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CONTRACTIVE AFFINE TRANSFORMATIONS OF COMPLEX PLANE AND APPLICATIONS

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Abstract. Some properties of affine automorphisms of complex plane $\mathbb{C}$ are considered. Special stress is set on contractions $\mathbb{C} \to \mathbb{C}$ and applications in creating fractal sets.

1. Introduction

Definition 1.1. Affine transformation f of the complex plane $\mathbb{C}$ is defined by

$$f(z) = Az + B\bar{z} + C, \tag{1.1}$$

where $z \in \mathbb{C}$, and A, B, C are complex constants. If $C = 0$, transformation (1.1) is linear.

The set of all affine transformations given by (1.1) will be denoted by $\mathcal{A}$. It is known that $\mathcal{A}$ endowed with composition of maps $(\mathcal{A}, \circ)$, forms a non-commutative group, having ratio (relation of collinear lengths) as invariance.

The following transformations from $\mathcal{A}$ deserve special attention:

Translation

$$f_{tr}(z) = z + C, \quad C \in \mathbb{C}; \tag{1.2}$$

Rotation (for angle θ anti-clockwise)

$$f_\theta(z) = e^{i\theta} z \quad \text{or} \quad f_\theta(z) = e^{i\theta + 2Arg\, z}\bar{z}, \quad \theta \neq 0,\ \theta \in \mathbb{R}; \tag{1.3}$$

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Homothety (or *homogenous scaling*)

$$f_s(z) = sz, \quad s \neq 1,\ s \in \mathbb{R}; \tag{1.4}$$

Stretch (or *non-homogenous scaling*)

$$f_t(z) = \frac{1+t}{2} z + \frac{1-t}{2} \bar{z}, \quad t \neq \pm 1,\ t \in \mathbb{R}; \tag{1.5}$$

Shear (or *skew*)

$$f_u(z) = \left(1 - i\frac{u}{2}\right) z + i\frac{u}{2} \bar{z}, \quad u \neq 0,\ u \in \mathbb{R}; \tag{1.6}$$

Symmetries, regarding real axis, imaginary axis and coordinate origin are given by

$$f_x(z) = \bar{z}, \quad f_y(z) = -\bar{z}, \quad f_O(z) = -z, \tag{1.7}$$

respectively.

Definition 1.2. Linear transform $f(z) = Az + B\bar{z}$ is unimodular if

$$|\ |A|^2 - |B|^2\ | = 1. \tag{1.8}$$

Unimodular affine transformations preserve area of planar figures being transformed. These transformations $(\mathcal{U}, \circ)$ form a sub-group of $(\mathcal{A}, \circ)$. Typical representative of $\mathcal{U}$ is shear transform, given by (1.6).

Definition 1.3. Affine transform $f(z) = Az + B\bar{z} + C$ is orthogonal if

$$A = s\left(\sqrt{1-a^2} + ia\right) \quad \text{or} \quad B = s\left(b + i\sqrt{1-b^2}\right), \quad a, b \in \mathbb{R}, \tag{1.9}$$

and $s \in \mathbb{R}$, C is arbitrary. Note that both constants A and B have alternative form $se^{i\theta}$, where $s, \theta \in \mathbb{R}$. Orthogonal transformations $(\mathcal{O}, \circ)$ have group property and it is a sub-group of $(\mathcal{A}, \circ)$.

Lemma 1.1. *Orthogonal transformations preserve inner product up to the multiplicative constant.*

Proof. The inner product in $\mathbb{C}$ is defined by

$$\langle z_1,\ z_2 \rangle = \frac{1}{2}(z_1 \bar{z}_2 + \bar{z}_1 z_2). \tag{1.10}$$

Then,

$$\begin{aligned} 2\langle f(z_1),\, f(z_2)\rangle &= f(z_1)\overline{f(z_2)} + \overline{f(z_1)}f(z_2) \\ &= se^{i\theta}z_1 \cdot se^{-i\theta}\bar{z}_2 + se^{-i\theta}\bar{z}_1 \cdot se^{i\theta}z_2 \\ &= s^2(z_1\bar{z}_2 + \bar{z}_1 z_2) = 2s^2\langle z_1,\, z_2\rangle, \end{aligned}$$

i.e., $\langle f(z_1),\, f(z_2)\rangle = s^2\langle z_1,\, z_2\rangle$. □

Definition 1.4. Affine transform $f(z) = Az + B\bar{z} + C$ is orthonormal (unitary) if it is orthogonal and $|s| = 1$.

Translations (1.2), rotations (1.3) and symmetries (1.7) are orthonormal transformations. Homothety (1.4), and combinations, like scaled translations, scaled rotations and scaled symmetries are orthogonal. Orthonormal transformations $(\mathcal{ON}, \circ)$ is included in $(\mathcal{A}, \circ)$ as its sub-group. This is also a sub-group of $(\mathcal{U}, \circ)$.

Orthogonal affine mappings preserve angles while orthonormal ones, besides angles, preserve distances as well.

The last important subset of $\mathcal{A}$ is the set $\mathcal{N}$ of non-homogenous scaling (1.5) $(\mathcal{N}, \circ)$ has structure of semigroup.

Here is hierarchy in the group $(\mathcal{A}, \circ)$:

Orthogonal	$\Longrightarrow$	Affine transform	$\Longleftarrow$	Unimodular
$\Uparrow$		$\Uparrow$		$\Uparrow$
Orthonormal		Non-homog. sc.		Orthonormal

or

$$\begin{array}{ccccc} (\mathcal{O}, \circ) & \subset & (\mathcal{A}, \circ) & \supset & (\mathcal{U}, \circ) \\ \cup & & \cup & & \cup \\ (\mathcal{ON}, \circ) & & (\mathcal{N}, \circ) & & (\mathcal{ON}, \circ) \end{array}$$

2. Binary Decomposition of the Group $(\mathcal{A}, \circ)$

Let

(2.1) $$\phi : z \mapsto \mathrm{A}z + \mathrm{B}\bar{z},$$

be the linear part of transformation (1.1), so $\phi \in (\mathcal{A}, \circ)$. Then, the following decomposition theorem takes place:

Theorem 2.1. *$\phi = \phi^{\perp} \circ \phi^{\Delta}$, where $\phi^{\perp}$ and ϕ^{Δ} are orthogonal and non-orthogonal component respectively given by*

$$\phi^{\perp}(z) = se^{i\theta}z, \quad \phi^{\Delta}(z) = \Big(\frac{1+t}{2} - i\frac{u}{2}\Big)z + \Big(\frac{1-t}{2} + i\frac{u}{2}\Big)\bar{z}, \tag{2.2}$$

Proof. It is proved in Lemma 1.1 that $\phi^{\perp}$ is orthogonal since it preserves inner product. On the other hand, $\phi^{\Delta}(z)$ does not since $\langle \phi^{\Delta}(z_1), \phi^{\Delta}(z_2)\rangle = \frac{1}{2}z_1 z_2$ which coincide with $\langle z_1, z_2\rangle$ if and only if z_1 and z_2 are real numbers. Therefore, transformation ϕ^{Δ} by no means can preserve orthogonality in complex plane. □

Remark 2.1. Note that $\phi^{\Delta}(z)$ is composition of stretch (1.5) and shear (1.6) transformation. In fact,

$$f_t(f_u(z)) = \frac{1}{2}\big[(1+t)f_u(z) + (1-t)\overline{f_u(z)}\big] = \frac{1}{2}\big[(1+t) - iu\big]z + \frac{1}{2}\big[(1-t) + iu\big]\bar{z} = \phi^{\Delta}(z).$$

But, $f_u(f_t(z)) = \big(1 - i\frac{u}{2}\big)f_t(z) + i\frac{u}{2}\overline{f_t(z)} = \phi^{\Delta}(z)$. Thus, $\phi^{\Delta}(z) = (f_t \circ f_u)(z) = (f_u \circ f_t)(z)$. Also, since $f(z) = \phi(z) + C$, the final decomposition form of (1.1) is

$$f(z) = se^{i\theta}\Big[\Big(\frac{1+t}{2} - i\frac{u}{2}\Big)z + \Big(\frac{1-t}{2} + i\frac{u}{2}\Big)\bar{z}\Big] + C, \tag{2.3}$$

where $C \in \mathbb{C}$ defines translation.

Definition 2.1. Real parameters s, θ, t and u will be called decomposition parameters of affine transformation (1.1).

Theorem 2.2. *For given A and B in* (1.1) *decomposition parameters are given by*

$$\begin{aligned} s &= \sqrt{(A+B)(\bar{A}+\bar{B})}\ \mathrm{sgn}\,(A+B+\bar{A}+\bar{B}) \\ \theta &= -i\ \tan^{-1}\Big(\frac{A+B-\bar{A}-\bar{B}}{A+B+\bar{A}+\bar{B}}\Big), \\ t &= \frac{\det \mathrm{A}}{(A+B)(\bar{A}+\bar{B})}, \quad u = i\frac{(A\bar{B}-B\bar{A})}{(A+B)(\bar{A}+\bar{B})}. \end{aligned} \tag{2.4}$$

Proof. According to Theorem 2.1, $\phi(z) = Az + B\bar{z}$, where

$$\begin{aligned} A &= \frac{s}{2}[(1+t)\cos\theta + u\sin\theta] + i\frac{s}{2}[(1+t)\sin\theta - u\cos\theta], \\ B &= \frac{s}{2}[(1-t)\cos\theta - u\sin\theta] + i\frac{s}{2}[(1-t)\sin\theta + u\cos\theta], \end{aligned}$$

or

$$A = \frac{s}{2}e^{i\theta}(1 + t - iu), \quad B = \frac{s}{2}e^{i\theta}(1 - t + iu). \tag{2.5}$$

The result (2.4) obtains by conjugating (2.5) and solving derived set of four nonlinear equations. □

Now, on the basis of Theorem 2.2, one can classify affine transformations in $\mathbb{C}$ plane in terms of decomposition parameters as follows:

$f(z)$	s	θ	t	u	A	B	C
translation	1	0	1	0	1	0	$C \neq 1$
rotation	1	$\theta \neq 1$	1	0	$e^{i\theta}$	0	0
homothety	$s \neq 1$	0	1	0	s	0	0
stretch	1	0	$t \neq 1$	0	$(1+t)/2$	$(1-t)/2$	0
shear	1	0	1	$u \neq 1$	$1 - iu/2$	$iu/2$	0
symmetry-x	1	0	-1	0	0	1	0
symmetry-y	-1	0	-1	0	0	-1	0
symmetry-O	1	π	1	0	-1	0	0

3. Contractions $\mathbb{C} \to \mathbb{C}$

Using the inner product defined by (1.10) one may to define metric in $\mathbb{C}$ in the usual way, $d(z_1, z_2) = \sqrt{\langle z_1, z_2 \rangle}$. Now $(\mathbb{C}, d)$ is a metric space.

Definition 3.1. Transformation of the metric space $(\mathbb{C}, d)$ into itself is contraction if and only if

$$d(f(z_1), f(z_2)) < d(z_1, z_2),$$

for all $z_1, z_2 \in \mathbb{C}$.

Theorem 3.1. *The map* (1.1) *is contraction provided* $|A| + |B| < 1$, *or*

$$|s|(\sqrt{(1+t)^2 + u^2} + \sqrt{(1-t)^2 + u^2}) < 2. \tag{3.1}$$

Proof. Let z_1 and z_2 $(z_1 \neq z_2)$ be two complex numbers. Then we have

$$\begin{aligned} |f(z_2) - f(z_1)| &= |Az_2 + B\bar{z}_2 + C - Az_1 - B\bar{z}_1 - C| \\ &= |A(z_2 - z_1) + B(\bar{z}_2 - \bar{z}_1)| \\ &\leq (|A| + |B|)|z_2 - z_1| < |z_2 - z_1|. \end{aligned}$$

So, $|A|+|B|<1$ implies contractivity of (1.1). Now, from (2.5) and Theorem 2.2, it follows that (1.1) is contraction if

$$\begin{aligned} |A|+|B| &= |s|\Big(\Big|\,\frac{1+t}{2}-i\frac{u}{2}\,\Big|+\Big|\,\frac{1-t}{2}+i\frac{u}{2}\,\Big|\Big) \\ &= |s|\Big(\sqrt{(1+t)^2+u^2}+\sqrt{(1-t)^2+u^2}\Big)\cdot\frac{1}{2}<1 \end{aligned}$$

or (3.1). □

The main application of contractive mapping in $\mathbb{C}$ is to produce self-affine fractal sets. Let

$$\{(\mathbb{C},d),\ f_1,\dots,f_n\},\quad n\ge 2 \tag{3.2}$$

be Iterated Function System (IFS) defined in metric space $(\mathbb{C},d)$, and let each f_k be a contraction in $(\mathbb{C},d)$.

Let $f(S)=\{f(z):\ z\in S\}$ and $W(S)=\cup_{i=1}^n f_i(S)$, $S\subset\mathbb{C}$ be Hutchinson operator [3]. According to [3] there exists a compact set $K\subset\mathbb{C}$ such that $W(K)=K$. In other words, the sequence K_0 (arbitrary finite set), $K_1,K_2,\dots$ where $K_i=W^{\circ i}(K_0)$, $i=1,2,\dots$ converge towards K, $\lim_{m\to+\infty}W^{\circ m}(K_0)=K$. The set K is known as attractor of the IFS which is called *hyperbolic* (3.2).

In order to make fractal constructions easier, it is advisable to have triangle to triangle transformation expressed in terms of coefficients A, B and C of (1.1).

Let two triangles $T_1=\{z_1,z_2,z_3\mid z_i\ne z_j,\ i\ne j\}$ and $T_2=\{w_1,w_2,w_3\mid w_i\ne w_j, i\ne j\}$ be given in the complex plane, and let $f:T_1\to T_2$. Then, we have:

Theorem 3.2. *Transformation $f:T_1\to T_2$ is contraction provided*

$$\begin{aligned} &|w_1(\bar z_2-\bar z_3)+w_2(\bar z_3-\bar z_1)+w_3(\bar z_1-\bar z_2)| \\ &|w_1(z_2-z_3)+w_2(z_3-z_1)+w_3(z_1-z_2)|+ \\ &\quad<|(z_3-z_2)\bar z_1+(z_1-z_3)\bar z_2+(z_2-z_1)\bar z_3|. \end{aligned}$$

Proof. Transformation f which maps T_1 into T_2 is determined by the coefficients

$$\begin{aligned} A &= \frac{w_1(\bar z_2-\bar z_3)+w_2(\bar z_3-\bar z_1)+w_3(\bar z_1-\bar z_2)}{z_1(\bar z_2-\bar z_3)+z_2(\bar z_3-\bar z_1)+z_3(\bar z_1-\bar z_2)}, \\ B &= \frac{w_1(z_3-z_2)+w_2(z_1-z_3)+w_3(z_2-z_1)}{z_1(\bar z_2-\bar z_3)+z_2(\bar z_3-\bar z_1)+z_3(\bar z_1-\bar z_2)}, \\ C &= \frac{w_1(z_2\bar z_3-z_3\bar z_2)+w_2(z_3\bar z_1-z_1\bar z_3)+w_3(z_1\bar z_2-z_2\bar z_1)}{z_1(\bar z_2-\bar z_3)+z_2(\bar z_3-\bar z_1)+z_3(\bar z_1-\bar z_2)}, \end{aligned} \tag{3.3}$$

which can be derived by solving the system of equations $f(z_k) = w_k,\ k = 1, 2, 3$. From Theorem 3.1 and (3.3) one gets the claim of this theorem. □

Since fractal constructions are affine invariant, in the sense that transforming the initial position of triangular configuration will result in transformation of the whole figure, one can restrict himself on special position of starting triangles, say by taking $z_1 = 0,\ z_2 = 1,\ z_3 = i$ and $w_1 = 0,\ w_2 = h$, $w_3 = p + iq$. Let this special triangle be denoted by T_1^* and T_2^* respectively. Then one has the following consequence of Theorem 3.2.

Corollary 3.1. *Transformation $f : T_1^* \to T_2^*$ with special choice as above is contraction in $\mathbb{C}$ if*

$$p^2 + (1 - h^2)q^2 < 1 - h^2, \quad |h| < 1. \tag{3.4}$$

In this case

$$A = \frac{1}{2}(h + q - ip),\ B = \frac{1}{2}(h - q + ip),\ C = 0. \tag{3.5}$$

Proof. The formula (3.5) can be obtained from (3.3) by substituting $z_1 = 0,\ z_2 = 1,\ z_3 = i,\ w_1 = 0,\ w_2 = h,\ w_3 = p + iq$. From (3.1), f is contraction $\mathbb{C} \to \mathbb{C}$ if

$$h^2 + p^2 + q^2 + \sqrt{p^2 + (h + q)^2} < 2$$

wherefrom, by algebraic transformations one gets

$$p^2 + (1 - h^2)q^2 < 1 - h^2. \tag{3.6}$$

Equality in (3.6) gives an ellipse in $(p, q)-$coordinate system with half-axes $a = \sqrt{1 - h^2},\ b = 1$. So, the solution of (3.6) is the open subset of $\mathbb{C}$ bounded by ellipse $p^2 + (1 - h^2)q^2 = 1 - h^2$. □

Many important fractal constructions, especially all "classic" constructions like Koch, Peano, Sierpinski, Cesaro, etc., can be considered as attractors of the IFS (3.2) that consists of orthogonal affine contractions $f(z) = se^{i\theta}z + C$. These self-similar fractal constructions are known under the common name *Initiator-Generator method* [5]. This method assumes pair of geometric objects *Initiator* and *Generator*. Usually, *Initiator* is linear segment, say $\{0, 1\}$ and *Generator* is a polygonal line $\{L_1, L_2, \ldots, L_n\} \subset \mathbb{C}$. Then,

$$f_k(z) = s_k e^{i\theta_k} z + C_k\ :\ \{0, 1\} \to L_k, \quad k = 1, \ldots, n. \tag{3.7}$$

Theorem 3.3. *Let S be an IFS* (3.2) *where mappings are given by* (3.7)*. Then, S is hyperbolic if $|L_k| < 1$, $k = 1, \ldots, n$, and f_k may be one of the following mappings*

$$z \mapsto \Delta w_k z + w_k, \tag{3.8}$$

$$z \mapsto -\Delta w_k z + w_{k+1}, \tag{3.9}$$

$$z \mapsto \Delta w_k \bar{z} + w_k, \tag{3.10}$$

$$z \mapsto -\Delta w_k \bar{z} + w_{k+1}, \tag{3.11}$$

where $L_k = \{w_k, w_{k+1}\}$, $\Delta w_k = w_{k+1} - w_k$, and $k = 1, 2, \ldots, n$.

Proof. There are four ways to map $\{0,1\}$ orthogonally to L_k: a) $\{0,1\}$ to $\{w_k, w_{k+1}\}$; b) $\{0,1\}$ to $\{w_{k+1}, w_k\}$; c) substitute z with $\bar{z}$ and apply a), and d) substitute z with $\bar{z}$ and apply b). Thus, the form of f_k will be $Az+C$ with $A = w_{k+1} - w_k$, $C = w_k$ or $A = w_k - w_{k+1}$, $C = w_{k+1}$ in the cases a) and b) respectively. By substituting $z \to \bar{z}$ the symmetry transform of the $\mathbb{C}$-plane is performing although preserving the segment $\{0,1\}$, and then map c) gives $A\bar{z} + C$ where A and C are the same as in the case a), so $A = w_{k+1} - w_k$, $C = w_k$, while in the case d) it is $A = w_k - w_{k+1}$, $C = w_{k+1}$. So, this explains equations (3.8)–(3.11). In all cases $|A| = |\Delta w_k| = |L_k| < 1$ is sufficient condition for all f_k to be orthogonal contractions in $\mathbb{C}$, thereby making IFS a hyperbolic one. □

Theorem 3.3 is a formal extension of the above mentioned Initiator-Generator method. By varying each mapping in orthogonal affine (hyperbolic) IFS for some old-known fractals, one may contribute with variety of new forms. To illustrate this, let us consider some examples.

4. Examples

Here we give five examples.

Example 4.1 (von Koch curve). One of the oldest constructions that later turned out to be the fractal construction goes back to Helge von Koch [4]. It consists of four contractive mappings that are orthogonal ($t = 1$, $u = 0$) so that, accordingly to Theorem 3.3 one gets:

$$\begin{aligned} f_1(z) &= \frac{1}{3}z, \quad f_2(z) = \frac{1}{6}(1 + i\sqrt{3})z + \frac{1}{3}, \\ f_3(z) &= \frac{1}{6}(1 - i\sqrt{3})z + \frac{1}{6}(3 + i\sqrt{3}), \quad f_4(z) = \frac{1}{3}z + \frac{2}{3}. \end{aligned}$$

All these maps are contractions, so IFS is hyperbolic. The attractor of this IFS is known as Koch's curve (Figure 1 (left-top)).

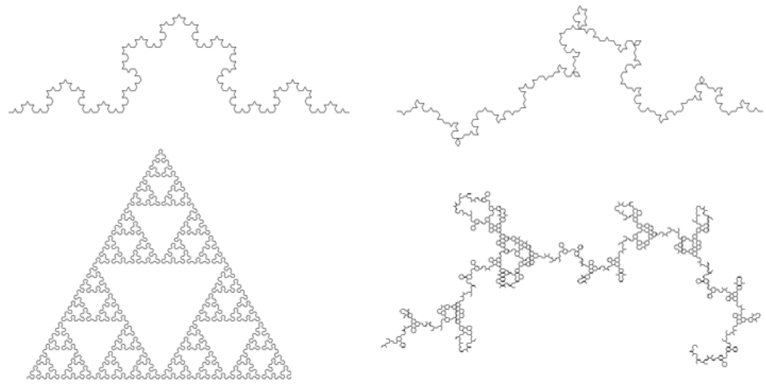

FIG. 1: Von Koch curve and its modification (top) and Sierpinski triangle and modification (bottom)

Example 4.2 (Variation on Koch curve). If z in any of f_i from Example 4.1 be replaced by $\overline{z}$ one gets symmetric mapping. More precise, if $f_i(z)$ maps $(0, 1)$ into (w_1, w_2), $f_i(\overline{z})$ will do the same but with the complex plane being flipped around the real axis. So, the IFS give by

$$f_1(z) = \frac{\overline{z}}{3}\ , \quad f_2(z) = \frac{1}{6}(1 + i\sqrt{3})\overline{z} + \frac{1}{3},$$

will produce fractal curve that is quite different from classic von Koch curve (Figure 1 (right-top)).

Example 4.3 (Peano curve). The attractor of the following IFS:

$$\begin{aligned} f_1(z) &= \frac{1}{3}z, \quad f_2(z) = \frac{1}{3}(iz+1), \quad f_3(z) = \frac{1}{3}(z+1+i), \\ f_4(z) &= \frac{1}{3}(-iz+2+i), \quad f_5(z) = \frac{1}{3}(-z+2), \quad f_6(z) = \frac{1}{3}(-iz+1), \\ f_7(z) &= \frac{1}{3}(z+1-i), \quad f_8(z) = \frac{1}{3}(z+2-i), \quad f_9(z) = \frac{1}{3}(z+2), \end{aligned}$$

is known as Peano's curve [6]. Note that in this case, all four possibilities offered by Theorem 3.3 leaves the attractor unchanged.

Example 4.4 (Sierpinski curve). This is attractor (Figure 1 (left-bottom)) of the IFS given by the following mappings:

$$\begin{aligned} f_1(z) &= \frac{1}{4}(1+i\sqrt{3})\overline{z}, \quad f_2(z) = \frac{1}{4}(2z+1+i\sqrt{3}), \\ f_3(z) &= \frac{1}{4}((1-i\sqrt{3})\overline{z}+3+i\sqrt{3}). \end{aligned}$$

Note that f_1 and f_3 are of the type (3.8) and f_2 of the (3.10).

Example 4.5 (Variation on Sierpinski curve). If f_2 in the Sierpinski IFS from previous example is replaced by $f_2(z) = \frac{1}{4}(2\bar{z} + 1 + i\sqrt{3})$ (type (3.10)) and f_3 by $f_3(z) = \frac{1}{4}((1 - i\sqrt{3})z + 3 + i\sqrt{3})$ (type (3.8)) the attractor dramatically changes into the form given in Figure 1 (right-bottom).

5. Conclusion

Using complex plane in studying fractal constructions is not a new idea. Some hints and particular examples are given in almost all "classics" of fractal themes ([5], [1], etc). But, somehow it lacks a systematic study of such simple thinks like affine automorphisms of complex plane. Here, we tried to fill this gap by mentioning some basic stuff about this issue, like classification of affine mappings, its group and the refinement on subgroups as well as their relations. Further, we offer decomposition of affine mappings $\mathbb{C} \to \mathbb{C}$ on orthogonal and non-orthogonal sub-mappings. Then, we discuss metric properties of these mappings and establish sufficient conditions for contractivity. After this, we apply such a theory on more concrete collections of contractive mappings known as IFS's like an extended Initiator-Generator method (Theorem 3.3). Finally, some examples of classic fractal sets are used to demonstrate possibility of their variations based on Theorem 3.3.

REFERENCES

1. M. F. Barnsley: *Fractals Everywhere*, Academic Press, 1988.
2. Y. Fisher (ed.): *Fractal Image Compression. Theory and Applications*, Springer, 1995.
3. J. E. Hutchinson: *Fractals and self similarity*, Indian J. Math., **30** (1981), 713–747.
4. H. von Koch: *Sur une courbe continue sans tangente, obtenue par une construction géometrique élémentaire*, Arkiv für Matematik **1** (1904), 681–704.
5. B. Mandelbrot: *The Fractal Geometry of Nature*, San Francisco, W.H. Freeman and Co., 1983.
6. G. Peano: *Sur une courbe, qui remplit toute une aire plaine*, Math. Annl. **36** (1890), 157–160.
7. R. S. Varga, *Matrix Iterative Analysis*, Prentice Hall, 1962.

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