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NEW DISCRETE OSTROWSKI-GRÜSS LIKE INEQUALITIES

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Abstract. The aim of this note is to establish new discrete Ostrowski-Grüss like inequalities involving two finite sequences and their forward differences by using the discrete version of the Montdomery identity.

1. Introduction

In 1938 A. M. Ostrowski [7] proved the following remarkable inequality (see also [6, p. 469]).

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e.,

$$\|f'\|_\infty = \sup_{t\in(a,b)} \left|f'(t)\right| < +\infty.$$

Then

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty,$$

for all $x \in [a, b]$.

Another celebrated inequality proved by G. Grüss [4] in 1935 can be stated as follows (see also [5, p. 296]).

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Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\phi \leq f(x) \leq \Phi$, $\gamma \leq g(x) \leq \Gamma$, for all $x \in [a, b]$, where ϕ, Φ, γ and Γ are real constants. Then

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \left(\frac{1}{b-a} \int_a^b f(x) \, dx \right) \left(\frac{1}{b-a} \int_a^b g(x) \, dx \right) \right| \leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma).$$

During the last few years, a great deal of research work has been devoted related to the above inequalities. We refer in particular to the books of Mitrinović, Pečarić and Fink [5, 6], Dragomir and Rassias [3] and also the papers appeared in RGMIA Research Report Collections. The main objective of the present note is to establish new discrete Ostrowski-Grüss like inequalities by using a fairly elementary analysis.

2. Statement of Results

In order to prove our main results we need the following discrete version of the well known Montgomery identity

$$x_k = \frac{1}{n} \sum_{i=1}^{n} x_i + \sum_{i=1}^{n-1} D_n(k, i) \Delta x_i, \tag{2.1}$$

where $\{x_k\}$ for $k = 1, \ldots, n$ be a finite sequence of real numbers, $\Delta x_i = x_{i+1} - x_i$ and

$$D_n(k, i) = \begin{cases} i/n, & 1 \leq i \leq k-1, \\ i/n - 1, & k \leq i \leq n. \end{cases} \tag{2.2}$$

For the proof of (2.1) and its further generalizations, see [1].

Our main results are given in the following theorems.

Theorem 2.1. *Let $\{u_k\}$, $\{v_k\}$ for $k = 1, \ldots, n$ be two finite sequences of real numbers such that $\max_{1 \leq k \leq n-1} \{|\Delta u_k|\} = A$, $\max_{1 \leq k \leq n-1} \{|\Delta v_k|\} = B$, where A, B are nonnegative constants. Then the following inequalities hold:*

$$\left| u_k v_k - \frac{1}{2n} \left[v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i \right] \right| \leq \frac{1}{2} [|v_k| A + |u_k| B] H_n(k) \tag{2.3}$$

and

$$\left|u_k v_k - \frac{1}{n}\left[v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i\right] + \frac{1}{n^2}\left(\sum_{i=1}^{n} u_i\right)\left(\sum_{i=1}^{n} v_i\right)\right| \leq AB\{H_n(k)\}^2, \tag{2.4}$$

for $k = 1, \ldots, n$, *where*

$$H_n(k) = \sum_{i=1}^{n-1} \left|D_n(k,i)\right|, \tag{2.5}$$

in which $D_n(k,i)$ *is defined by* (2.2).

Remark 2.1. By taking $v_k = 1$ and hence $\Delta v_k = 0$ for $k = 1, \ldots, n$ in (2.3) and by simple calculation, we get

$$\left|u_k - \frac{1}{n}\sum_{i=1}^{n} u_i\right| \leq H_n(k) \max_{1 \leq k \leq n-1}\{|\Delta u_k|\}, \tag{2.6}$$

for $k = 1, \ldots, n$. By elementary computation (see [2]]) we have

$$H_n(k) = \frac{1}{n}\left[\frac{n^2-1}{4} + \left(k - \frac{n+1}{2}\right)^2\right].$$

In fact, the inequality (2.6) is established by Dragomir [2, Theorem 3.1] in a normed linear space.

Theorem 2.2. *Assume that the hypotheses of Theorem 2.1 hold. Then*

$$\left|J_n(u_k, v_k)\right| \leq \frac{1}{2n}\sum_{k=1}^{n}\left[\left|v_k\right| A + \left|u_k\right| B\right] H_n(k) \tag{2.7}$$

and

$$\left|J_n(u_k, v_k)\right| \leq \frac{AB}{n}\sum_{k=1}^{n}\left(H_n(k)\right)^2, \tag{2.8}$$

where

$$J_n(u_k, v_k) = \frac{1}{n}\sum_{k=1}^{n} u_k v_k - \left(\frac{1}{n}\sum_{k=1}^{n} u_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} v_k\right),$$

and $H_n(k)$ *is given by* (2.5).

Remark 2.2. We note that in [10] the present author has established inequalities similar to those of given above by using somewhat different representation. For several other discrete inequalities of the Ostrowski-Grüss type we refer the interested readers to [5, 6, 8, 9].

3. Proofs of Theorems 2.1 and 2.2

From the hypotheses, we have the following identities (see [1]):

$$u_k - \frac{1}{n}\sum_{i=1}^{n} u_i = \sum_{i=1}^{n-1} D_n(k,i)\,\Delta u_i \tag{3.1}$$

and

$$v_k - \frac{1}{n}\sum_{i=1}^{n} v_i = \sum_{i=1}^{n-1} D_n(k,i)\,\Delta v_i, \tag{3.2}$$

for $k = 1, \ldots, n$. Multiplying (3.1) by v_k and (3.2) by u_k, adding the resulting identities and rewriting we get

$$\begin{aligned} u_k v_k - \frac{1}{2n}\left[v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i\right] \\ = \frac{1}{2}\left[v_k \sum_{i=1}^{n-1} D_n(k,i)\,\Delta u_i + u_k \sum_{i=1}^{n-1} D_n(k,i)\,\Delta v_i\right]. \end{aligned} \tag{3.3}$$

From (3.3) and using the properties of modulus we have

$$\begin{aligned} &\left| u_k v_k - \frac{1}{2n}\left[v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i\right]\right| \\ &\quad\le \frac{1}{2}\left[|v_k| \sum_{i=1}^{n-1} |D_n(k,i)|\,|\Delta u_i| + |u_k| \sum_{i=1}^{n-1} |D_n(k,i)|\,|\Delta v_i|\right] \\ &\quad\le \frac{1}{2}\left[|v_k|\,A + |u_k|\,B\right] \sum_{i=1}^{n-1} |D_n(k,i)| = \frac{1}{2}\left[|v_k|\,A + |u_k|\,B\right] H_n(k). \end{aligned}$$

This is the required inequality in (2.3).

Multiplying the left sides and right sides of (3.1) and (3.2) we get

$$\begin{aligned} u_k v_k - \frac{1}{n}\left[v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i\right] + \frac{1}{n^2}\left(\sum_{i=1}^{n} u_i\right)\left(\sum_{i=1}^{n} v_i\right) \\ = \left[\sum_{i=1}^{n-1} D_n(k,i)\Delta u_i\right]\left[\sum_{i=1}^{n-1} D_n(k,i)\Delta v_i\right]. \end{aligned} \tag{3.4}$$

From (3.4) and using the properties of modulus we have

$$\begin{aligned}\left|u_k v_k - \frac{1}{n}\left[v_k \sum_{i=1}^{n} u_i + u_k \sum_{i=1}^{n} v_i\right] + \frac{1}{n^2}\left(\sum_{i=1}^{n} u_i\right)\left(\sum_{i=1}^{n} v_i\right)\right| \\ \leq \left[\sum_{i=1}^{n-1} |D_n(k,i)| \, |\Delta u_i|\right]\left[\sum_{i=1}^{n-1} |D_n(k,i)| \, |\Delta v_i|\right] \\ \leq AB\left[\sum_{i=1}^{n-1} |D_n(k,i)|\right]^2 = AB \left\{H_n(k)\right\}^2,\end{aligned}$$

which is the desired inequality in (2.4). The proof is complete.

Summing both sides of (3.3) and (3.4) over k from 1 to n and rewriting we get

$$J_n(u_k, v_k) = \frac{1}{2n} \sum_{k=1}^{n} \left[v_k \sum_{i=1}^{n-1} D_n(k,i) \, \Delta u_i + u_k \sum_{i=1}^{n-1} D_n(k,i) \, \Delta v_i\right], \tag{3.5}$$

and

$$J_n(u_k, v_k) = \frac{1}{n} \sum_{k=1}^{n} \left[\sum_{i=1}^{n-1} D_n(k,i) \, \Delta u_i\right]\left[\sum_{i=1}^{n-1} D_n(k,i) \, \Delta v_i\right]. \tag{3.6}$$

From (3.5) and (3.6), using the properties of modulus and closely looking at the proofs of (2.3) and (2.4) we get the required inequalities in (2.7) and (2.8).

REFERENCES

1. A. Aglić Aljinović and J. Pečarić: *A discrete Euler identity.* J. Inequal. Pure and Appl. Math **5**(3) Art. 58, 2004.
2. S. S. Dragomir: *The discrete version of Ostrowski's inequality in normed linear spaces.* J. Inequal. Pure and Appl. Math. **3**(1) Art. 2, 2002.
3. S. S. Dragomir and Th. M. Rassias (Eds.): *Ostrowski type Inequalities and Applications in Numerical Integration.* Kluwer Academic Publishers, Dordrecht, 2002.
4. G. Grüss: *Über das maximum des absoluten Bertrages von* $\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx$. Math. Z. **39** (1935), 215–226.
5. D. S. Mitrinović, J. E. Pečarić and A. M. Fink: *Classical and New Inequalities in Analysis.* Kluwer Academic Publishers, Dordrecht, 1993.

6. D. S. Mitrinović, J. E. Pečarić and A. M. Fink: *Inequalities for Functions and Their Integrals and Derivatives*. Kluwer Academic Publishers, Dordrecht, 1994.

7. A. M. Ostrowski: *Über die Absolutabweichung einer differentiebaren Functions van ihrem Integralmittelwert*. Comment. Math. Hel. **10** (1938), 226–227.

8. B. G. Pachpatte: *A note on Ostrowski type inequalities*. Demonstratio Math. **35** (2002), 27–30.

9. B. G. Pachpatte: *On multidimensional Ostrowski and Grüss type finite difference inequalities*. J. Inequal. Pure and Appl. Math. **4**(1) Art. 7, 2003.

10. B. G. Pachpatte: *A note on Ostrowski and Grüss type discrete inequalities*. Tamkang J. Math. **35**(1) (2004), 61–65.

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