{1 \leq k \neq j \leq n} w(A_j) \|A_k\| \\ &= \sum_{i,j=1}^{n} w(A_j) w(A_k) - \sum_{k=1}^{n} w(A_j) \|A_k\| \\ &= \sum_{k=1}^{n} w(A_k) \cdot \sum_{k=1}^{n} \|A_k\| - \sum_{k=1}^{n} w(A_k) \|A_k\|. \end{aligned} \tag{3.23}$$

Now, utilising (3.19) and (3.23), we deduce the desired result (3.22). $\square$

REFERENCES

1. S.S. Dragomir: *Some Schwarz type inequalities for sequences of operators in Hilbert spaces*. Bull. Austral. Math. Soc. **73** (2006), 17–26.
2. S.S. Dragomir: *Norm inequalities for sequences of operators related to the Schwarz inequality*. RGMIA Res. Rep. Coll. **9** (2006), Supplement, Article 2 [Online `http://rgmia.vu.edu.au/v9(E).html`].
3. K.E. Gustafson and D.K.M. Rao: *Numerical Range*. Springer-Verlag, New York, Inc., 1997.
4. P.R. Halmos: *A Hilbert Space Problem Book*. Springer-Verlag, New York, Heidelberg, Berlin, Second Edition, 1982.
5. F. Kittaneh: *Norm inequalities for sums and differences of positive operators*. Lin. Alg. & Appl. **383** (2004), 85–91.
6. F. Kittaneh: *Numerical radius inequalities for Hilbert space operators*. Studia Math. **168** (2005), No. 1, 73–80.

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