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STOCHASTIC KOROVKIN THEORY GIVEN QUANTITATIVELY

George A. Anastassiou

Abstract. We introduce and study very general stochastic positive linear operators induced by general positive linear operators that are acting on continuous functions. These are acting on the space of real differentiable stochastic processes. Under some very mild, general and natural assumptions on the stochastic processes we produce related stochastic Shisha–Mond type inequalities of L^q-type $1 \leq q < \infty$ and corresponding stochastic Korovkin type theorems. These are regarding the stochastic q-mean convergence of a sequence of stochastic positive linear operators to the stochastic unit operator for various cases. All convergences are produced with rates and are given via the stochastic inequalities involving the stochastic modulus of continuity of the $n-$th derivative of the engaged stochastic process, $n \geq 0$. The impressive fact is that the basic real Korovkin test functions assumptions are enough for the conclusions of our stochastic Korovkin theory. We give an application.

1. Introduction

Motivation for this work are [1], [2], [7], [8]. We introduce the stochastic positive linear operator M, see (2.1), based on a general positive linear operator $\tilde{L}$ from $C([a,b])$ into itself. The operator M is acting on a wide space of differentiable real valued stochastic processes X.

We give the definition of q-mean first modulus of continuity, $1 \leq q < \infty$, see (2.4), and we prove important properties of it, such as in Proposition 2.2. Here we assume that $X^{(n)}(x,\omega)$ is continuous in $x \in [a,b]$, uniformly with

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respect to $\omega \in \Omega$—the probability space, $n \geq 0$. We assume also the integrability conditions (2.7) or the one in Assumption 3. We first give the pointwise stochastic Shisha–Mond type inequalities, see (2.8), (2.11), (2.15) and (2.18). Then we derive the corresponding uniform stochastic Shisha–Mond type inequalities (2.9), (2.12), (2.16) and (2.19). From these we establish the stochastic Korovkin type Theorems 2, 4, 6 and 8. These are regarding the q-mean convergence of a sequence of stochastic positive linear operators $\{M_N\}_{N\in\mathbb{N}}$ as in (2.1) to the stochastic unit operator I.

The impressive thing here is that the basic Korovkin real assumptions are enough to enforce our conclusions at the stochastic setting. So our stochastic inequalities that involve the q-mean first modulus of continuity of $X^{(n)}$ describe quantitatively and with rates the above convergence. At the end we give an application regarding the stochastic Bernstein operators where we apply the stochastic inequality (2.19).

2. Results

Concepts 1. Let $\tilde{L}$ be a positive linear operator from $C([a,b])$ into itself. Let $X(t,\omega)$ be a stochastic process from $[a,b]\times(\Omega,\mathcal{B},P)$ into $\mathbb{R}$, where $(\Omega,\mathcal{B},P)$ is a probability space. Here we assume that $X(\cdot,\omega)\in C^n([a,b])$, for each $\omega\in\Omega$ and $X^{(k)}(t,\cdot)$ is measurable for all $k=0,1,\ldots,n$, for each $t\in[a,b]$, $n\geq 0$.

Define

$$M(X)(t,\omega) := \tilde{L}(X(\cdot,\omega))(t), \quad \omega\in\Omega, \quad t\in[a,b], \tag{2.1}$$

and assume that it is a random variable in ω. Clearly M is a positive linear operator on stochastic processes.

We make

Remark 2.1. By the Riesz representation theorem we have that there exists μ_t unique, completed Borel measure on $[a,b]$ with

$$m_t := \mu_t([a,b]) = \tilde{L}(1)(t) \geq 0,$$

such that

$$\tilde{L}(f)(t) = \int_{[a,b]} f(x)d\mu_t(x),$$

for each $t\in[a,b]$ and all $f\in C([a,b])$. Consequently we have that

$$M(X)(t,\omega) = \int_{[a,b]} X(x,\omega)d\mu_t(x), \quad (t,\omega)\in[a,b]\times\Omega,$$

and X as above.

We make

Remark 2.2. Let $n \geq 1$. Using the Taylor formula with $t \in [a,b]$ fixed momentarily, we get

$$X(s,\omega) = \sum_{k=0}^{n} \frac{X^{(k)}(t,\omega)}{k!}(s-t)^k + \int_t^s \left(X^{(n)}(x,\omega) - X^{(n)}(t,\omega)\right)\frac{(s-x)^{n-1}}{(n-1)!}dx,$$

for all $s \in [a,b]$. Therefore we obtain

$$\begin{aligned} M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t) &= \int_{[a,b]} X(s,\omega)\mu_t(ds) - X(t,\omega)\tilde{L}(1)(t) \\ &= \sum_{k=1}^{n} \frac{X^{(k)}(t,\omega)}{k!}\left(\tilde{L}((\cdot - t)^k)(t)\right) \\ &\quad + \int_{[a,b]} \left(\int_t^s \left(X^{(n)}(x,\omega) - X^{(n)}(t,\omega)\right)\frac{(s-x)^{n-1}}{(n-1)!}dx\right)\mu_t(ds), \end{aligned}$$

for each $t \in [a,b]$.

Furthermore we have

$$\begin{aligned} |M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)| &\leq \sum_{k=1}^{n} \frac{|X^{(k)}(t,\omega)|}{k!}|\tilde{L}((\cdot - t)^k)(t)| \\ &+ \frac{1}{(n-1)!}\int_{[a,b]} \left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right|\mu_t(ds), \end{aligned} \tag{2.2}$$

for each $t \in [a,b]$.

We also make

Remark 2.3. Here we are working on the remainder of (2.2). Let $p,q > 1$: $1/p + 1/q = 1$, i.e. $p = q/(q-1)$. We notice by Hölder's inequality that

$$\begin{aligned} &\left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right| \\ &\qquad \leq \left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|^q dx\right|^{1/q} \frac{|t-s|^{\frac{qn-1}{q}}(q-1)^{\frac{q-1}{q}}}{(qn-1)^{\frac{q-1}{q}}}. \end{aligned}$$

Hence we have

$$\begin{aligned} &\left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right|^q \\ &\qquad \leq \left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|^q dx\right| \frac{|t-s|^{qn-1}(q-1)^{q-1}}{(qn-1)^{q-1}}. \end{aligned} \tag{2.3}$$

Applying again Hölder's inequality we obtain

$$\begin{aligned}\Gamma \quad &:= \quad \frac{1}{(n-1)!}\left(\int_\Omega\left\{\int_{[a,b]}\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|\right.\right.\right.\\ &\qquad\qquad \left.\left.\left.\times\, |s-x|^{n-1}dx\right|\mu_t(ds)\right\}^q P(d\omega)\right)^{1/q}\\ &\le \quad \frac{(\tilde{L}(1)(t))^{\frac{q-1}{q}}}{(n-1)!}\left(\int_\Omega\left\{\int_{[a,b]}\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|\right.\right.\right.\\ &\qquad\qquad \left.\left.\left.\times\, |s-x|^{n-1}dx\right|^q \mu_t(ds)\right\}P(d\omega)\right)^{1/q}\\ &\overset{\text{by (9)}}{\le} \quad c_0(t,q,n)\left(\int_\Omega\left(\int_{[a,b]}\left(\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|^q dx\right|\right.\right.\right.\\ &\qquad\qquad \left.\left.\left.\times\, |t-s|^{qn-1}\right)\mu_t(ds)\right)P(d\omega)\right)^{1/q} =: (*),\end{aligned}$$

where

$$c_0(t,q,n)=\frac{1}{(n-1)!}\cdot\left(\frac{\tilde{L}(1)(t)(q-1)}{qn-1}\right)^{1-1/q}.$$

Here $\varphi(x,\omega) := |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|^q \ge 0$, is a real valued random variable for each $x\in[a,b]$, as well continuous in x, and thus by [3, Proposition 3.3(i)], it is jointly measurable in (x,ω). And from the proof of [3, Proposition 3.3], the integral $\int_t^s \varphi(x,\omega)dx$ is a real valued random variable.

Hence

$$\lambda(s,\omega) := \left|\int_t^s \varphi(x,\omega)dx\right| |t-s|^{qn-1}$$

is a real valued random variable, which is continuous in $s\in[a,b]$, i.e. it is Borel measurable on $[a,b]$. Again by [3, Proposition 3.3(i)], $\lambda(s,\omega)$ is jointly measurable in (s,ω).

Therefore by applying Tonelli–Fubini's theorem, see [4, p. 104], we get that

$$\begin{aligned}(*) \quad &= \quad c_0(t,q,n)\left(\int_{[a,b]}\left(\int_\Omega\left(\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|^q dx\right|\right.\right.\right.\\ &\qquad\qquad \left.\left.\left.\times\, |t-s|^{qn-1}\right)P(d\omega)\right)\mu_t(ds)\right)^{1/q}\\ &= \quad c_0(t,q,n)\left(\int_{[a,b]}\left(\int_\Omega\left(\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|^q dx\right|P(d\omega)\right)\right.\right.\end{aligned}$$

$$\times |t-s|^{qn-1}\mu_t(ds)\Bigg)^{1/q}$$

$$= \quad c_0(t,q,n)\left(\int_{[a,b]}\left(\left|\int_t^s\left(\int_\Omega |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|^q P(d\omega)\right)dx\right|\right)\right.$$
$$\left.\times |t-s|^{qn-1}\right)\mu_t(ds)\Bigg)^{1/q}.$$

Thus so far we have proved that

Lemma 2.1. *It holds*

$$\Gamma \quad := \quad \frac{1}{(n-1)!}\left(\int_\Omega\left(\int_{[a,b]}\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|\right.\right.\right.$$
$$\left.\left.\left.\times |s-x|^{n-1}dx\right|\mu_t(ds)\right)^q P(d\omega)\right)^{1/q}$$

$$\leq \quad c_0(t,q,n)\left(\int_{[a,b]}\left(\left|\int_t^s\left(\int_\Omega |X^{(n)}(x,\omega)- \ X^{(n)}(t,\omega)|^q P(d\omega)\right)dx\right|\right)\right.$$
$$\left.\times |t-s|^{qn-1}\right)\mu_t(ds)\Bigg)^{1/q}, \quad q>1,\ n\geq 1.$$

We give

Definition 2.1. We define the q-mean first modulus of continuity of X by

$$\text{(2.4)} \quad \Omega_1(X,\delta)_{L^q} := \sup\left\{\left(\int_\Omega |X(x,\omega)-X(y,\omega)|^q P(d\omega)\right)^{1/q} : \right.$$
$$\left. x,y\in[a,b],\ |x-y|\leq\delta\right\}, \quad \delta>0,\ 1\leq q<\infty.$$

Definition 2.2. Let $1\leq q<\infty$. Let $X(x,\omega)$ be a real stochastic process. We call X a q-mean uniformly continuous stochastic process (or random function) over $[a,b]$, iff $\forall\varepsilon>0\ \exists\delta>0$: whenever $|x-y|\leq\delta$; $x,y\in[a,b]$ implies that

$$\int_\Omega |X(x,s)-X(y,s)|^q P(ds)\leq\varepsilon.$$

We denote it as $X\in C_{\mathbb{R}}^{Uq}([a,b])$.

It holds

Proposition 2.1. *Let $X \in C_{\mathbb{R}}^{Uq}([a,b])$, then $\Omega_1(X,\delta)_{L^q} < \infty$, any $\delta > 0$.*

Proof. Similar to the proof of [3, Proposition 3.1]. □

Also it holds

Proposition 2.2. *Let $X(t,\omega)$ be a stochastic process from $[a,b] \times (\Omega, \mathcal{B}, P)$ into $\mathbb{R}$. The following are true:*

(i) *$\Omega_1(X,\delta)_{L^q}$ is nonnegative and nondecreasing in $\delta > 0$.*

(ii) *$\lim_{\delta \downarrow 0} \Omega_1(X,\delta)_{L^q} = \Omega_1(X,0)_{L^q} = 0$, iff $X \in C_{\mathbb{R}}^{Uq}([a,b])$.*

(iii) *$\Omega_1(X,\delta_1+\delta_2)_{L^q} \le \Omega_1(X,\delta_1)_{L^q} + \Omega_1(X,\delta_2)_{L^q}$, $\delta_1,\delta_2 > 0$.*

(iv) *$\Omega_1(X,n\delta)_{L^q} \le n\Omega_1(X,\delta)_{L^q}$, $\delta > 0$, $n \in \mathbb{N}$.*

(v)
$$\Omega_1(X,\lambda\delta)_{L^q} \le \lceil\lambda\rceil\Omega_1(X,\delta)_{L^q} \le (\lambda+1)\Omega_1(X,\delta)_{L^q},$$
$\lambda > 0$, $\delta > 0$, where $\lceil\cdot\rceil$ is the ceiling of the number.

(vi) *$\Omega_1(X+Y,\delta)_{L^q} \le \Omega_1(X,\delta)_{L^q} + \Omega_1(Y,\delta)_{L^q}$, $\delta > 0$.*

(vii) *$\Omega_1(X,\cdot)_{L^q}$ is continuous on $\mathbb{R}_+$ for $X \in C_{\mathbb{R}}^{Uq}([a,b])$.*

Proof. Obvious. □

We give

Remark 2.4. By Proposition 2.2 (v) we get

$$\Omega_1(X,|x-y|)_{L^q} \le \left\lceil \frac{|x-y|}{\delta} \right\rceil \Omega_1(X,\delta)_{L^q}, \quad x,y \in [a,b],\ \delta > 0. \tag{2.5}$$

Assumption 1. Let $n \ge 0$.

Here we assume that $X^{(n)}(x,\omega)$ is *continuous in $x \in [a,b]$, uniformly with respect to $\omega \in \Omega$*. I.e. $\forall\varepsilon > 0\ \exists\delta > 0$: whenever $|x-y| \le \delta$; $x,y \in [a,b]$, then

$$|X^{(n)}(x,\omega) - X^{(n)}(y,\omega)| \le \varepsilon, \quad \omega \in \Omega.$$

We denote this by $X^{(n)} \in C_{\mathbb{R}}^{U}([a,b])$, *the space of continuous in x, uniformly with respect to ω, stochastic processes.*

Hence here $X^{(n)}(\cdot,\omega) \in C([a,b])$, $\omega \in \Omega$ and $X^{(n)}$ is q-mean uniformly continuous in $t \in [a,b]$, that is $X^{(n)} \in C_{\mathbb{R}}^{Uq}([a,b])$, for any $1 \le q < \infty$.

We make

Remark 2.5. We continue work on the remainder of (2.2). We observe the following $(q > 1)$,

$$c_0(t,q,n)\left(\int_{[a,b]}\left(\left|\int_t^s\left(\int_\Omega |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|^q P(d\omega)\right)dx\right|\right)\right.$$
$$\left.\times\, |t-s|^{qn-1}\Big)\mu_t(ds)\right)^{1/q}$$
$$\leq\ c_0(t,q,n)\left(\int_{[a,b]}(\left(\left|\int_t^s \left(\Omega_1^q(X^{(n)},|x-t|_{L^q}\right)dx\right|\right)|t-s|^{qn-1}\Big)\mu_t(ds)\right)^{1/q}$$

(let $h > 0$)

(by (2.5))

$$\leq\ c_0(t,q,n)\left(\int_{[a,b]}\left(\left|\int_t^s\left(\left\lceil\frac{|x-t|}{h}\right\rceil^q \Omega_1^q(X^{(n)},h)_{L^q}\right)dx\right|\right)\right.$$
$$\left.\times\, |t-s|^{qn-1}\Big)\mu_t(ds)\right)^{1/q}$$
$$\leq\ \Omega_1(X^{(n)},h)_{L^q}c_0(t,q,n)\left(\int_{[a,b]}\left(\left|\int_t^s\left(1+\frac{|x-t|}{h}\right)^q dx\right|\right)\right.$$
$$\left.\times\, |t-s|^{qn-1}\Big)\mu_t(ds)\right)^{1/q} =:\ (**).$$

Call $\tau := 2^{1-1/q}c_0(t,q,n)\Omega_1(X^{(n)},h)_{L^q}$. Hence we have

$$(**)\ \leq\ \tau\left(\int_{[a,b]}\left(\left|\int_t^s\left(1+\frac{|x-t|^q}{h^q}\right)dx\right|\right)|t-s|^{qn-1}\Big)\mu_t(ds)\right)^{1/q}$$
$$\leq\ \tau\left(\int_{[a,b]}\left(|s-t|+\frac{1}{h^q}\left|\int_t^s|x-t|^q dx\right|\right)|t-s|^{qn-1}\Big)\mu_t(ds)\right)^{1/q}$$
$$=\ \tau\left(\int_{[a,b]}\left(|s-t|+\frac{1}{h^q}\frac{|t-s|^{q+1}}{(q+1)}\right)|t-s|^{qn-1}\Big)\mu_t(ds)\right)^{1/q}$$
$$=\ \tau\left(\left(\int_{[a,b]}|s-t|^{qn}\mu_t(ds)\right)+\frac{1}{h^q(q+1)}\left(\int_{[a,b]}|t-s|^{q(n+1)}\mu_t(ds)\right)\right)^{1/q}$$
$$\leq\ \tau\left[m_t^{1/n+1}\left(\int_{[a,b]}|s-t|^{q(n+1)}\mu_t(ds)\right)^{n/n+1}\right.$$
$$\left.+\frac{1}{h^q(q+1)}\left(\int_{[a,b]}|t-s|^{q(n+1)}\mu_t(ds)\right)\right)^{1/q} =: (***).$$

We set and assume that

$$\begin{aligned} h &:= \left(\frac{1}{(q+1)} \int_{[a,b]} |t-s|^{q(n+1)} \mu_t(ds) \right)^{1/q(n+1)} \\ &= \left(\frac{1}{(q+1)} \tilde{L}(|t-\cdot|^{q(n+1)})(t) \right)^{1/q(n+1)} > 0. \end{aligned}$$

I.e.

$$h^{q(n+1)} = \frac{1}{(q+1)} \left(\int_{[a,b]} |t-s|^{q(n+1)} \mu_t(ds) \right) > 0.$$

Therefore

$$(***) = \tau \left[m_t^{1/n+1} h^{qn} (q+1)^{n/n+1} + h^{qn} \right]^{1/q} = \tau h^n \left[m_t^{1/n+1} (q+1)^{n/n+1} + 1 \right]^{1/q}.$$

We have proved that

$$\begin{aligned} \Gamma &= \frac{1}{(n-1)!} \left(\int_\Omega \left(\int_{[a,b]} \left| \int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)| \right. \right. \right. \\ &\qquad \left. \left. \left. \times |s-x|^{n-1} dx \right| \mu_t(ds) \right)^q P(d\omega) \right)^{1/q} \\ &\leq \tau h^n \left[m_t^{1/n+1} (q+1)^{n/n+1} + 1 \right]^{1/q}. \end{aligned}$$

We have established

Lemma 2.2. *It holds*

$$\begin{aligned} \Gamma &\leq [((\tilde{L}(1))(t))^{1/(n+1)} (q+1)^{n/(n+1)} + 1]^{1/q} \frac{1}{(n-1)!(q+1)^{\frac{n}{q(n+1)}}} \\ &\quad \times ((\tilde{L}(|\cdot - t|^{q(n+1)})(t))^{\frac{n}{q(n+1)}} \cdot \left(\frac{2(q-1)\tilde{L}(1)(t)}{qn-1} \right)^{1-\frac{1}{q}} \\ &\quad \times \Omega_1 \left(X^{(n)}, \frac{1}{(q+1)^{\frac{1}{q(n+1)}}} \cdot (\tilde{L}(|\cdot - t|^{q(n+1)})(t))^{\frac{1}{q(n+1)}} \right)_{L^q}, \end{aligned}$$

$q > 1$, $n \geq 1$.

We make

Remark 2.6. Here we see that

$$|M(X)(t,\omega)-X(t,\omega)| \leq |M(X)(t,\omega)-X(t,\omega)\tilde{L}(1)(t)| + |X(t,\omega)|\,|\tilde{L}(1)(t)-1|. \tag{2.6}$$

Combining (2.6) with (2.2) we have that

$$|M(X)(t,\omega)-X(t,\omega)| \leq |X(t,\omega)|\,|\tilde{L}(1)(t)-1| + \sum_{k=1}^{n} \frac{|X^{(k)}(t,\omega)|}{k!}|\tilde{L}((\cdot-t)^k)(t)| + \frac{1}{(n-1)!}\left(\int_{[a,b]}\left|\int_t^s |X^{(n)}(x,\omega)-X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right|\mu_t(ds)\right),$$

for all $t \in [a,b]$.

We need

Definition 2.3. Denote by

$$(EX)(t) := \int_\Omega X(t,\omega)P(d\omega), \quad t \in [a,b],$$

the expectation operator.

We make

Assumption 2. We assume that

$$(E|X^{(k)}|^q)(t) < \infty, \quad t \in [a,b] \tag{2.7}$$

and for all $k = 0, 1, \ldots, n$; $n \geq 0$.

Based on all the above it holds

Theorem 2.1. *Suppose Concepts* 1, $1 < q < \infty$, *Assumptions* 1 *and* 2, $n \geq 1$. *Then*

$$\begin{aligned}
&(E(|M(X)-X|^q)(t))^{1/q} \\
&\leq ((E|X|^q)(t))^{1/q}|\tilde{L}(1)(t)-1| + \sum_{k=1}^{n} \frac{((E|X^{(k)}|^q)(t))^{1/q}}{k!}|\tilde{L}((\cdot-t)^k)(t)| \\
&\quad + \left(\frac{2(q-1)\tilde{L}(1)(t)}{qn-1}\right)^{1-\frac{1}{q}} \cdot \frac{1}{(n-1)!(q+1)^{\frac{n}{q(n+1)}}} \\
&\quad \times \left[(\tilde{L}(1)(t))^{1/(n+1)}(q+1)^{n/(n+1)}+1\right]^{1/q} \cdot \left(\tilde{L}(|\cdot-t|^{q(n+1)})(t)\right)^{\frac{n}{q(n+1)}} \\
&\quad \times \Omega_1\left(X^{(n)}, \frac{1}{(q+1)^{\frac{1}{q(n+1)}}}\left(\tilde{L}(|\cdot-t|^{q(n+1)})(t)\right)^{\frac{1}{q(n+1)}}\right)_{L^q}, \quad t \in [a,b].
\end{aligned} \tag{2.8}$$

Note 1. If $\tilde{L}(|\cdot - t|^{q(n+1)})(t) = 0$, then (2.8) holds trivially as equality.

We further present

Corollary 2.1. *Suppose Concepts* 1, $1 < q < \infty$, *Assumptions* 1 *and* 2, $n \geq 1$. *Then*

$$
\begin{aligned}
(2.9)\ &\|E(|M(X) - X|^q)\|_\infty^{1/q} \\
&\leq \|E(|X^q|)\|_\infty^{1/q} \|\tilde{L}1 - 1\|_\infty + \sum_{k=1}^{n} \frac{\|E(|X^{(k)}|^q)\|_\infty^{1/q}}{k!} \|\tilde{L}((\cdot - t)^k)(t)\|_\infty \\
&\quad + \left(\frac{2(q-1)\|\tilde{L}(1)\|_\infty}{qn-1}\right)^{1-\frac{1}{q}} \frac{1}{(n-1)!(q+1)^{n/q(n+1)}} \\
&\quad \times (\|(\tilde{L}(1))^{\frac{1}{n+1}}(q+1)^{\frac{n}{n+1}} + 1\|_\infty)^{1/q} \|\tilde{L}(|\cdot - t|^{q(n+1)})(t)\|_\infty^{n/q(n+1)} \\
&\quad \times \Omega_1\left(X^{(n)}, \frac{1}{(q+1)^{\frac{1}{q(n+1)}}} \|\tilde{L}(|\cdot - t|^{q(n+1)})(t)\|_\infty^{\frac{1}{q(n+1)}}\right)_{L^q}.
\end{aligned}
$$

We present our first Korovkin ([5]) type theorem for stochastic processes in our general setting.

Theorem 2.2. *Let* $\{\tilde{L}_N\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators and the induced sequence of positive linear operators* $\{M_N\}_{N\in\mathbb{N}}$, *on stochastic processes all as in Concepts* 1, $1 < q < \infty$, *Assumptions* 1 *and* 2, $n \geq 1$. *Additionally assume that* $\{\tilde{L}_N(1)\}_{N\in\mathbb{N}}$ *is bounded and* $\|\tilde{L}_N(|\cdot - t|^{q(n+1)})(t)\|_\infty \to 0$, *along with* $\tilde{L}_N 1 \xrightarrow{u} 1$, *as* $N \to \infty$. *Then* $\|E(|M_N(X) - X|^q)\|_\infty \to 0$, *as* $N \to \infty$, *for all* X *as in Concepts* 1 *and Assumptions* 1, 2, $n \geq 1$. *I.e.*

$$
M_N \underset{N\to\infty}{\overset{\text{“}q\text{-mean”}}{\longrightarrow}} I,
$$

the unit operator, with rates and in our setting.

Proof. By Corollary 2.1 and the fact

$$
(2.10)\quad \|\tilde{L}_N((\cdot - t)^k)(t)\|_\infty \leq \|\tilde{L}_N(1)\|_\infty^{\frac{q(n+1)-k}{q(n+1)}} \|\tilde{L}_N(|\cdot - t|^{q(n+1)})(t)\|_\infty^{\frac{k}{q(n+1)}},
$$

for $k = 1, \ldots, n$. $\square$

We need

Lemma 2.3. *Let $\varphi(s,x) \not\equiv 0$ jointly continuous in $(s,x) \in [a,b]^2$. Consider*

$$\gamma(s) := \int_t^s \varphi(s,x)dx,$$

where t is fixed in $[a,b]$. Then $\gamma(s)$ is continuous in $s \in [a,b]$.

Proof. Easy. $\square$

We make

Remark 2.7. Let $n \geq 1$. By (2.2) we get

$$\int_\Omega |M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)|P(d\omega) \leq \sum_{k=1}^n \frac{E|X^{(k)}|(t)}{k!}|\tilde{L}((\cdot - t)^k)(t)|$$
$$+\frac{1}{(n-1)!}\left(\int_\Omega \left(\int_{[a,b]}\left|\int_t^s |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|\,|s-x|^{n-1}dx\right|\mu_t(ds)\right)P(d\omega).$$

The integrand function is jointly continuous in (x,s) and measurable in ω, therefore is jointly measurable in (s,ω) and also nonnegative. Use also Lemma 2.3. Therefore we can apply twice Tonelli–Fubini's theorem to obtain

$$\int_\Omega |M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)|P(d\omega) \leq \sum_{k=1}^n \frac{E|X^{(k)}|(t)}{k!}|\tilde{L}((\cdot - t)^k)(t)|$$
$$+\frac{1}{(n-1)!}\left(\int_{[a,b]}\left|\int_t^s \left(\int_\Omega |X^{(n)}(x,\omega) - X^{(n)}(t,\omega)|P(d\omega)\right)|s-x|^{n-1})dx\right|\mu_t(ds)\right)$$
$$\leq \mathcal{J} + \frac{1}{(n-1)!}\left(\int_{[a,b]}\left|\int_t^s \Omega_1(X^{(n)}, |x-t|)_{L^1}|s-x|^{n-1}dx\right|\mu_t(ds)\right)$$
$$\underset{(h>0)}{\leq} \mathcal{J} + \frac{\Omega_1(X^{(n)},h)_{L^1}}{(n-1)!}\left(\int_{[a,b]}\left|\int_t^s \left(1+\frac{|x-t|}{h}\right)|s-x|^{n-1}dx\right|\mu_t(ds)\right)$$
$$\leq \mathcal{J} + \frac{\Omega_1(X^{(n)},h)_{L^1}}{(n-1)!}\left(\int_{[a,b]}\left\{\left|\int_t^s |s-x|^{n-1}dx\right|\right.\right.$$
$$\left.\left.+\frac{1}{h}\left|\int_t^s |x-t|\,|s-x|^{n-1}dx\right|\right\}\mu_t(ds)\right)$$
$$= \mathcal{J} + \frac{\Omega_1(X^{(n)},h)_{L^1}}{(n-1)!}\left(\int_{[a,b]}\left(\frac{|t-s|^n}{n} + \frac{1}{h}\frac{|t-s|^{n+1}}{n(n+1)}\right)\mu_t(ds)\right)$$
$$= \mathcal{J} + \frac{\Omega_1(X^{(n)},h)_{L^1}}{(n-1)!}\left[\frac{\tilde{L}(|\cdot - t|^n)(t)}{n} + \frac{\tilde{L}(|\cdot - t|^{n+1})(t)}{hn(n+1)}\right]$$

$$\leq \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1}}{(n-1)!}\left[\frac{1}{n}(\tilde{L}(1)(t))^{1/(n+1)}((\tilde{L}(|\cdot - t|^{n+1}))(t))^{n/(n+1)} + \frac{1}{hn(n+1)}(\tilde{L}(|\cdot - t|^{n+1})(t))\right],$$

with

$$\mathcal{J} := \sum_{k=1}^{n} \frac{(E|X^{(k)}|)(t)}{k!} |\tilde{L}((\cdot - t)^k)(t)|\Big).$$

Now take $h := (\tilde{L}(|\cdot - t|^{n+1})(t))^{1/(n+1)} > 0$, i.e.

$$h^{n+1} = \tilde{L}(|\cdot - t|^{n+1})(t)) = \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1} h^n}{n!}\left[(\tilde{L}(1)(t))^{1/(n+1)} + \frac{1}{(n+1)}\right].$$

We have proved that

$$\int_\Omega |M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)| P(d\omega) \leq \mathcal{J} + \frac{\Omega_1(X^{(n)}, h)_{L^1}}{n!} h^n \left(((\tilde{L}(1))(t))^{1/(n+1)} + \frac{1}{n+1}\right).$$

Also by (2.6) we find

$$\int_\Omega |M(X)(t,\omega) - X(t,\omega)| P(d\omega) \leq (E|X|)(t)|\tilde{L}(1)(t) - 1| + \int_\Omega |M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t)| P(d\omega).$$

Assumption 3. *Here we suppose* $(E|X^{(k)}|)(t) < \infty$, $\forall t \in [a,b]$, *all* $k = 0, 1, \ldots, n$, $n \geq 0$.

From the above is derived

Theorem 2.3. *Suppose Concepts* 1 *and Assumptions* 1, 3, $n \geq 1$. *Then*

$$\begin{aligned}(2.11)\quad & E(|M(X) - X|)(t) \\ & \leq (E|X|)(t)|\tilde{L}(1)(t) - 1| + \sum_{k=1}^{n} \frac{(E|X^{(k)}|)(t)}{k!} |\tilde{L}((\cdot - t)^k)(t)| \\ & + \frac{1}{n!}\left(((\tilde{L}(1))(t))^{1/(n+1)} + \frac{1}{n+1}\right)(\tilde{L}(|\cdot - t|^{n+1})(t))^{n/(n+1)} \\ & \times \Omega_1(X^{(n)}, (\tilde{L}(|\cdot - t|^{n+1})(t))^{1/(n+1)})_{L^1}, \quad t \in [a,b].\end{aligned}$$

Note 2. If $\tilde{L}(|\cdot - t|^{n+1})(t) = 0$, then (2.11) holds trivially as equality.

We further present

Corollary 2.2. *Suppose Concepts* 1 *and Assumptions* 1, 3, $n \geq 1$. *Then*

$$\begin{aligned}(2.12)\quad \|E(|M(X) - X)|)\|_\infty &\leq \|E|X|\,\|_\infty \|\tilde{L}1 - 1\|_\infty \\ &+ \sum_{k=1}^{n} \frac{\|E(|X^{(k)}|)\|_\infty}{k!} \|\tilde{L}((\cdot - t)^k)(t)\|_\infty \\ &+ \frac{1}{n!} \left\| (\tilde{L}(1))^{1/(n+1)} + \frac{1}{n+1} \right\|_\infty \|\tilde{L}(|\cdot - t|^{n+1})(t)\|_\infty^{n/(n+1)} \\ &\times \Omega_1(X^{(n)}, \|\tilde{L}(|\cdot - t|^{n+1})(t)\|_\infty^{1/(n+1)})_{L^1}.\end{aligned}$$

The following Korovkin type theorem for stochastic processes in our general setting is valid.

Theorem 2.4. *Let* $\{\tilde{L}_N\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators and the induced sequence of positive linear operators* $\{M_N\}_{N\in\mathbb{N}}$ *on stochastic processes, all as in Concepts* 1, *Assumptions* 1 *and* 3, $n \geq 1$. *Additionally assume that* $\{\tilde{L}_N(1)\}_{N\in\mathbb{N}}$ *is bounded and* $\|\tilde{L}_N(|\cdot - t|^{n+1})(t)\|_\infty \to 0$, *along with* $\tilde{L}_n 1 \xrightarrow{u} 1$, *as* $N \to \infty$. *Then* $\|E(|M_N(X) - X|)\|_\infty \to 0$, *as* $N \to \infty$, *for all* X *as in Concepts* 1 *and Assumptions* 1, 3, $n \geq 1$. *I.e.* $M_N \underset{N\to+\infty}{\overset{\text{“1-mean”}}{\longrightarrow}} I$ *with rates.*

Proof. By Corollary 2.2 and the fact

$$\|\tilde{L}_N((\cdot - t)^k)(t)\|_\infty \leq \|\tilde{L}_N(1)\|_\infty^{1-\frac{k}{n+1}} \|\tilde{L}_N(|\cdot - t|^{n+1})(t)\|_\infty^{\frac{k}{n+1}},$$

for $k = 1, \ldots, n$. $\square$

Note 3. We observe that $M_N \overset{\text{“}q\text{-mean”}}{\longrightarrow} I$ implies $M_N \overset{\text{“1-mean”}}{\longrightarrow} I$, according to Theorems 2.2 and 2.4, $n \geq 1$.

Next we specialize in the $n = 0$ case. We do first the subcase $q > 1$. For that we make

Remark 2.8. We have that

$$(2.13)\quad \Delta(t,\omega) := M(X)(t,\omega) - X(t,\omega)\tilde{L}(1)(t) = \int_{[a,b]} (X(s,\omega) - X(t,\omega))\mu_t(ds).$$

Let $q > 1$, then by Hölder's inequality we have

$$\begin{aligned} |\Delta(t,\omega)|^q &\leq \left(\int_{[a,b]} |X(s,\omega) - X(t,\omega)| \mu_t(ds) \right)^q \\ &\leq m_t^{q-1} \int_{[a,b]} |X(s,\omega) - X(t,\omega)|^q \mu_t(ds). \end{aligned}$$

Therefore we obtain

$$\left(\int_\Omega |\Delta(t,\omega)|^q P(d\omega) \right)^{1/q} \leq m_t^{1-\frac{1}{q}} \left(\int_\Omega \left(\int_{[a,b]} |X(s,\omega) - X(t,\omega)|^q \mu_t(ds) \right) P(d\omega) \right)^{1/q}.$$

The integrand function is nonnegative, continuous in s, measurable in ω, therefore jointly measurable in (s,ω) and by Tonelli–Fubini's theorem we get

$$\begin{aligned} &\left(\int_\Omega |\Delta(t,\omega)|^q P(d\omega) \right)^{1/q} \\ &\quad \leq m_t^{1-\frac{1}{q}} \left(\int_{[a,b]} \left(\int_\Omega |X(s,\omega) - X(t,\omega)|^q P(d\omega) \right) \mu_t(ds) \right)^{1/q} \\ &\quad \leq m_t^{1-\frac{1}{q}} \left(\int_{[a,b]} \Omega_1^q(X, |s-t|)_{L^q} \mu_t(ds) \right)^{1/q} \\ &\qquad \text{(take } h > 0) \\ &\quad \leq m_t^{1-\frac{1}{q}} \Omega_1(X,h)_{L^q} \left(\int_{[a,b]} \left(1 + \frac{|s-t|}{h} \right)^q \mu_t(ds) \right)^{1/q} \\ &\quad \leq 2^{1-\frac{1}{q}} m_t^{1-\frac{1}{q}} \Omega_1(X,h)_{L^q} \left(m_t + \frac{1}{h^q} \int_{[a,b]} |s-t|^q \mu_t(ds) \right)^{1/q} \\ &\quad = 2^{1-\frac{1}{q}} m_t^{1-\frac{1}{q}} \Omega_1 \left(X, \left(\int_{[a,b]} |s-t|^q d\mu_t(s) \right)^{1/q} \right)_{L^q} (m_t + 1)^{1/q}, \end{aligned}$$

where we choose

$$h := \left(\int_{[a,b]} |s-t|^q d\mu_t(s) \right)^{1/q} > 0. \tag{2.14}$$

We have established

Theorem 2.5. *Suppose Concepts* 1 *and Assumptions* 1, 2 *for* $n = 0$, $1 < q < \infty$. *Then*

$$\begin{aligned} &(E(|M(X) - X|^q)(t))^{1/q} \\ &\quad \leq (E(|X|^q)(t))^{1/q} |\tilde{L}(1)(t) - 1| \\ &\qquad + (2\tilde{L}(1)(t))^{1-\frac{1}{q}} ((\tilde{L}(1))(t) + 1)^{1/q} \Omega_1(X, (\tilde{L}(|\cdot - t|^q)(t))^{1/q})_{L^q}, \end{aligned} \tag{2.15}$$

for all $t \in [a,b]$.

Note 4. Inequality (2.15) is trivially true and holds as equality when (see (2.14)) $h = 0$.

We give

Corollary 2.3. *Suppose Concepts* 1 *and Assumptions* 1, 2 *for* $n = 0$, $1 < q < \infty$. *Then*

$$\|E(|M(X) - X|^q)\|_\infty^{1/q} \leq \|E(|X|^q)\|_\infty^{1/q}\|\tilde{L}1 - 1\|_\infty + (2\|\tilde{L}(1)\|_\infty)^{1-\frac{1}{q}}\|\tilde{L}(1) + 1\|_\infty^{1/q}\Omega_1(X, \|\tilde{L}(|\cdot - t|^q)(t)\|_\infty^{1/q})_{L^q}. \tag{2.16}$$

We present the next Korovkin type result.

Theorem 2.6. *Let* $\{\tilde{L}_N\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators and the induced sequence of positive linear operators* $\{M_N\}_{N\in\mathbb{N}}$ *on stochastic processes, all as in Concepts 1,* $1 < q < \infty$, *Assumptions 1, 2 for* $n = 0$. *Additionally assume that* $\{\tilde{L}_N(1)\}_{N\in\mathbb{N}}$ *is bounded and* $\|\tilde{L}_N(|\cdot - t|^q)(t)\|_\infty \to 0$, *along with* $\tilde{L}_N 1 \xrightarrow{u} 1$, *as* $N \to \infty$. *Then* $\|E(|M_N(X) - X|^q)\|_\infty \to 0$, *as* $N \to \infty$, *for all* X *as in Concepts* 1 *and Assumptions* 1, 2, $n = 0$. *I.e.* $M_N \underset{N\to\infty}{\overset{\text{“q-mean”}}{\longrightarrow}} I$ *with rates in our setting.*

Note 5. The rate of convergence in Theorem 2.2 is much higher than of Theorem 2.6 because of the assumed differentiability of X, see and compare inequalities (2.9), (2.10) and (2.16).

We make

Remark 2.9. Let $\Delta(t,\omega)$ as in (2.13). Then

$$\begin{aligned}
\int_\Omega |\Delta(t,\omega)|P(d\omega) &\leq \int_\Omega \left(\int_{[a,b]} |X(s,\omega) - X(t,\omega)|\mu_t(ds)\right)P(d\omega) \\
&\qquad\qquad \text{(by Tonelli–Fubini's theorem)} \\
&= \int_{[a,b]} \left(\int_\Omega |X(s,\omega) - X(t,\omega)|P(d\omega)\right)\mu_t(ds) \\
&\leq \int_{[a,b]} \Omega_1(X, |s-t|)_{L^1}\mu_t(ds) \\
&\leq \Omega_1(X,h)_{L^1}\int_{[a,b]}\left(1 + \frac{|s-t|}{h}\right)\mu_t(ds)
\end{aligned}$$

$$
\begin{aligned}
&= \Omega_1(X,h)_{L^1}\left(m_t + \frac{1}{h}\int_{[a,b]} |s-t|\mu_t(ds)\right)\\
&\le \Omega_1(X,h)_{L^1}\left(m_t + \frac{1}{h} m_t^{1/2}\left(\int_{[a,b]}(s-t)^2\mu_t(ds)\right)^{1/2}\right)\\
&= \Omega_1(X,h)_{L^1}(m_t + \sqrt{m_t}),
\end{aligned}
$$

where we choose

$$h := \left(\int_{[a,b]}(s-t)^2\mu_t(ds)\right)^{1/2} > 0 \tag{2.17}$$

I.e. we got

$$\int_\Omega |\Delta(t,\omega)|P(d\omega) \le \big(\tilde{L}(1)(t) + \sqrt{\tilde{L}(1)(t)}\big)\Omega_1\big(X, ((\tilde{L}(\cdot - t)^2)(t))^{1/2}\big)_{L^1}.$$

We have proved

Theorem 2.7. *Suppose Concepts* 1 *and Assumptions* 1, 3 *for* $n = 0$. *Then*

$$
\begin{aligned}
(E(|M(X) - X|))(t) &\le (E|X|)(t)|\tilde{L}(1)(t) - 1|\\
&+(\tilde{L}(1)(t) + \sqrt{\tilde{L}(1)(t)})\Omega_1(X, ((\tilde{L}(\cdot - t)^2)(t))^{1/2})_{L^1}, \quad t \in [a,b].
\end{aligned}
\tag{2.18}
$$

Note 6. Inequality (2.18) is trivially true and holds as equality when (see (2.17)) $h = 0$.

We give (see also [6])

Corollary 2.4. *Suppose Concepts* 1 *and Assumptions* 1, 3 *for* $n = 0$. *Then*

$$
\begin{aligned}
\|E(|M(X) - X|)\|_\infty &\le \|E(X)\|_\infty \|\tilde{L}1 - 1\|_\infty\\
&+\|\tilde{L}1 + \sqrt{\tilde{L}1}\|_\infty \Omega_1(X, \|(\tilde{L}((\cdot - t)^2))(t)\|_\infty^{1/2})_{L^1}.
\end{aligned}
\tag{2.19}
$$

We present a final Korovkin (see [5]) type result.

Theorem 2.8. *Let* $\{\tilde{L}_N\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators and the induced sequence of positive linear operators* $\{M_N\}_{N\in\mathbb{N}}$ *on stochastic processes, all as in Concepts* 1 *and Assumptions* 1, 3 *for* $n = 0$. *Additionally assume that* $\{\tilde{L}_N(1)\}_{N\in\mathbb{N}}$ *is bounded and*

$$\tilde{L}_N 1 \xrightarrow{u} 1, \quad \tilde{L}_N id \xrightarrow{u} id, \quad \tilde{L}_N id^2 \xrightarrow{u} id^2, \quad as \ \ N \to \infty. \tag{2.20}$$

Then $\|E(|M_N(X)-X|)\|_\infty \to 0$, *as* $N \to \infty$, *for all* X *as in Concepts* 1 *and Assumptions* 1, 3 *for* $n=0$. *I.e.* $M_N \underset{N\to\infty}{\overset{\text{“1-mean”}}{\longrightarrow}} I$ *with rates in our setting.*

Proof. We use Corollary 2.4. By [6] we have that

$$\|(\tilde{L}_N((\cdot - t)^2))(t)\|_\infty \leq \|\tilde{L}_N(x^2)(t) - t^2\|_\infty + 2c\|L_N(x)(t) - t\|_\infty + c^2\|L_N(1)(t) - 1\|_\infty, \tag{2.21}$$

where $c := \max(|a|, |b|)$, for all $N \in \mathbb{N}$. Thus by assuming the basic Korovkin conditions (2.20) we get by (2.21) that $\|(\tilde{L}_N((\cdot - t)^2))(t)\|_\infty \to 0$, as $N \to \infty$, etc. □

We make also

Remark 2.10. 1) If $X^{(n)}$ fulfills a Lipschitz type condition then our results become more specific and simplify.

2) In the special important case of $\tilde{L}(1)(t) = 1$, $\forall t \in [a,b]$, all of our results here simplify a lot and take an elegant form. Furthermore in this case, supposing Assumption 2 we need to impose (2.7) only for $k = 1, \dots, n$ and supposing Assumption 3 we need to impose it only for $k = 1, \dots, n$, $n \geq 1$.

We finish by giving

Application 1. Let $f \in C([0,1])$ and the Bernstein polynomial

$$B_N(f)(t) := \sum_{k=0}^{n} f\left(\frac{k}{N}\right)\binom{N}{k} t^k (1-t)^{N-k}, \quad t \in [0,1],\ N \in \mathbb{N}.$$

We have that

$$B_N((\cdot - t)^2)(t) = \frac{t(1-t)}{N}, \quad t \in [0,1],$$

and

$$\|B_N((\cdot - t)^2)(t)\|_\infty^{1/2} \leq \frac{1}{2\sqrt{N}}, \quad N \in \mathbb{N}.$$

Clearly B_N is an example of an $\tilde{L}_N$ as in Concepts 1. Define the corresponding application of M_N by

$$\begin{aligned} \tilde{B}_N(X)(t,\omega) &:= B_N(X(\cdot,\omega))(t) \\ &= \sum_{k=0}^{N} X\left(\frac{k}{N}, \omega\right)\binom{N}{k} t^k (1-t)^{N-k}, \quad t \in [0,1], \end{aligned}$$

for all $\omega \in \Omega$, $N \geq 1$, where X is as in Concepts 1 and Assumptions 1, 3 for $n = 0$. Since $B_N(1)(t) = 1$ by (2.19) we get that

$$\|E(|\tilde{B}_N(X) - X|)\|_\infty \leq 2\Omega_1 \left(X, \frac{1}{2\sqrt{N}} \right)_{L^1}, \quad N \geq 1,$$

for all X as above. Thus as $N \to \infty$ we obtain

$$\|E(|\tilde{B}_N(X) - X|)\|_\infty \to 0,$$

i.e. $\tilde{B}_N \overset{\text{"1-mean"}}{\longrightarrow} I$ with rates, which is the expected conclusion given by Theorem 2.8. If X is of Lipschitz type of order 1 i.e. if $\Omega_1(X, \delta)_{L^1} \leq K\delta$, where $K > 0$, $\forall \delta > 0$, then

$$\|E(|\tilde{B}_N(X) - X|)\|_\infty \leq \frac{K}{\sqrt{N}}, \quad N \geq 1.$$

One can give many similar other applications of the above theory.

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Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, U.S.A.

ganastss@memphis.edu