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SUPERCRITICAL HOPF BIFURCATION IN DELAY DIFFERENTIAL EQUATIONS – AN ELEMENTARY PROOF OF EXCHANGE OF STABILITY

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Abstract. The local Hopf Bifurcation theorem states that under supercritical conditions, stability passes from the trivial branch to the bifurcated one. We give here an "elementary" proof of this result, based on the following steps: i) reduction to a two-dimensional system via the center manifold technic; ii) estimation of the distance between solutions of the original and the reduced equations. Thus there is no reference to the Floquet theory. Incidently, we obtain an estimate of the stability region.

1. Generalities and the Reduced System

We first recall the Hopf Bifurcation theorem for retarded functional differential equations. Consider the system

$$\dot{u}(t) = Lu_t + R(\alpha, u_t), \tag{1.1}$$

with the following hypotheses:

$(\mathbf{H_0})$ $L : C \to \mathbb{R}^n$ is a linear continuous operator, $R : \mathbb{R} \times C \to \mathbb{R}^n$ is C^{k+1}, $k \geq 1$, such that $R(\alpha, 0) = 0$, for all $\alpha \in \mathbb{R}$ and $D_2R(\alpha_0, 0) = 0$, for some $\alpha_0 \in \mathbb{R}$; where $C = C([-r, 0], \mathbb{R}^n)$, $n \in \mathbb{N}^*$, $u_t(\theta) = u(t + \theta)$ for $\theta \in [-r, 0]$.

$(\mathbf{H_1})$ The characteristic equation

$$\Delta(\lambda, \alpha) = 0, \quad \text{with} \quad \Delta(\lambda, \alpha) = \det\left(\lambda Id - (L + D_2R(\alpha, 0))e^{\lambda(\cdot)}Id\right) \tag{1.2}$$

of the linearized equation of equation (1.1) at the equilibrium point 0, has only roots with a negative real parts for $\alpha \leq \alpha_0$, and exactly two roots cross

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the imaginary axis at $\alpha = \alpha_0$, at points $i\nu_0$ and $-i\nu_0$, with multiplicity 1, $\nu_0 > 0$.

More precisely, if $\lambda(\alpha)$ is the characteristic root of (1.2), bifurcated from $i\nu_0$, we assume that:

(H₂) $\frac{d}{d\alpha} \operatorname{Re} \lambda(\alpha)\Big|_{\alpha=\alpha_0} > 0.$

Denote by $(T_\alpha(t))_{t\geq 0}$ the semi group associated with the linearized equation of (1.1) and by A_α it's infinitesimal generator. To the continuous branch of eigenvalues $\lambda(\alpha)$, we can associate a continuous branch of eigenvectors $a(\alpha) + ib(\alpha)$.

Let $N_\alpha = \operatorname{span}\{a(\alpha), b(\alpha)\}$ and $\Phi_\alpha = (a(\alpha), b(\alpha))$. There exists a 2×2 matrix $B(\alpha)$ of the following form

$$B(\alpha) = \begin{pmatrix} r(\alpha) & \beta(\alpha) \\ -\beta(\alpha) & r(\alpha) \end{pmatrix},$$

where $\lambda(\alpha) = r(\alpha) + i\beta(\alpha)$ such that $A_\alpha\Phi_\alpha = \Phi_\alpha B(\alpha)$. Using the formal adjoint theory for FDEs in [5], $N_\alpha = \{\Phi_\alpha\langle\Psi_\alpha, \varphi\rangle, \varphi \in C\}$ and C can be decomposed as $C = N_\alpha \oplus S_\alpha$, where $\Psi_\alpha = \operatorname{col}(e(\alpha), f(\alpha))$ is a basis of the adjoint space N_α^*, such that $\langle\Psi_\alpha, \Phi_\alpha\rangle = I$ (the 2×2 identity matrix) and $\langle\cdot,\cdot\rangle$ is the bilinear form on $C^* \times C$ associated with the adjoint equation of the linearized equation of (1.1) around 0. Using the decomposition $u_t = \Phi_\alpha x(t) + y_t$, $x(t) \in \mathbb{R}^2$, $y_t \in S_\alpha$ in the variation of constants formula introduced in [5], equation (1.1) reads as

$$\begin{cases} x(t) = e^{B(\alpha)t}x_0 + \displaystyle\int_0^t e^{B(\alpha)(t-\tau)}\Psi_\alpha(0)F(\alpha, \Phi_\alpha x(\tau), y_\tau)d\tau, \\ y_t = T_\alpha(t)y_0 + \displaystyle\int_0^t d[K_\alpha(t,\tau)^S]F(\alpha, \Phi_\alpha x(\tau), y_\tau), \end{cases} \tag{1.3}$$

where $F(\alpha, \varphi, \psi) = R(\alpha, \varphi + \psi) - D_2R(\alpha, \varphi + \psi)$, $K_\alpha(t,\tau)^S = K_\alpha(t,\tau) - \Phi_\alpha\langle\Psi_\alpha, K_\alpha(t,\tau)\rangle$, $K_\alpha(t,\tau) = \int_0^\tau X_\alpha(t+\theta-\nu)d\nu$, X_α is the $n\times n$ fundamental matrix solution of the linearized equation of (1.1).

Looking for solutions defined on the whole real axis and uniformly bounded on their domain, the second identity in (1.3) can be written starting from any initial point σ as $y_t = T_\alpha(t-\sigma)y_\sigma + \int_\sigma^t d[K_\alpha(t,\tau)^S]F(\alpha, \Phi_\alpha x(\tau), y_\tau)$.

Letting $\sigma \to -\infty$, the terms $T_\alpha(t-\sigma)y(\sigma)$ approach to 0, while the integral has a limit, which yields an expression where the projection y_0 has been eliminated and the system (1.3) has the form:

$$(1.4)\quad \begin{cases} x(t) = e^{B(\alpha)t}x_0 + \displaystyle\int_0^t e^{B(\alpha)(t-\tau)}\Psi_\alpha(0)F(\alpha, \Phi_\alpha x(\tau), y_\tau)d\tau, \\ y_t = \displaystyle\int_{-\infty}^t d[K_\alpha(t,\tau)^S]F(\alpha, \Phi_\alpha x(\tau), y_\tau). \end{cases}$$

In view of the center manifold theorem (see [5]), there exists a map $g(\alpha, \cdot)$ defined from a neighborhood of 0 in N_α into a neighborhood of 0 in S_α such that $y_t = g(\alpha, \Phi_\alpha x(t))$ is the equation of a local center manifold, for α close to α_0, where g is C^{k+1}, $g(\alpha, 0) = 0$ and $D_2 g(\alpha_0, 0) = 0$. Then the system (1.4) on the center manifold can be reduced to the following problem

$$(1.5)\quad x(t) = e^{B(\alpha)t}x_0 + \int_0^t e^{B(\alpha)(t-\tau)}\Psi_\alpha(0)F(\alpha, \Phi_\alpha x(\tau), g(\alpha, \Phi_\alpha x(\tau))d\tau,$$

through this transformation:

$(x(t), y_t)$ is the solution of (1.4) on the center manifold $\Leftrightarrow$ $x(t)$ is the solution of (1.5) and $y_t = g(\alpha, \Phi_\alpha x(t))$.

In polar coordinates $x = (\rho\cos\phi, \rho\sin\phi)$, system (1.5) reads as

$$(1.6)\quad \begin{cases} \dot\rho = r(\alpha)\rho + h_1(\alpha, \rho, \phi), \\ \dot\phi = \beta(\alpha) + h_2(\alpha, \rho, \phi), \end{cases}$$

where $h_1(\alpha, \rho, \phi)$ and $h_2(\alpha, \rho, \phi)$ have the form:

$$\begin{aligned} h_1(\alpha, \rho, \phi) &= \cos\phi f_1(\alpha, \rho\cos\phi, \rho\sin\phi) + \sin\phi f_2(\alpha, \rho\cos\phi, \rho\sin\phi), \\ h_2(\alpha, \rho, \phi) &= \frac{1}{\rho}\left[-\sin\phi f_1(\alpha, \rho\cos\phi, \rho\sin\phi) + \cos\phi f_2(\alpha, \rho\cos\phi, \rho\sin\phi)\right], \end{aligned}$$

f_1, f_2 are the components of $\Psi_\alpha(0)F(\alpha, \Phi_\alpha x(t), g(\alpha, \Phi_\alpha x(t)))$ in $\mathbb{R}^2$.

For ρ near to 0, we have $\phi \simeq \nu_0 t$ (in view of the second equation in (1.6)). Therefore t is diffeomorphic to ϕ, we can eliminate t between the two equations in (1.6). We obtain

$$(1.7)\quad \frac{d\rho}{d\phi} = \frac{r(\alpha)\rho + h_1(\alpha, \rho, \phi)}{\beta(\alpha) + h_2(\alpha, \rho, \phi)}.$$

Let $\rho(\phi, c)$ be the solution of the last equation with initial data c.

The periodic solution with initial value c correspond to finding c and α such that

$$G(\alpha, c) = 0, \tag{1.8}$$

where G is the Hopf bifurcation function

$$G(\alpha, c) = \int_0^{2\pi} \frac{r(\alpha)\rho(s,c) + h_1(\alpha, \rho(s,c), s)}{\beta(\alpha) + h_2(\alpha, \rho(s,c), s)} ds. \tag{1.9}$$

From the Hopf bifurcation theorem (see [2], [3] and [4]), there exists $\varepsilon_0 > 0$, and functions $P(\varepsilon)$, $\omega(\varepsilon)$ and $\alpha(\varepsilon)$ ($\varepsilon \in [0, \varepsilon_0)$) sufficiently regular such that $P(\varepsilon)$ is $\omega(\varepsilon)-$periodic solution of equation (1.5) for $\alpha = \alpha(\varepsilon)$. Furthermore, $\omega(0) := \omega_0 = 2\pi/\nu_0$, $\alpha(0) = \alpha_0$, $\alpha^{'}(0) = 0$ and the bifurcation is subcritical if $\alpha^{''}(0) < 0$, and supercritical if $\alpha^{''}(0) > 0$.

Now, we introduce the assumption on supercritical Hopf bifurcation:

($\mathbf{H_3}$) $\quad \alpha^{''}(0) > 0$.

Remark 1.1. a) ($\mathbf{H_3}$) means that the bifurcated branch lies in the parameter region where the trivial solution is unstable, so, we will find out, as is well known ([7]), that the system stabilizes around the nontrivial periodic solutions.
b) ($\mathbf{H_3}$) implies that α is increasing along the bifurcated branch, occasionally, we will use α instead of ε as an independent variable along the branch.

Information about elements of bifurcation (α, ω, P) are needed here. In fact, the following classical features will be enough for our purposes (see [8] and [9]):

$$\begin{cases} \alpha = \alpha_0 + \varepsilon^2\alpha_2 + \varepsilon^4\alpha_4 + \cdots \\ \omega = \omega_0 + \varepsilon^2\omega_2 + \varepsilon^4\omega_4 + \cdots \\ P(\varepsilon)(t) = \varepsilon \tilde{P}(t). \end{cases} \tag{1.10}$$

Note that the elements of Hopf bifurcation in (1.10) are obtained formally by inserting Taylor expansions of α, ω, ρ:

$$\alpha = \alpha(\varepsilon) = \sum \alpha_i \varepsilon^i, \quad \omega = \omega(\varepsilon) = \sum \omega_i \varepsilon^i, \quad \rho = \varepsilon(\sum \rho_i \varepsilon^i)$$

into the equation (1.8) and the integrated form of (1.6), with $\rho(0) = \varepsilon$, $\phi(0) = 0$, $\rho(\omega) = \varepsilon$, $\phi(\omega) = 2\pi$, and equating like powers of ε.

2. Stability, Asymptotic Stability Along the Branch of Bifurcation

From now on, we assume all of the hypotheses introduced before: $(\mathbf{H_0})$ through $(\mathbf{H_3})$. Our first step is to give (1.4) in another form, by centering it around $y_t = g(\alpha, \Phi_\alpha x(t))$. We set $y_t = z_t + g(\alpha, \Phi_\alpha x(t))$, that is, we consider (x, z) instead of (x, y). Inserting this into system (1.4), we obtain:

$$\text{(2.1)} \qquad \begin{cases} x(t) = e^{B(\alpha)t}x_0 + \displaystyle\int_0^t e^{B(\alpha)(t-\tau)} R_1(\alpha, x(\tau), z_\tau)d\tau, \\ z_t = \displaystyle\int_{-\infty}^t d[K_\alpha(t,\tau)^S] R_2(\alpha, x(\tau), z_\tau), \end{cases}$$

where

$$\text{(2.2)} \qquad R_1(\alpha, x(t), z_t) = \Psi_\alpha(0)F(\alpha, \Phi_\alpha x(t), z_t + g(\alpha, \Phi_\alpha x(t))),$$

and

$$R_2(\alpha, x(t), z_t) = F(\alpha, \Phi_\alpha x(t), z_t + g(\alpha, \Phi_\alpha x(t))) - F(\alpha, \Phi_\alpha x(t), g(\alpha, \Phi_\alpha x(t))).$$

Note that the bifurcating periodic solutions of (2.1) lie on $z = 0$, that is, are in fact the same as bifurcating solutions of (1.5). Extending the notations of Section 1, we denote such solutions by $p = (P, 0)$. We will normalize these solutions, considering that $P(0) = P_0 = (\varepsilon, 0)$, $\varepsilon > 0$.

Let (x_0, z_0) be a data near $(P_0, 0)$, with $x_0 = (c, 0)$, $c > 0$.

We will denote by x^* the solution of (1.5) such that $x^*(0) = x_0$, and by $(x^\#, z)$ the solution of (2.1) such that $x^\#(0) = x_0$, $z_{t=0} = z_0$, and let $\omega^* = \omega^*(\alpha, x_0)$ denotes the first return-time for x^*, that is $x^*(\omega^*) = (\zeta, 0)$, $\zeta > 0$ and $\omega^* > 0$ is the solution of the equation

$$\text{(2.3)} \qquad \phi(\omega) = 2\pi;$$

$\omega^\# = \omega^\#(\alpha, x_0, z_0)$ denotes the first return-time for $x^\#$, that is the solution of the analogous equation of (2.3) resulting from polar coordinates in the first equation in (2.1).

The main result of the paper is the following one.

Theorem 2.1. *Under hypotheses* $(\mathbf{H_0})$ *through* $(\mathbf{H_3})$*, there exists* $\varepsilon_1 > 0$, $\eta > 0$, $K > 0$ *such that for each* ε, $0 < \varepsilon < \varepsilon_1$; θ, $0 \le \theta \le 1$, *the relations* $P_0 = (\varepsilon, 0)$, $|\varepsilon - c| \le \theta\varepsilon^2$ *and* $\|z_0\| \le \eta\theta\varepsilon^3$ *imply*

i) $\left\|x^\#(\omega^\#) - P_0\right\| \le \theta(\varepsilon^2 - K\varepsilon^4)$; ii) $\|z_{\omega^\#}\| \le \eta\theta\varepsilon^3$.

Corollary 2.1. *Assume* $(\mathbf{H_0})$ *through* $(\mathbf{H_3})$*. Then the bifurcated orbits of equation* (1.1) *for* α *close enough to* α_0 *are asymptotically stable.*

Proof. In view of the transformations considered above, we only have to prove the same fact for the equation (2.1). Starting with (x_0, z_0) and θ such that

$$|\varepsilon - c| \leq \theta\varepsilon^2, \quad \|z_0\| \leq \eta\theta\varepsilon^3,$$

denoting by $(c^1, 0) = x^{\#}(\omega^{\#})$, $z_0^1 = z_{\omega^{\#}}$ and c^j, z_0^j the corresponding terms after j rotations $(c^j, 0) = x^{\#}(\omega_j^{\#})$ and $\phi^{\#}(\omega_j^{\#}) = 2j\pi$, we obtain:

$$\left|\varepsilon - c^j\right| \leq \theta(1 - K\varepsilon^2)^j\varepsilon^2, \quad \left\|z_0^j\right\| \leq \eta\theta\varepsilon^3.$$

Thus $(x(t), z_t)$ approaches exponentially the orbit of p. □

Theorem 2.1 is a consequence of two intermediate results. We will first state these as lemmas, and derive the Theorem from them.

Lemma 2.1. *Assume the hypotheses of Theorem 2.1. Then there exists* $K_1 > 0$ *such that* $\|x^*(\omega^*) - P_0\| \leq \theta(\varepsilon^2 - K_1\varepsilon^4)$.

Remark 2.1. Lemma 2.1 states the stability result for a two-dimensional system. There no doubt however that the estimation in Lemma 2.1 is the control estimate for Theorem 2.1. Precisely, our purpose here is to connect in an elementary way the result for such systems to the corresponding stability result for higher dimensional systems.

Lemma 2.2. *Assume the hypotheses of Theorem 2.1 and fix*

$$\omega_1 > \max(\omega_0, \omega^*, \omega^{\#}).$$

Then there exists $K_2 > 0$, *such that:*

i) $\left\|x^{\#}(t) - P(t)\right\| \leq \theta K_2\varepsilon^2$, $0 \leq t \leq \omega_1$;

ii) $\|z_t\| \leq \theta\varepsilon^3 K_2\eta$, $0 \leq t \leq \omega_1$;

iii) $\left\|x^{\#}(t) - x^*(t)\right\| \leq \theta K_2\varepsilon^4\eta$, $0 \leq t \leq \omega_1$.

Proof of Lemma 2.1. In polar coordinates, let (ρ, ϕ) corresponding to the $\omega-$periodic solution P (of (1.6)) guaranteed by Hopf Bifurcation theorem and (ρ^*, ϕ^*) corresponding to x^* with $\rho(0) = \varepsilon$ and $\rho^*(0) = c$.

From (1.9) we obtain $\|x^*(\omega^*) - P_0\| = |\rho^*(2\pi) - \varepsilon| = |c - \varepsilon + G(\alpha, c)|$.

On one hand,

$$G(\alpha, c) = G(\alpha, \varepsilon) + (c - \varepsilon)D_c G(\alpha, \varepsilon) + \frac{1}{2}(c - \varepsilon)^2 D^2_{cc} G(\alpha, \varepsilon + \nu(c - \varepsilon)),$$

for some $\nu \in [0, 1]$.

In view of (1.8), $G(\alpha, \varepsilon) = 0$. By differentiation with respect to ε (with $\alpha = \alpha(\varepsilon)$), we obtain

$$D_c G(\alpha, \varepsilon) = -\alpha'(\varepsilon) D_\alpha G(\alpha, \varepsilon) = -\varepsilon \alpha'(\varepsilon) D_\alpha \tilde{G} (\alpha, \varepsilon),$$

where

$$\tilde{G} (\alpha, \varepsilon) = \begin{cases} \frac{1}{\varepsilon} G(\alpha, \varepsilon), & \text{for } \varepsilon \neq 0, \\ D_c G(\alpha, 0), & \text{for } \varepsilon = 0. \end{cases}$$

Setting

$$K(c, \varepsilon) = \frac{\alpha'(\varepsilon)}{\varepsilon} D_\alpha \tilde{G} (\alpha, \varepsilon) - \frac{1}{2\varepsilon^2}(c - \varepsilon) D^2_{cc} G(\alpha, \varepsilon + \nu(c - \varepsilon)),$$

we have $G(\alpha, c) = -(c - \varepsilon)\varepsilon^2 K(c, \varepsilon)$. Our next task is to look at $\lim\limits_{\substack{c \to 0 \\ \varepsilon \to 0}} K(c, \varepsilon)$.

Claim 2.1. $\lim\limits_{\substack{c \to 0 \\ \varepsilon \to 0}} K(c, \varepsilon) = \dfrac{\alpha''(0) 2\pi r'(\alpha_0)}{\omega_0} > 0.$

Proof. In view of (1.10) and the hypotheses of Theorem 2.1 we have $\alpha'(\varepsilon) = O(\varepsilon)$ and $c - \varepsilon = O(\varepsilon^2)$. Moreover, by (1.9), we have:

$$\frac{\alpha'(\varepsilon)}{\varepsilon} D_\alpha \tilde{G} (\alpha, \varepsilon) \to \alpha''(0) \frac{2\pi r'(\alpha_0)}{\omega_0} \quad \text{as} \quad \varepsilon \to 0$$

(because for $c = 0$ the corresponding solution $\rho = 0$ and $h_i = o(\rho)$, $i = 1, 2$).

Using the Taylor expansion of h_1 and h_2 in terms of ρ, we have

$$\begin{aligned} h_1(\alpha, \rho, s) &= \rho^2 C_3(\alpha, s) + \rho^3 C_4(\alpha, s) + \cdots \\ h_2(\alpha, \rho, s) &= \rho D_3(\alpha, s) + \rho^2 D_4(\alpha, s) + \cdots \end{aligned}$$

where C_j and D_j are polynomials of degree j with respect to $\cos(s)$, $\sin(s)$.

Substituting these expansions for h_1 and h_2 in the integral expression of (1.7) we obtain

$$\begin{aligned} (2.4)\quad \rho^*(\phi) &= c + \frac{1}{\beta(\alpha)} \int_0^{\phi} \Big[r(\alpha)\rho^* + \big(C_3 - D_3' r(\alpha)\big)(\rho^*)^2 \\ &\quad + \big(C_4 - C_3 D_3' + r(\alpha)\big((D_3')^2 - D_4'\big)\big)(\rho^*)^3 + \cdots \Big] ds, \end{aligned}$$

where for each j, $D_j' = D_j/\beta(\alpha)$.

In view of (2.4), we have

$$\begin{aligned} D_{cc}^2 G(\alpha, c) = \frac{1}{\beta(\alpha)} \int_0^{2\pi} &\Big[r(\alpha) D_{cc}^2 \rho^* + \big(C_3 - D_3' r(\alpha)\big) D_{cc}^2 (\rho^*)^2 \\ &+ \Big(C_4 - C_3 D_3' + r(\alpha)\big((D_3')^2 - D_4'\big)\Big) D_{cc}^2 (\rho^*)^3 + \cdots \Big] ds. \end{aligned}$$

From (2.4) once again, we can deduce that

$$\lim_{\substack{c \to 0 \\ \varepsilon \to 0}} D_c \rho^* = 1; \quad \lim_{\substack{c \to 0 \\ \varepsilon \to 0}} D_{cc}^2 (\rho^*)^2 = 2, \quad \lim_{\substack{c \to 0 \\ \varepsilon \to 0}} D_{cc}^2 (\rho^*)^j = 0 \ \text{ for } \ j \geq 3$$

(since $r(\alpha_0) = 0$ and $\lim_{\substack{c \to 0 \\ \varepsilon \to 0}} \rho^* = 0$).

Therefore, we may conclude that

$$\begin{aligned} \lim_{\substack{c \to 0 \\ \varepsilon \to 0}} D_{cc}^2 G(\alpha, c) &= \frac{1}{\beta(\alpha_0)} \int_0^{2\pi} C_3(\alpha_0, s) D_{cc}^2 (\rho^*)^2 \Big|_{\substack{\varepsilon = 0 \\ c = 0}} ds \\ &= \frac{2}{\beta(\alpha_0)} \int_0^{2\pi} C_3(\alpha_0, s) ds = 0 \end{aligned}$$

(since C_3 is a polynomial in $(\cos(s), \sin(s))$ of degree 3).

Coming back to K, this yields the result. □

By continuity property of K, we may deduce that

$$(**) \qquad K(c, \varepsilon) \geq K_1 > 0,$$

for (c, ε) in a neighborhood of $(0, 0)$ and some $K_1 > 0$. We may conclude the proof of Lemma 2.1: taking ε_1 small enough, we can assert that for each pair (c, ε) satisfying the requirements stated in Theorem 2.1, condition $(**)$ holds.

Therefore, we have $\|x^*(\omega^*) - P_0\| \le |c-\varepsilon|\,(1-\varepsilon^2 K_1)$. This complete the proof of Lemma 2.1. $\square$

Proof of Lemma 2.2. According to (1.5) and the first equation of (2.1), we have

$$x^{\#}(t) - P(t) = e^{B(\alpha)t}(x_0 - P_0) + \int_0^t e^{B(\alpha)(t-\tau)}\left[R_1(\alpha, x^{\#}(\tau), z_\tau) - R_1(\alpha, P(\tau), 0)\right] d\tau, \tag{2.5}$$

$$\begin{aligned} &\left\|R_1(\alpha, x^{\#}(\tau), z_\tau) - R_1(\alpha, P(\tau), 0)\right\| \\ &\quad \le \left\|R_1(\alpha, x^{\#}(\tau), z_\tau) - R_1(\alpha, x^{\#}(\tau), 0)\right\| \\ &\qquad + \left\|R_1(\alpha, x^{\#}(\tau), 0) - R_1(\alpha, P(\tau), 0)\right\|, \end{aligned} \tag{2.6}$$

$$\left\|R_1(\alpha, x^{\#}(\tau), z_\tau) - R_1(\alpha, x^{\#}(\tau), 0)\right\| \le \left\|D_3 R_1(\alpha, x^{\#}(\tau), \mu z_\tau)\right\| \|z_\tau\|, \tag{2.7}$$

and

$$\begin{aligned} &\left\|R_1(\alpha, x^{\#}(\tau), 0) - R_1(\alpha, P(\tau), 0)\right\| \\ &\quad \le \left\|D_2 R_1(\alpha, P(\tau) + \nu(x^{\#}(\tau) - P(\tau)), 0)\right\| \left\|x^{\#}(\tau) - P(\tau)\right\| \end{aligned} \tag{2.8}$$

for some $\mu, \nu \in [0,1]$.

From the hypotheses of Theorem 2.1, if $\varepsilon \to 0$ then $(x_0, z_0) \to (0,0)$ and $\lim\limits_{\varepsilon\to 0} x^{\#}(t) = 0$, $\lim\limits_{\varepsilon\to 0} z_t = 0$. Therefore, $x^{\#}(t) = o(1)$, $z_t = o(1)$ as $\varepsilon \to 0$ and

$$\begin{cases} D_3 R_1(\alpha, x^{\#}(t), \mu z_t) = o(1), \\ D_2 R_1(\alpha, P(t) + \nu(x^{\#}(t) - P(t)), 0) = o(1). \end{cases} \tag{2.9}$$

According to (2.5) through (2.9), with $0 \le t \le \omega_1$, we obtain

$$\|x^{\#}(t) - P(t)\| \le M_1 \left\{ \|x_0 - P_0\| + o(1) \max_{0\le\tau\le t} \|x^{\#}(\tau) - P(\tau)\| + o(1) \max_{0\le\tau\le t} \|z_\tau\| \right\}$$

with $M_1 = \max\limits_{0\le t\le\omega_1} \|e^{B(\alpha)t}\|$. We conclude that

$$\|x^{\#}(t) - P(t)\| \le M \left\{ \|x_0 - P_0\| + o(1) \max_{0\le\tau\le t} \|z_\tau\| \right\}, \tag{2.10}$$

where $M > 0$ is a constant.

The second equation in (2.1) gives

$$\|z_t\| \leq \|T_\alpha(t)z_0\| + \int_0^t \left\| d[K_\alpha(t,\tau)^S] R_2(\alpha, x^{\#}(\tau), z_\tau) \right\|.$$

$$\begin{aligned} \left\| R_2(\alpha, x^{\#}(\tau), z_\tau) \right\| &= \left\| R_2(\alpha, x^{\#}(\tau), z_\tau) - R_2(\alpha, x^{\#}(\tau), 0) \right\| \\ &\leq \left\| D_3 R_2(\alpha, x^{\#}(\tau), \mu z_\tau) \right\| \|z_\tau\|. \end{aligned}$$

Then $\|z_t\| \leq N_1 \left\{ \|z_0\| + o(1) \max_{0\leq\tau\leq t} \|z_\tau\| \right\}$, where $N_1 = \max_{0\leq t\leq \omega_1} \|T_\alpha(t)\|$, and we deduce

(2.11) $$\|z_t\| \leq N \|z_0\|.$$

With the estimate given in Theorem 2.1, (2.10) and (2.11) lead to i) and ii) of Lemma 2.2.

Now, from (1.5) and the first equation in (2.1), we obtain:

(2.12) $$\begin{aligned} &\left\| x^{\#}(t) - x^*(t) \right\| \\ &\leq \int_0^t \left\| e^{B(\alpha)(t-\tau)} (R_1(\alpha, x^{\#}(\tau), z_\tau) - R_1(\alpha, x^*(\tau), 0)) \right\| d\tau. \end{aligned}$$

$$\begin{aligned} \left\| R_1(\alpha, x^{\#}(\tau), z_\tau) - R_1(\alpha, x^*(\tau), 0) \right\| &\leq \left\| D_3 R_1(\alpha, x^{\#}(\tau), \mu z_\tau) \right\| \|z_\tau\| \\ &+ \left\| D_2 R_1(\alpha, x^*(\tau) + \nu(x^{\#}(\tau) - x^*(\tau)), 0) \right\| \left\| x^{\#}(\tau) - x^*(\tau) \right\| \end{aligned}$$

for some μ, $\nu \in [0,1]$.

Finally, from (2.9), there exists $K' > 0$ such that

$$\begin{aligned} &\left\| R_1(\alpha, x^{\#}(\tau), z_\tau) - R_1(\alpha, x^*(\tau), 0) \right\| \\ &\qquad \leq K'\varepsilon \left[\max_{0\leq\tau\leq t} \left\| x^{\#}(\tau) - x^*(\tau) \right\| + \max_{0\leq\tau\leq t} \|z_\tau\| \right] \end{aligned}$$

with a constant $K' > 0$, and from (2.12), we obtain

$$\left\| x^{\#}(t) - x^*(t) \right\| \leq \varepsilon K'' \max_{0\leq\tau\leq t} \|z_\tau\| \qquad (\text{with } K'' > 0),$$

which, together with ii), gives iii). □

Proof of Theorem 2.1. i) In inequality

$$\|x^{\#}(\omega^{\#}) - P_0\| \leq \|x^{\#}(\omega^{\#}) - x^*(\omega^{\#})\| + \|x^*(\omega^{\#}) - x^*(\omega^*)\| + \|x^*(\omega^*) - P_0\| \tag{2.13}$$

the first and the third addend on the right hand side can be estimated using Lemma 2.1 and Lemma 2.2, respectively. In order to estimate the second addend, we need to evaluate $\left|\omega^{\#} - \omega^*\right|$.

Claim 2.2. $\left|\omega^* - \omega^{\#}\right| \leq \eta\theta K'''\varepsilon^3$.

Proof. Near the bifurcation point, equation (1.5) varies more slowly in magnitude than in direction. This is a classical observation, best perceived in polar coordinates representation: $\frac{d}{dt}\phi \simeq \beta(\alpha_0)$; $\frac{d}{dt}\rho = o(\rho)$. So, for some $\varepsilon_0 > 0$, $C, C', C'' > 0$ we have:

$$C\,\|x_0\| \leq \|x^*(t)\| \leq C'\,\|x_0\|\,, \qquad \left|\frac{d}{dt}\phi\right| \geq C'', \tag{2.14}$$

for $0 \leq t \leq \omega_1$, $0 \leq \varepsilon \leq \varepsilon_0$.

Define $\psi = \operatorname{angle}(x^{\#}(\omega^{\#}), x^*(\omega^{\#})) = \operatorname{angle}(x^*(\omega^*), x^*(\omega^{\#})) = \phi^*(\omega^{\#})$. Using Lemma 2.2 iii) and (2.14), we have

$$|\tan\psi| = \tan\phi^*(\omega^{\#}) \simeq \frac{\|x^*(\omega^{\#}) - x^{\#}(\omega^{\#})\|}{\|x^*(\omega^*)\|} \leq \frac{\theta K_2\varepsilon^3\eta}{C}.$$

On the other hand $|\tan\psi| \simeq \phi^*(\omega^{\#}) \geq C''\left|\omega^* - \omega^{\#}\right|$. This yields the desired estimate: $\left|\omega^* - \omega^{\#}\right| \leq \eta\theta K'''\varepsilon^3$. □

The second addend of (2.13) can now be estimated using

$$\left\|x^*(\omega^{\#}) - x^*(\omega^*)\right\| \leq \sup_{0<t<\omega_1}\left\|\frac{dx^*(t)}{dt}\right\|\left|\omega^* - \omega^{\#}\right|.$$

From equation (2.2) we have $\|R_1(\alpha, x, 0)\| = o(x)$, and from the proof of Claim 2.2 we have, $\left\|\frac{dx^*(t)}{dt}\right\| \leq C'C'''\varepsilon$, for some constant C'''.

Thus, in view of the claim 2.2, we obtain

$$\left\|x^*(\omega^{\#}) - x^*(\omega^*)\right\| \leq \eta\theta\varepsilon^4 K_4.$$

To reach the conclusion of i) in Theorem 2.1, choose η small enough for $K_2\eta < K_1/4$ (Lemma 2.1 and Lemma 2.2) and $K_4\eta < K_1/4$; define $K = K_1/2$, and replace addend of right hand side of (2.13) by their corresponding estimates. From Lemma 2.2 we conclude ii). □

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