FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 21 (2006), 77–86

ON THE GUARANTEED CONVERGENCE OF THE CHEBYSHEV-LIKE METHOD FOR COMPUTING POLYNOMIAL ZEROS*

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Abstract. The construction of computationally verifiable initial conditions which provide both the guaranteed and fast convergence of the numerical root-finding algorithm is one of the most important problems in solving nonlinear equations. In this paper we establish initial convergence conditions that guarantee the convergence of the fourth order method for the simultaneous determination of polynomial zeros, proposed in [M.S. Petković, S. Tričković, Đ. Herceg, Japan J. Idustr. Appl. Math. 15 (1998), 295–315]. The stated initial conditions are of significant practical importance since they are computationally verifiable; they depend only on the coefficients of a given polynomial, its degree and initial approximations to polynomial zeros.

One of the most important problems in solving equations of the form $f(z) = 0$ is the construction of such initial conditions which (i) provide the guaranteed convergence of the considered iterative root-finding method and (ii) depend only on the attainable data. In the case of algebraic polynomials, these initial conditions should depend only on the polynomial coefficients, initial approximations to the zeros and the polynomial degree. First and fundamental results devoted to this subject are due to Smale [9], [10], who developed so-called point estimation theory. This approach has been applied later to iterative methods for the simultaneous determination of polynomial zeros, see, e.g., [1]–[7], [10]–[12]. In this paper we will study the guaranteed convergence of the fourth order simultaneous method for finding simple zeros of a given polynomial, proposed in [8].

Received December 1, 2006.
2000 *Mathematics Subject Classification.* Primary 65H05.
*This work was supported by the Serbian Ministry of Science under grant 144024.

We consider monic algebraic polynomials of the form

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \quad (a_i \in \mathbb{C})$$

having only simple (real or complex) zeros. Throughout this paper we will always assume that the polynomial degree n is ≥ 3. For $m = 0, 1, \ldots$ let

$$d^{(m)} = \min_{\substack{1\leq i,j\leq n \\ j\neq i}} |z_i^{(m)} - z_j^{(m)}|$$

be the minimal distance between approximations obtained in the mth iteration, and let

$$W_i^{(m)} = \frac{P(z_i^{(m)})}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (z_i^{(m)} - z_j^{(m)})}, \quad w^{(m)} = \max_{1\leq j\leq n} |W_j^{(m)}|,$$

$$G_{k,i}^{(m)} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{W_j^{(m)}}{(z_i^{(m)} - z_j^{(m)})^k} \quad (k = 1, 2).$$

For simplicity, we will omit sometimes the iteration index m and denote quantities in the latter $(m+1)$-st iteration by $\hat{}$ ("hat").

In this paper we study convergence properties of recently developed Chebyshev-like iterative method for the simultaneous determination of simple complex zeros of a polynomial [8]

$$\hat{z}_i = z_i - \frac{W_i}{1 + G_{1,i}}\Big(1 - \frac{W_i G_{2,i}}{(1 + G_{1,i})^2}\Big) \quad (i \in I_n := \{1, \ldots, n\}). \tag{1}$$

Here z_i is a current approximation and $\hat{z}_i$ is a new approximation to the wanted zero. It was proved in [8] that the order of convergence of the iterative method (1) is equal to four. The aim of this paper is to state computationally verifiable initial conditions which guarantee the convergence of the method (1); namely, they depend only on initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ to the zeros $\zeta_1, \ldots, \zeta_n$ of a polynomial P, its degree n and the polynomial coefficients $a_0, a_1, \ldots, a_{n-1}$.

Before stating the main result concerned with the guaranteed convergence of the simultaneous method (1), we give a convergence theorem which can be applied to a general class of simultaneous methods of the form

$$z_i^{(m+1)} = z_i^{(m)} - C_i(z_1^{(m)}, \ldots, z_n^{(m)}) \quad (i \in I_n;\ m = 0, 1, \ldots), \tag{2}$$

where $z_1^{(m)}, \ldots, z_n^{(m)}$ are some distinct approximations to the simple zeros $\zeta_1, \ldots, \zeta_n$ respectively, obtained in the m-th iterative step by the method (2). In what follows the term

$$C_i^{(m)} = C_i(z_1^{(m)}, \ldots, z_n^{(m)}) \quad (i \in I_n)$$

will be called the *correction.*

The following theorem (see [4], [7]), involving corrections C_i and the function g defined by

$$g(t) = \begin{cases} 1 + 2t, & 0 < t \le \dfrac{1}{2} \\ \dfrac{1}{1-t}, & \dfrac{1}{2} < t < 1 \end{cases}$$

plays the key role in our convergence analysis of the simultaneous method (1).

Theorem 1. *Let $z_1^{(0)}, \ldots, z_n^{(0)}$ be distinct initial approximations to the zeros of P. If there exists a real number $\beta \in (0, 1)$ such that the following two inequalities*

(i) $|C_i^{(m+1)}| \le \beta |C_i^{(m)}| \quad (m = 0, 1, \ldots),$

(ii) $|z_i^{(0)} - z_j^{(0)}| > g(\beta)\Big(|C_i^{(0)}| + |C_j^{(0)}|\Big) \quad (i \ne j, \quad i, j \in I_n),$

are valid, then the iterative method (2) *is convergent.*

First we give three lemmas which are concerned with some necessary bounds and estimates.

Lemma 1. *Let $z_1, \ldots, z_n$ be distinct approximations to the zeros $\zeta_1, \ldots, \zeta_n$ of a polynomial P, and let $\hat{z}_1, \ldots, \hat{z}_n$ be new approximations obtained by the iterative formula* (1)*. If the inequality*

$$w < c_n d, \quad c_n = \frac{2}{5n+3} \quad (n \ge 3), \tag{3}$$

holds, then for all $i \in I_n$ we have

(i) $\dfrac{3n+5}{5n+3} < |1 + G_{1,i}| < \dfrac{7n+1}{5n+3};$

(ii) $|\hat{z}_i - z_i| \le \dfrac{\lambda_n}{c_n} |W_i| \le \lambda_n d, \quad$ *where* $\lambda_n = \dfrac{2(9n^2 + 34n + 21)}{(3n+5)^3}.$

Proof. According to the definition of the minimal distance d and the inequality (3), we have

$$(4)\quad |G_{1,i}| \le \sum_{j\ne i} \frac{|W_j|}{|z_i - z_j|} < (n-1)c_n, \quad |G_{2,i}| \le \sum_{j\ne i} \frac{|W_j|}{|z_i - z_j|^2} < \frac{(n-1)c_n}{d}$$

so that we find

$$|1 + G_{1,i}| \ge 1 - \sum_{j\ne i} \frac{|W_j|}{|z_i - z_j|} > 1 - (n-1)c_n = \frac{3n+5}{5n+3}$$

and

$$|1 + G_{1,i}| \le 1 + \sum_{j\ne i} \frac{|W_j|}{|z_i - z_j|} < 1 + (n-1)c_n = \frac{7n+1}{5n+3}.$$

Therefore, the assertion (i) of Lemma 1 is proved.

Using Lemma 1(i) and (4) we estimate

$$(5)\qquad \Big|\frac{W_i}{1+G_{1,i}}\Big| < \frac{w}{1-(n-1)c_n} < \frac{2}{3n+5}d$$

and

$$(6)\quad \Big|\frac{W_i G_{2,i}}{(1+G_{1,i})^2}\Big| < \frac{w\dfrac{(n-1)c_n}{d}}{(1-(n-1)c_n)^2} < \frac{c_n^2(n-1)}{(1-(n-1)c_n)^2} \le \frac{4(n-1)}{(3n+5)^2}.$$

By Lemma 1(i), (5) and (6) we obtain the bound (ii):

$$\begin{aligned}
|\hat{z}_i - z_i| &= |C_i| = \Big|\frac{W_i}{1+G_{1,i}}\Big(1 - \frac{W_i G_{2,i}}{(1+G_{1,i})^2}\Big)\Big| \\
&\le \frac{|W_i|}{|1+G_{1,i}|}\Big(1 + \frac{|W_i G_{2,i}|}{|1+G_{1,i}|^2}\Big) \\
&< |W_i| \cdot \frac{5n+3}{3n+5}\Big(1 + \frac{4(n-1)}{(3n+5)^2}\Big) \\
&= |W_i| \frac{(5n+3)(9n^2+34n+21)}{(3n+5)^3} \\
&< \frac{2(9n^2+34n+21)}{(3n+5)^3} d = \lambda_n d. \ \square
\end{aligned}$$

Lemma 2. *For distinct complex numbers* $z_1, \dots, z_n, \hat{z}_1, \dots, \hat{z}_n$ *let*

$$d = \min_{\substack{1\le i,j\le n\\ i\ne j}} |z_i - z_j|, \quad \hat{d} = \min_{\substack{1\le i,j\le n\\ i\ne j}} |\hat{z}_i - \hat{z}_j| \quad (i \in I_n).$$

If (3) *holds, then*

$$|\hat{z}_i - z_j| > (1-\lambda_n)d \quad (i,j \in I_n), \tag{7}$$

$$|\hat{z}_i - \hat{z}_j| > (1-2\lambda_n)d \quad (i,j \in I_n), \tag{8}$$

and

$$\left|\prod_{j\neq i} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| < \left(1 + \frac{\lambda_n}{1-2\lambda_n}\right)^{n-1}. \tag{9}$$

The proofs of the inequalities (7), (8) and (9) are derived by using the triangle inequality and the inequality (ii) of Lemma 1. We observe that the necessary condition $\lambda_n < 1/2$ is satisfied under the condition (3).

Let us note that, since (8) is valid for arbitrary pair $i, j \in I_n$ and $\lambda_n < 1/2$ if (3) holds, then there follows

$$\hat{d} = \min_{j\neq i} |\hat{z}_i - \hat{z}_j| > (1-2\lambda_n)d \tag{10}$$

and (8) is well defined.

Lemma 3. *If the inequality* (3) *holds, then*

(i) $|\widehat{W}_i| < 0.22|W_i|$;

(ii) $\widehat{w} < \dfrac{2}{5n+3}\hat{d}$.

Proof. For distinct points $z_1, \ldots, z_n$ we use the Lagrangean interpolation and obtain the following representation of the polynomial P :

$$P(z) = \left(\sum_{j=1}^{n} \frac{W_j}{z - z_j} + 1\right) \prod_{j=1}^{n} (z - z_j).$$

Putting $z = \hat{z}_i$ in the last relation we find

$$P(\hat{z}_i) = (\hat{z}_i - z_i) A_i \prod_{j\neq i} (\hat{z}_i - z_j),$$

where

$$A_i = \sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1 = \frac{W_i}{\hat{z}_i - z_i} + \sum_{j\neq i} \frac{W_j}{\hat{z}_i - z_j} + 1.$$

After dividing with $\prod\limits_{j\neq i}(\hat{z}_i-\hat{z}_j)$, we get

$$\widehat{W}_i=\frac{P(\hat{z}_i)}{\prod\limits_{j\neq i}(\hat{z}_i-\hat{z}_j)}=(\hat{z}_i-z_i)A_i\prod_{j\neq i}\left(\frac{\hat{z}_i-z_j}{\hat{z}_i-\hat{z}_j}\right). \tag{11}$$

From the iterative formula (1) we have

$$\frac{W_i}{\hat{z}_i-z_i}=\frac{-(1+G_{1,i})}{1-\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}}=-1-\frac{G_{1,i}+\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}}{1-\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}}. \tag{12}$$

According to this we find

$$\begin{aligned}
A_i &= -1-\frac{G_{1,i}+\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}}{1-\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}}+\sum_{j\neq i}\frac{W_j}{\hat{z}_i-z_j}+1\\
&= \frac{-\sum\limits_{j\neq i}\dfrac{W_j}{z_i-z_j}+\sum\limits_{j\neq i}\dfrac{W_j}{\hat{z}_i-z_j}-\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}-\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}\sum\limits_{j\neq i}\dfrac{W_j}{\hat{z}_i-z_j}}{1-\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}}\\
&= \frac{-(\hat{z}_i-z_i)\sum\limits_{j\neq i}\dfrac{W_j}{(z_i-z_j)(\hat{z}_i-z_j)}-\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}\Big(1+\sum\limits_{j\neq i}\dfrac{W_j}{\hat{z}_i-z_j}\Big)}{1-\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}}.
\end{aligned}$$

From the last formula we obtain by (6), (7), the definition of the minimal distance and (ii) of Lemma 1

$$\begin{aligned}
|A_i| &\leq \frac{|\hat{z}_i-z_i|\sum\limits_{j\neq i}\dfrac{|W_j|}{|z_i-z_j||\hat{z}_i-z_j|}+\Big|\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}\Big|\Big(1+\sum\limits_{j\neq i}\dfrac{|W_j|}{|\hat{z}_i-z_j|}\Big)}{1-\Big|\dfrac{W_iG_{2,i}}{(1+G_{1,i})^2}\Big|}\\
&< \frac{8(135n^5+594n^4+646n^3-292n^2-821n-262)}{(5n+3)(9n^2+26n+29)(27n^3+117n^2+157n+83)}=y_n.
\end{aligned}$$

Now, taking into account the last estimate and the assertions of Lemma 1 and Lemma 2, we start from formula (11) and obtain

$$|\widehat{W}_i| \leq |\hat{z}_i-z_i|\Big|\sum_{j=1}^{n}\frac{W_j}{\hat{z}_i-z_j}+1\Big|\prod_{j\neq i}\Big(1+\frac{|\hat{z}_j-z_j|}{|\hat{z}_i-\hat{z}_j|}\Big)$$

$$< \frac{\lambda_n}{c_n}|W_i|y_n\Big(1+\frac{\lambda_n}{1-2\lambda_n}\Big)^{n-1} = \phi(n)|W_i|.$$

Using the symbolic computation in the programming package *Mathematica* 5.2, we find that the function ϕ achieves its maximum for $n = 5$,

$$\phi(n) \le \phi(5) < 0.22, \quad \text{for all } n \ge 3.$$

Therefore, $|\widehat{W}_i| < 0.22|W_i|$ and the assertion (i) is valid.

According to this, (10) and the inequality

$$\frac{0.22(3n+5)^3}{27n^3+99n^2+89n+41} \le 0.32 < 1,$$

we find

$$|\widehat{W}_i| < 0.22|W_i| < \frac{0.22\cdot 2}{5n+3}d < \frac{0.22\cdot 2}{5n+3}\cdot\frac{(3n+5)^3}{27n^3+99n^2+89n+41}\hat{d},$$

wherefrom

$$\hat{w} < \frac{2}{5n+3}\hat{d},$$

which proves the assertion (ii) of Lemma 3. □

Now we are able to establish the main convergence theorem for the iterative method (1).

Theorem 2. *If the initial approximations* $z_1^{(0)},\ldots,z_n^{(0)}$ *satisfy the initial condition*

$$w^{(0)} < c_n d^{(0)}, \quad c_n = \frac{2}{5n+3} \quad (n \ge 3), \tag{13}$$

then the iterative method (1) *is convergent.*

Proof. It is sufficient to prove that the inequalities (i) and (ii) of Theorem 1 are valid for the corrections

$$C_i^{(m)} = \frac{W_i^{(m)}}{1+G_{1,i}^{(m)}}\left(1-\frac{W_i^{(m)}G_{2,i}^{(m)}}{(1+G_{1,i}^{(m)})^2}\right) \quad (i \in I_n),$$

which appears in the considered method (1).

Using Lemma 1 (i) and (6), we find

$$\begin{aligned} |C_i| &= |\hat{z}_i - z_i| = \Big|\frac{W_i}{1+G_{1,i}}\Big(1-\frac{W_iG_{2,i}}{(1+G_{1,i})^2}\Big)\Big| \\ &< \frac{W_i}{\frac{3n+5}{5n+3}}\Big(1+\frac{4(n-1)}{(3n+5)^2}\Big) = \frac{(5n+3)(9n^2+34n+21)}{(3n+5)^3}|W_i| \\ &= x_n|W_i|. \end{aligned}$$

It is easy to show that the sequence $\{x_n\}_{n=3,4,\ldots}$ is monotonically increasing and $x_n < x(\infty) = 5/3$, wherefrom

$$|C_i| < \frac{5}{3}|W_i|. \tag{14}$$

In Lemma 3 (assertion (ii)) the implication $w < c_n d \Rightarrow \hat{w} < c_n \hat{d}$ has been proved. Using a similar procedure, we prove by induction that the initial condition (13) implies the inequality $w^{(m)} < c_n d^{(m)}$ for each $m = 1, 2, \ldots$. Therefore, by (i) of Lemma 3 we obtain

$$|W_i^{(m+1)}| < 0.22|W_i^{(m)}| \quad (i \in I_n; m = 0, 1, \ldots).$$

According to this and by the inequalities (i) of Lemma 1 and (14), we obtain (omitting iteration indices)

$$\begin{aligned}
|\widehat{C}_i| &= \frac{5}{3}|\widehat{W}_i| < \frac{5}{3} \cdot 0.22|W_i| \\
&= \frac{1.1}{3}\left|\frac{W_i}{1+G_{1,i}}\Big(1 - \frac{W_i G_{2,i}}{(1+G_{1,i})^2}\Big)\right| \cdot \left|\frac{1+G_{1,i}}{1 - \dfrac{W_i G_{2,i}}{(1+G_{1,i})^2}}\right| \\
&< \frac{1.1}{3}|C_i| \frac{\dfrac{7n+1}{5n+3}}{1 - \dfrac{4(n-1)}{(3n+5)^2}} < 0.52|C_i|.
\end{aligned}$$

In this manner we have proved that the inequality $|C_i^{(m+1)}| < 0.52|C_i^{(m)}|$ holds for each $i = 1, \ldots, n$ and $m = 0, 1, \ldots$. Furthermore, comparing this result with (i) of Theorem 1, we see that $\beta = 0.52$. This furnishes the first part of the theorem. In addition, according to (4) we note that the following is valid:

$$|G_{1,i}| < (n-1)c_n = \frac{2(n-1)}{5n+3} \leq \frac{2}{9} < 1,$$

which means that $0 \notin 1 + G_{k,i}$. Using mathematical induction and the assertion (ii) of Lemma 3 we prove that $0 \notin 1 + G_{k,i}^{(m)}$ for arbitrary iteration index m. Therefore, under the condition (13) the iterative method (1) is well defined in each iteration.

To prove (ii) of Theorem 2, we first note that $\beta = 0.52$ yields $g(\beta) = 1/(1 - 0.52) \approx 2.08$. It remains to prove the disjunctivity of the inclusion discs

$$S_1 = \{z_1^{(0)}; 2.08|C_1^{(0)}|\}, \ldots, S_n = \{z_n^{(0)}; 2.08|C_n^{(0)}|\}.$$

By virtue of (14) we have $|C_i^{(0)}| < \dfrac{5}{3}w^{(0)}$, wherefrom

$$\begin{aligned} d^{(0)} &> \frac{5n+3}{2}w^{(0)} > \frac{5n+3}{2}\cdot\frac{3}{5}|C_i^{(0)}| \geq \frac{3(5n+3)}{20}(|C_i^{(0)}|+|C_j^{(0)}|) \\ &> g(0.52)(|C_i^{(0)}|+|C_j^{(0)}|). \end{aligned}$$

This means that

$$|z_i^{(0)} - z_j^{(0)}| \geq d^{(0)} > g(0.52)(|C_i^{(0)}|+|C_j^{(0)}|) = \operatorname{rad} S_i + \operatorname{rad} S_j.$$

Therefore, the inclusion discs $S_1, \ldots, S_n$ are disjoint, which completes the proof of Theorem 2. $\square$

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