NEW DISCRETE OSTROWSKI-GRÜSS LIKE INEQUALITIES

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Abstract. The aim of this note is to establish new discrete Ostrowski-Grüss like inequalities involving two finite sequences and their forward differences by using the discrete version of the Montdomery identity.

1. Introduction

In 1938 A. M. Ostrowski [7] proved the following remarkable inequality (see also [6, p. 469]).

Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous on \( [a, b] \) and differentiable on \( (a, b) \) whose derivative \( f' : (a, b) \rightarrow \mathbb{R} \) is bounded on \( (a, b) \), i.e.,

\[
\|f'\|_{\infty} = \sup_{t \in (a,b)} |f'(t)| < +\infty.
\]

Then

\[
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b - a) \|f'\|_{\infty},
\]

for all \( x \in [a, b] \).

Another celebrated inequality proved by G. Grüss [4] in 1935 can be stated as follows (see also [5, p. 296]).

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Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be two integrable functions such that \( \phi \leq f(x) \leq \Phi, \gamma \leq g(x) \leq \Gamma \), for all \( x \in [a, b] \), where \( \phi, \Phi, \gamma \) and \( \Gamma \) are real constants. Then

\[
\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \left( \frac{1}{b-a} \int_a^b f(x) \, dx \right) \left( \frac{1}{b-a} \int_a^b g(x) \, dx \right) \right| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma).
\]

During the last few years, a great deal of research work has been devoted related to the above inequalities. We refer in particular to the books of Mitrinović, Pečarić and Fink [5, 6], Dragomir and Rassias [3] and also the papers appeared in RGMIA Research Report Collections. The main objective of the present note is to establish new discrete Ostrowski-Grüss like inequalities by using a fairly elementary analysis.

### 2. Statement of Results

In order to prove our main results we need the following discrete version of the well known Montgomery identity

\[(2.1) \quad x_k = \frac{1}{n} \sum_{i=1}^n x_i + \sum_{i=1}^{n-1} D_n(k,i) \Delta x_i ,\]

where \( \{x_k\} \) for \( k = 1, \ldots, n \) be a finite sequence of real numbers, \( \Delta x_i = x_{i+1} - x_i \) and

\[(2.2) \quad D_n(k,i) = \begin{cases} i/n, & 1 \leq i \leq k - 1, \\ i/n - 1, & k \leq i \leq n. \end{cases} \]

For the proof of (2.1) and its further generalizations, see [1].

Our main results are given in the following theorems.

**Theorem 2.1.** Let \( \{u_k\}, \{v_k\} \) for \( k = 1, \ldots, n \) be two finite sequences of real numbers such that \( \max_{1 \leq k \leq n-1} \{|\Delta u_k|\} = A \), \( \max_{1 \leq k \leq n-1} \{|\Delta v_k|\} = B \), where \( A, B \) are nonnegative constants. Then the following inequalities hold:

\[(2.3) \quad \left| u_kv_k - \frac{1}{2n} \left[ v_k \sum_{i=1}^n u_i + u_k \sum_{i=1}^n v_i \right] \right| \leq \frac{1}{2} |v_k| A + |u_k| B H_n(k) \]
and

\[ u_k v_k - \frac{1}{n} \left[ v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i \right] + \frac{1}{n^2} \left( \sum_{i=1}^{n} u_i \right) \left( \sum_{i=1}^{n} v_i \right) \leq AB \{ H_n(k) \}^2, \tag{2.4} \]

for \( k = 1, \ldots, n \), where

\[ H_n(k) = \sum_{i=1}^{n-1} |D_n(k, i)|, \tag{2.5} \]

in which \( D_n(k, i) \) is defined by (2.2).

**Remark 2.1.** By taking \( v_k = 1 \) and hence \( \Delta v_k = 0 \) for \( k = 1, \ldots, n \) in (2.3) and by simple calculation, we get

\[ \left| u_k - \frac{1}{n} \sum_{i=1}^{n} u_i \right| \leq H_n(k) \max_{1 \leq k \leq n-1} \{ |\Delta u_k| \}, \tag{2.6} \]

for \( k = 1, \ldots, n \). By elementary computation (see [2]) we have

\[ H_n(k) = \frac{1}{n} \left[ \frac{n^2-1}{4} + \left( k - \frac{n+1}{2} \right)^2 \right]. \]

In fact, the inequality (2.6) is established by Dragomir [2, Theorem 3.1] in a normed linear space.

**Theorem 2.2.** Assume that the hypotheses of Theorem 2.1 hold. Then

\[ |J_n(u_k, v_k)| \leq \frac{1}{2n} \sum_{k=1}^{n} \max \{ |v_k| A + |u_k| B | H_n(k) \} \tag{2.7} \]

and

\[ |J_n(u_k, v_k)| \leq \frac{AB}{n} \sum_{k=1}^{n} (H_n(k))^2, \tag{2.8} \]

where

\[ J_n(u_k, v_k) = \frac{1}{n} \sum_{k=1}^{n} u_k v_k - \left( \frac{1}{n} \sum_{k=1}^{n} u_k \right) \left( \frac{1}{n} \sum_{k=1}^{n} v_k \right), \]

and \( H_n(k) \) is given by (2.5).

**Remark 2.2.** We note that in [10] the present author has established inequalities similar to those of given above by using somewhat different representation. For several other discrete inequalities of the Ostrowski-Grüss type we refer the interested readers to [5, 6, 8, 9].
3. Proofs of Theorems 2.1 and 2.2

From the hypotheses, we have the following identities (see [1]):

\[ u_k - \frac{1}{n} \sum_{i=1}^{n} u_i = \sum_{i=1}^{n-1} D_n(k, i) \Delta u_i \]

and

\[ v_k - \frac{1}{n} \sum_{i=1}^{n} v_i = \sum_{i=1}^{n-1} D_n(k, i) \Delta v_i, \]

for \( k = 1, \ldots, n \). Multiplying (3.1) by \( v_k \) and (3.2) by \( u_k \), adding the resulting identities and rewriting we get

\[ u_k v_k - \frac{1}{2n} \left[ v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i \right] = \frac{1}{2} \left[ v_k \sum_{i=1}^{n-1} D_n(k, i) \Delta u_i + u_k \sum_{i=1}^{n-1} D_n(k, i) \Delta v_i \right]. \]

From (3.3) and using the properties of modulus we have

\[ \left| u_k v_k - \frac{1}{2n} \left[ v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i \right] \right| \]
\[ \leq \frac{1}{2} \left[ \left| v_k \right| \sum_{i=1}^{n-1} |D_n(k, i)| |\Delta u_i| + |u_k| \sum_{i=1}^{n-1} |D_n(k, i)| |\Delta v_i| \right] \]
\[ \leq \frac{1}{2} \left| v_k \right| A + |u_k| B \sum_{i=1}^{n-1} |D_n(k, i)| = \frac{1}{2} \left| v_k \right| A + |u_k| B H_n(k). \]

This is the required inequality in (2.3).

Multiplying the left sides and right sides of (3.1) and (3.2) we get

\[ u_k v_k - \frac{1}{n} \left[ v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i \right] = \frac{1}{n^2} \left( \sum_{i=1}^{n} u_i \right) \left( \sum_{i=1}^{n} v_i \right) \]
\[ = \left[ \sum_{i=1}^{n-1} D_n(k, i) \Delta u_i \right] \left[ \sum_{i=1}^{n-1} D_n(k, i) \Delta v_i \right]. \]
From (3.4) and using the properties of modulus we have
\[
\left| u_k v_k - \frac{1}{n} \left[ v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i \right] + \frac{1}{n^2} \left( \sum_{i=1}^{n} u_i \right) \left( \sum_{i=1}^{n} v_i \right) \right|
\leq \sum_{i=1}^{n-1} |D_n (k, i)| \left| \Delta u_i \right| \sum_{i=1}^{n-1} |D_n (k, i)| \left| \Delta v_i \right|
\leq AB \left[ \sum_{i=1}^{n-1} |D_n (k, i)| \right]^2 = AB \left\{ H_n (k) \right\}^2 ,
\]
which is the desired inequality in (2.4). The proof is complete.

Summing both sides of (3.3) and (3.4) over \( k \) from 1 to \( n \) and rewriting we get
\[
(3.5) \quad J_n (u_k, v_k) = \frac{1}{2n} \sum_{k=1}^{n} \left[ v_k \sum_{i=1}^{n-1} D_n (k, i) \Delta u_i + u_k \sum_{i=1}^{n-1} D_n (k, i) \Delta v_i \right] ,
\]
and
\[
(3.6) \quad J_n (u_k, v_k) = \frac{1}{n} \sum_{k=1}^{n} \left[ \sum_{i=1}^{n-1} D_n (k, i) \Delta u_i \right] \left[ \sum_{i=1}^{n-1} D_n (k, i) \Delta v_i \right] .
\]
From (3.5) and (3.6), using the properties of modulus and closely looking at the proofs of (2.3) and (2.4) we get the required inequalities in (2.7) and (2.8).

REFERENCES

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