# CONSISTENT ESTIMATING IN UAR MODELS\*

## Biljana Č. Popović and Miroslav M. Ristić

**Abstract.** We consider the first and the second order autoregressive time series with uniform marginal distribution. The asymptotic properties of the estimators of their parameters are discussed. We prove the consistency of these estimators.

#### 1. Introduction

At the beginning of ninth decade of the previous sanctuary the autoregressive time series with the uniform marginal distribution were focussed. Chernick [2] was the first who had defined the first order autoregressive uniformly distributed process on (0, 1),  $\{U_n\}$ :

$$U_n = \frac{1}{k} U_{n-1} + \varepsilon_n, \ n \in \mathbb{Z},$$

where  $\mathbb{Z}$  was the set of all integers,  $k \geq 2$  and  $k \in \mathbb{Z}$ ,  $\{\varepsilon_n\}$  was the sequence of i.i.d. random variables defined as

$$\left(\begin{array}{ccc} 0 & 1/k & \dots & (k-1)/k \\ 1/k & 1/k & \dots & 1/k \end{array}\right)$$

and random variables  $U_m$  and  $\varepsilon_n$  were independent if and only if m < n. The properties of the uniformly distributed processes were explored by Ristić and Popović in [7]. They estimated the unknown parameter of the model by means of the method of least squares and the method of maximum of

Received May 4, 2006

 $<sup>2000\</sup> Mathematics\ Subject\ Classification.\ 62 M10.$ 

<sup>\*</sup>This work was partially supported by Grant of MNTR 1834.

ratios of the process' values. According to the sample  $(U_1, U_2, \ldots, U_N)$ , the estimator which follows the second mentioned method is

(1.1) 
$$\hat{k}_N^+ = \max_{2 \le i \le N} \left\{ \frac{U_{i-1}}{U_i} \right\}$$

and its asymptotic properties will be discussed in Section 2.

In 1982 Chernick and Davis [3] defined the first order uniform autoregressive process with negative autocorrelations:

$$U_n = -\frac{1}{k}U_{n-1} + \varepsilon_n, \ n \in \mathbb{Z},$$

where  $k \geq 2$  is an integer,  $\{\varepsilon_n\}$  is the sequence of i.i.d. random variables defined as

$$\left(\begin{array}{ccc} 1/k & \dots & (k-1)/k & 1\\ 1/k & \dots & 1/k & 1/k \end{array}\right)$$

and the random variables  $U_m$  and  $\varepsilon_n$  are independent iff m < n. The unknown parameter of this model was estimated in [7] by Ristić and Popović also. The methods used were the same as mentioned above and the estimator produced by the method of maximum of values of ratios was

(1.2) 
$$\hat{k}_N^- = \max_{2 \le i \le N} \left\{ \frac{U_{i-1}}{1 - U_i} \right\}$$

and its asymptotic properties will be discussed in Section 2 of this paper.

Ristić and Popović defined the new uniform autoregressive process of the first order (NUAR(1)) in [6]. It was the generalization of just mentioned two processes and it was defined as

$$U_n = \begin{cases} \alpha U_{n-1}, & \text{w.p. } \alpha, \\ \beta U_{n-1} + \varepsilon_n, & \text{w.p. } 1 - \alpha, \end{cases}$$

where  $0 < \alpha, \beta < 1$ , but  $(1-\alpha)/\beta \ge 2$  was an integer,  $\{\varepsilon_n\}$  was the sequence of i.i.d. random variables defined by

$$\begin{pmatrix} \alpha & \alpha + \beta & \dots & 1 - \beta \\ \beta/(1 - \alpha) & \beta/(1 - \alpha) & \dots & \beta/(1 - \alpha) \end{pmatrix}$$

and the random variables  $U_m$  and  $\varepsilon_n$  were independent iff m < n. In the same paper, the authors estimated parameter  $\alpha$  of this model by means of minimum of the ratios of values. The considered estimator was

(1.3) 
$$\hat{\alpha}_N = \min_{2 \le i \le N} \left\{ \frac{U_i}{U_{i-1}} \right\}.$$

The second parameter  $\beta$  of this model is estimated in Section 2 of this paper. In the same section the asymptotic properties of these two estimators are discussed.

The following uniform autoregressive process of the second order (UAR(2)) was defined by Ristić and Popović in [8]

$$U_n = \begin{cases} \alpha U_{n-1} + \varepsilon_n, & \text{w.p. } \frac{\alpha}{\alpha - \beta}, \\ \beta U_{n-2} + \varepsilon_n, & \text{w.p. } -\frac{\beta}{\alpha - \beta}, \end{cases}$$

where  $\alpha$  and  $\beta$  were parameters,  $\alpha \in [0, 1)$  and  $\beta \in (-1, 0]$ , such that  $\alpha - \beta > 0$ ,  $\{\varepsilon_n\}$  was the sequence of i.i.d. random variables with the distribution

$$P\{\varepsilon_n = j(\alpha - \beta) - \beta\} = \alpha - \beta \quad , \quad j = 0, 1, \dots, k - 1 \, , \quad k = \frac{1}{\alpha - \beta}$$

and the random variables  $U_m$  and  $\varepsilon_n$  were independent iff m < n. The parameters of this model were estimated by the method of conditional least square in the same paper. In Section 3 of this paper the same parameters are estimated by the method of ratios of values of the process and their asymptotic properties are discussed.

### 2. Consistent Estimators of the Parameters of the First Order Autoregression

In this section the estimators of the first order uniform autoregressive processes will be determined by the method of the ratios of values. The consistency of these estimators will be proved also.

Let us consider the first order uniformly distributed autoregressive time series defined by Chernick [2] and the estimator  $\hat{k}_N^+$  defined in (1.1). Let us assign  $G_N^+(x) = P\{\hat{k}_N^+ \leq x\}$ , the distribution function of the estimator  $\hat{k}_N^+$ . From the definition of the estimator itself, it follows that

$$G_N^+(x) = P\left\{\bigcap_{i=2}^N \left\{U_{i-1} \le xU_i\right\}\right\} = P\left\{\bigcap_{i=2}^N \left\{\left(1 - \frac{x}{k}\right)U_{i-1} \le x\varepsilon_i\right\}\right\}.$$

Let be  $x \ge k$ . Then, we have  $(1 - \frac{x}{k}) U_{i-1} \le x \varepsilon_i$  almost sure for all  $i = 2, 3, \ldots, N$ . That is because of the negative sign of the left side and the

positive sign of the right side of this inequality. It follows that  $G_N^+(x) = 1$  for any  $x \ge k$ . Let us consider now x < k. Then it will be

$$G_N^+(x) = \frac{1}{k^{N-1}} \sum_{j_2=0}^{k-1} \cdots \sum_{j_N=0}^{k-1} P\left\{ \bigcap_{i=2}^N \left\{ \left(1 - \frac{x}{k}\right) U_{i-1} \le \frac{xj_i}{k} \right\} \right\} \le \left(\frac{k-1}{k}\right)^{N-1},$$

because of the fact that  $P\left\{\left(1-\frac{x}{k}\right)U_{i-1} \leq \frac{xj_i}{k}\right\} = 0$  whenever  $j_i = 0$ . If  $N \to +\infty$ , then

$$G_N^+(x) \to G^+(x) = \begin{cases} 0, & x < k, \\ 1, & x \ge k \end{cases}$$

It means that  $\hat{k}_N^+$  converges to k in distribution when  $N \to +\infty$ . This implies the convergence in probability of the same estimator and so, the defined estimator is the consistent estimator of k.

The third process that we shall consider here is the new uniform autoregressive process of the first order defined by Ristić and Popović in [6]. The parameter  $\beta$  can be estimated by means of the minimum of the ratios also. The estimator will be

(2.1) 
$$\hat{\beta}_N = \min_{2 \le i \le N} \left\{ \frac{1 - U_i}{1 - U_{i-1}} \right\}.$$

Let us consider the estimator for the parameter  $\alpha$  defined by (1.3). Let's use the notation  $G_N^*(x) = P\{\hat{\alpha}_N > x\}$ . It follows from the definition of  $\hat{\alpha}_N$  that

$$G_N^*(x) = P\left\{\bigcap_{i=2}^N \{U_i > xU_{i-1}\}\right\}.$$

Further on we have

$$G_N^*(x) = \alpha P \left\{ \bigcap_{i=2}^{N-1} \{ U_i > x U_{i-1} \}, \alpha U_{N-1} > x U_{N-1} \right\}$$
  
+  $(1 - \alpha) P \left\{ \bigcap_{i=2}^{N-1} \{ U_i > x U_{i-1} \}, \beta U_{N-1} + \varepsilon_N > x U_{N-1} \right\}.$ 

If  $x < \alpha$ , then  $\alpha U_{N-1} > xU_{N-1}$  and  $\beta U_{N-1} + \varepsilon_N > xU_{N-1}$  almost sure. As a consequence, it will be  $G_N^*(x) = G_{N-1}^*(x) = \cdots = G_2^*(x) = 1$ . If  $x = \alpha$ , then  $P\{\alpha U_{N-1} > xU_{N-1}\} = 0$  and  $\beta U_{N-1} + \varepsilon_N > xU_{N-1}$  almost sure, so that  $G_N^*(\alpha) = (1 - \alpha)G_{N-1}^*(\alpha) = \cdots = (1 - \alpha)^{N-2}G_2^*(\alpha) = (1 - \alpha)^{N-1}$ . As

60

 $G_N^*(x)$  is the decreasing function, it follows that  $G_N^*(x) \leq (1-\alpha)^{N-1}$  for all  $x \geq \alpha$ . So, when  $N \to +\infty$ ,

$$G_N^*(x) \to G^*(x) = \begin{cases} 1, & x < \alpha, \\ 0, & x \ge \alpha, \end{cases}$$

and it completes the proof that  $\hat{\alpha}_N$  is the consistent estimator for  $\alpha$ .

Let us prove now that  $\hat{\beta}_N$  is the consistent estimator for  $\beta$ . Further on we will use the notation  $H_N^*(x) = P\{\hat{\beta}_N > x\}$ . From the definition of the proposed estimator, it follows that

$$H_N^*(x) = P\left\{\bigcap_{i=2}^N \{1 - U_i > x(1 - U_{i-1})\}\right\}.$$

If  $x < \beta$ , then  $H_N^*(x) = H_{N-1}^*(x) = \cdots = H_2^*(x) = 1$ . If  $x = \beta$ , then  $H_N^*(\beta) = (1-\beta)H_{N-1}^*(\beta) = \cdots = (1-\beta)^{N-2}H_2^*(\beta) = (1-\beta)^{N-1}$ . Since the function  $H_N^*(x)$  is the decreasing one, the inequality  $H_N^*(x) \le (1-\beta)^{N-1}$  will be fulfilled for any  $x \ge \beta$ . So, when  $N \to +\infty$ ,

$$H_N^*(x) \to H^*(x) = \begin{cases} 1, & x < \beta, \\ 0, & x \ge \beta. \end{cases}$$

The proof of the consistency of the estimator  $\hat{\beta}_N$  is completed.

## 3. Consistent Estimators of the Parameters of the Second Order Autoregression

The unknown parameters will be estimated by method of ratios the first. From the definition of the process UAR(2), it follows that the following inequalities are valid

$$\beta U_{n-2} + \varepsilon_n \le U_n \le \alpha U_{n-1} + \varepsilon_n$$

for any integer n. From the other side, the innovation random variable  $\varepsilon_n$  satisfies the condition

$$-\beta \le \varepsilon_n \le 1 - \alpha$$

for any integer n. So, the previous inequalities become

$$\beta(U_{n-2} - 1) \le U_n \le \alpha(U_{n-1} - 1) + 1$$

for all integer indexes n. Solving these inequalities with respect to  $\alpha$  and  $\beta$ , we will find out that

$$\alpha \leq \frac{1-U_n}{1-U_{n-1}},$$
  
$$\beta \geq \frac{-U_n}{1-U_{n-2}}.$$

It means that two the following statistics:

$$\hat{\alpha}_N = \min_{3 \le n \le N} \left\{ \frac{1 - U_n}{1 - U_{n-1}} \right\},$$
$$\hat{\beta}_N = \max_{3 \le n \le N} \left\{ \frac{-U_n}{1 - U_{n-2}} \right\}$$

will be the estimators for  $\alpha$  and  $\beta$  respectively.

Let us examine the asymptotic properties of these statistics. We will use the notations  $G_N^*(x) = P\{\hat{\alpha}_N > x\}$  and  $H_N(x) = P\{\hat{\beta}_N \le x\}$ . It follows, from the definition of  $\hat{\alpha}_N$ , that

$$G_N^*(x) = G_{N-1}^*(x) = \dots = G_3^*(x) = 1, \ x < \alpha,$$

and  $G_N^*(\alpha) = (1-\alpha)^{N-2}$ . Since  $G_N^*(x)$  is the decreasing function,  $G_N^*(x) \leq (1-\alpha)^{N-2}$ , for all  $x \geq \alpha$ . The analogue procedure as above completes the proof of consistency.

Finally, let us consider the estimator  $\hat{\beta}_N$ . It is easy to verify that  $H_N(x) = \cdots = H_3(x) = 1$  for all  $x \ge \beta$ . If  $x < \beta$ , then

$$H_N(x) \le (1+\beta)^{N-2} \to 0, \ N \to +\infty.$$

So, when  $N \to +\infty$ 

$$H_N(x) \to H(x) = \begin{cases} 1, \ x \le \beta, \\ 0, \ x < \beta \end{cases}$$

and it completes the proof that  $\hat{\beta}_N$  is the consistent estimator for  $\beta$ .

#### REFERENCES

1. P. BROCKWELL and R. DAVIS: *Time Series: Theory and Methods*. Springer-Verlag, New York, 1987.

62

- M. R. CHERNICK: A limit theorem for the maximum of autoregressive processes with uniform marginal distributions. Ann. Probab. (9) 1 (1981), 145– 149.
- M. R. CHERNICK and R. A. DAVIS: Extremes in autoregressive processes with uniform marginal distributions. Statistics and Probability Letters 1 (1982), 85–88.
- 4. J. L. DOOB: Stochastic Processes. Wiley, New York, 1953.
- A. J. LAWRANCE: Uniformly distributed first-order autoregressive time series models and multiplicative congruential random number generators. J. Appl. Prob. 29 (1992), 896–903.
- M. M. RISTIĆ and B. Č. POPOVIĆ: A new uniform AR(1) time series model (NUAR(1)). Publ. Inst. Math. (Beograd) (N.S.) 68 (82) (2000), 145– 152.
- M. M. RISTIĆ and B. Č. POPOVIĆ: Parameter estimation for uniform autoregressive processes. Novi Sad J. Math. 30 (1) (2000), 89–95.
- M. M. RISTIĆ and B. Č. POPOVIĆ: The uniform autoregressive process of the second order (UAR(2)). Stat. Prob. Letters 57 (2002), 113–119.

Faculty of Sciences and Mathematics University of Niš 18000 Niš, Serbia