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AN ESTIMATION OF THE SINGULAR VALUES OF INTEGRAL OPERATOR WITH LOGARITHMIC KERNEL

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Abstract. The present paper focuses on a problem of estimating the singular values of operators with the kernel of the form $k(x) = \log^{\beta}(1/x)$ and $\beta > 0$. In the special case, when $\beta = 1$ and kernel $k(x) = \log(1/x)$, the exact asymptotic of singular values of operator is obtained in [5].

1. Introduction

Suppose $\mathcal{H}$ is complex Hilbert space and T is a compact operator on $\mathcal{H}$. The singular values of the operator T ($s_n(T)$) are the eigenvalues of the operator $(T^*T)^{1/2}$ (or $(TT^*)^{1/2}$). The eigenvalues of $(T^*T)^{1/2}$ arranged in a decreasing order and repeated according to their multiplicity, form a sequence $s_1, s_2, \ldots$ tending to zero.

Denote the set of compact operators on $\mathcal{H}$ by C_∞.

The operator T is Hilbert-Schmidt one ($T \in C_2$) if

$$\sqrt{\sum_{n=1}^{+\infty} s_n^2(T)} = \|T\|_2 < +\infty.$$

If $T \in C_\infty$ is an integral operator on $L^2(\Omega)$ defined by

$$Tf(x) = \int_\Omega k(x,y) f(y)\, dy,$$

then (see [7])

$$\|T\|_2^2 = \int_\Omega \int_\Omega |k(x,y)|^2\, dxdy.$$

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2. Main Result

Theorem 2.1. *For the operator $A : L^2(0,1) \to L^2(0,1)$, defined by*

$$Af(x) = \int_0^x \log^\beta \frac{1}{x-y}\, f(y)\, dy, \quad \beta > 0,$$

we have, for the singular values $s_n(A)$ of the operator A, the inequalities

$$c\,\frac{\log^{\beta-1} n}{n} \leqslant s_n(A) \leqslant o(n^{-1/2}), \quad n \in \mathbb{N},$$

where $c = \text{const} > 0$.

Proof. In the case when $\beta = 1$ and kernel $k(x) = \log(1/x)$ the exact asymptotic of singular values of the operator A is known (see [2]). In [9] M. Kac heuristically deduced the asymptotic formula for the eigenvalues of one unbounded operator (which appeared in the theory of thinly tab) and which is closely to operator $A + A^*$ in case $\beta = 1$. In [8] H.M. Hogan and L.A. Sahnovič gave a strong proof of this asymptotic formula. The obtained asymptotic formula $s_n(A) \asymp c/n$ is according to inequalities for the generalized case.

Note that

$$Bf(x) = (A + A^*)f(x) = \int_0^1 \log^\beta \frac{1}{|x-y|} f(y)\, dy.$$

Let the function $\omega \in C_0^\infty(\mathbb{R})$ satisfies:

$$0 \leqslant \omega \leqslant 1, \quad \omega(x) = 1 \quad \text{for} \quad |x| \leqslant \varepsilon.$$

Define the operator $B_0 : L^2(0,1) \to L^2(0,1)$ by

$$B_0 f(x) = \int_0^1 \omega(|x-y|) \log^\beta \frac{1}{|x-y|} f(y) dy.$$

From the previous we get that $B_0 f = \int_0^1 \widetilde{k}(x-y) f(y)\, dy$, where the kernel $\widetilde{k}(x) = \omega(|x|)\,(\log(1/|x|))^\beta$. Consider the function

$$H(x,y) = \sum_{n=-\infty}^{+\infty} \left[\widetilde{k}(x-y+4n) - \widetilde{k}(x+y+4n+2)\right], \quad x, y \in [-1,1].$$

Taking $\varphi_n(x) = \sin(n\pi(1+x)/2)$ for $n \in \mathbb{N}$, the system of functions $\{\varphi_n\}_{n=1}^{+\infty}$ is an orthonormal basis of $L^2(-1,1)$. By a direct computation it will be proved (see [1]) that

$$\int_{-1}^{1} H(x,y)\varphi_n(y)\,dy = \widehat{K}\left(\frac{n\pi}{2}\right)\varphi_n(x), \quad x \in [-1,1],\ n = 1,2,\dots,$$

where $\widehat{K}(\xi) = \int_{-\infty}^{+\infty} e^{it\xi}\,\widetilde{k}(t)\,dt$.

The operator $\mathcal{K} : L^2(-1,1) \to L^2(-1,1)$, defined by

$$\mathcal{K}f(x) = \int_{-1}^{1} H(x,y)f(y)\,dy,$$

is selfadjoint and $\{\lambda_n(\mathcal{K})\}_{n\geqslant 1}$ are its singular values. We have that

$$\lambda_n(\mathcal{K}) = \widehat{K}\left(\frac{n\pi}{2}\right).$$

Since

$$\begin{aligned}
\widehat{K}(\lambda) &= \int_{\mathbb{R}} e^{i\lambda t}\,\widetilde{k}(t)\,dt \\
&= \int_{\mathbb{R}} \cos\lambda t\ \omega(|t|)\left(\log\frac{1}{|t|}\right)^{\beta} dt + i\int_{\mathbb{R}} \sin\lambda t\ \omega(|t|)\left(\log\frac{1}{|t|}\right)^{\beta} dt,
\end{aligned}$$

then from $\int_{\mathbb{R}} \sin\lambda t\ \omega(|t|)\,(\log(1/|t|))^{\beta}\,dt = 0$, we get

$$\begin{aligned}
\widehat{K}(\lambda) &= 2\int_0^{+\infty} \cos\lambda t\ \omega(t)\left(\log\frac{1}{t}\right)^{\beta} dt \\
&= 2\int_0^{+\infty} \frac{e^{i\lambda t}+e^{-i\lambda t}}{2}\omega(t)\left(\log\frac{1}{t}\right)^{\beta} dt \\
&= \int_0^{+\infty} e^{i\lambda t}\omega(t)\left(\log\frac{1}{t}\right)^{\beta} dt + \overline{\int_0^{+\infty} e^{i\lambda t}\omega(t)\left(\log\frac{1}{t}\right)^{\beta} dt}.
\end{aligned}$$

The direct application of Theorem 1.11 from [6] to the function

$$F(\lambda) = \int_0^{+\infty} e^{i\lambda x}\sum_{m=0}^{+\infty} c_m x^{\alpha_m - 1}\left(\log\frac{1}{x}\right)^{\beta_m} dx,$$

we get

$$F(\lambda) = c_0 J(\alpha_0,\beta_0,\lambda) + o(\lambda^{-q}), \quad \text{for all} \quad q \in \mathbb{N},$$

and

$$J(\alpha_0, \beta_0, \lambda) \sim e^{i\alpha_0\pi/2}\lambda^{-\alpha_0} \sum_{k=0}^{+\infty} c_k(\alpha_0, \beta_0)\, (\log \lambda)^{\beta_0 - k}\,.$$

Putting here $c_0 = 1, \alpha_0 = 1,\ \beta_0 = \beta$, and

$$f(t) = \omega(t) \left(\log \frac{1}{t}\right)^{\beta} \sim (-\log t)^{\beta}\,, \quad t \to 0+,$$

we have

$$\begin{aligned} F(\lambda) &= J(1, \beta, \lambda) + o(\lambda^{-q}) \\ &= e^{i\pi/2}\lambda^{-1} \sum_{k=0}^{+\infty} c_k(1, \beta)\, (\log \lambda)^{\beta-k} = \frac{i}{\lambda} \sum_{k=0}^{+\infty} c_k(1, \beta)\, (\log \lambda)^{\beta-k} \\ &= \frac{i}{\lambda} \Big(c_0(1, \beta)\, (\log \lambda)^{\beta} + c_1(1, \beta)\, (\log \lambda)^{\beta-1} + \cdots \Big)\,. \end{aligned}$$

Since $\widehat{K}(\lambda) = F(\lambda) + \overline{F(\lambda)}$, then, from the previous relation, it follows

$$\begin{aligned} \widehat{K}(\lambda) &= \frac{i}{\lambda} \Big(c_0(1, \beta)\, (\log \lambda)^{\beta} + c_1(1, \beta)\, (\log \lambda)^{\beta-1} + \cdots \Big) \\ &- \frac{i}{\lambda} \Big(\overline{c_0(1, \beta)}\, (\log \lambda)^{\beta} + \overline{c_1(1, \beta)}\, (\log \lambda)^{\beta-1} + \cdots \Big)\,. \end{aligned}$$

By Theorem 1.11 from [6] we obtain

$$c_k(1, \beta) = (-1)^k \sum_{s=0}^{k} \binom{k}{s} \Gamma^{(s)}(1) \left(\frac{i\pi}{2}\right)^{k-s},$$

where $\Gamma(\cdot)$ is Euler Gamma function. For $k = 0$ we have $c_0(1, \beta) = \Gamma(1) = 1$, and for $k = 1$ we obtain

$$\begin{aligned} c_1(1, \beta) &= (-1) \sum_{s=0}^{1} \binom{1}{s} \Gamma^{(s)}(1) \left(\frac{i\pi}{2}\right)^{1-s} \\ &= (-1) \left[\binom{1}{0} \Gamma(1) \frac{i\pi}{2} + \binom{1}{1} \Gamma'(1) \right] = -\left(\frac{i\pi}{2} + \Gamma'(1) \right). \end{aligned}$$

Then, it follows immediately from these relations that

$$\widehat{K}(\lambda) = \frac{i}{\lambda} \left(c_1(1, \beta) - \overline{c_1(1, \beta)} \right) (\log \lambda)^{\beta-1} + O\left(\frac{(\log \lambda)^{\beta-2}}{\lambda} \right),$$

so that

$$\widehat{K}(\lambda) \sim \frac{i}{\lambda}\,(2\,i\,\mathrm{Im}(c_1(1,\beta)))\,(\log\lambda)^{\beta-1} = -\frac{2}{\lambda}\left(-\frac{\pi}{2}\right)(\log\lambda)^{\beta-1}.$$

The last relation implies

$$\widehat{K}(\lambda) \sim \frac{\pi}{\lambda}(\log\lambda)^{\beta-1}, \quad \lambda \to +\infty,$$

thus, according to the method exposed in [3], we have that

$$s_n(B_0) \sim \frac{\pi}{n}(\log n)^{\beta-1}, \quad n \in \mathbb{N}. \tag{2.1}$$

Since

$$Bf = \int_0^1 \log^\beta \frac{1}{|x-y|} f(y)\,dy,$$

and

$$B_0 f = \int_0^1 \omega(|x-y|) \log^\beta \frac{1}{|x-y|} f(y)\,dy,$$

we have that $B = B_0 - \overline{B}$, where $\overline{B}$ is the operator with kernel $\overline{k}(t) \in C^\infty(\mathbb{R})$. By Theorem about a gowning of the singular values of the operator with smooth kernel (see [7]) we have that the operator $\overline{B} \in C_p$, for every $p > 0$, so for its singular values we get $s_n(\overline{B}) = o(n^{-1/p})$. From this and relation (2.1), according to Theorem of Ky-Fan (see [7]) we have

$$s_n(B) \sim \frac{\pi}{n}(\log n)^{\beta-1}, \quad n \in \mathbb{N}.$$

On the other side, from some properties of the singular values, we obtain

$$s_{2n}(B) \leqslant s_n(A) + s_n(A^*) = 2s_n(A),$$

i.e.,

$$s_n(A) \geqslant \frac{c_1}{2}\frac{(\log n)^{\beta-1}}{n},$$

where c_1 is a constant independent of n.

According to

$$\int_0^1\int_0^1 |k(x-y)|^2 dx\,dy < +\infty,$$

A is a Hilbert-Schmidt operator and we have

$$s_n(A) = o(n^{-1/2}). \qquad \square$$

Remark 2.1. From this inequalities we can see that when $\beta > 0$ this operator does not belong to the class C_p, for $0 < p < 1$.

Remark 2.2. Let us consider the kernels

$$k_i(x) = \left(\log \frac{1}{x}\right)^{\beta} (1 + r_i(x)), \quad i = 1, 2,$$

with $r_i \in C^3[0,1]$, $\dfrac{d^k r_i}{dx^k}(0) = 0$ for $k \in \{0, 1, 2\}$, and the operators

$$A_i f(x) = \int_0^x k_i(x-y) f(y)\, dy.$$

Then from [2] we have

$$\lim_{n\to+\infty} \frac{s_n(A_1)}{s_n(A_2)} = 1.$$

Consider the case $k_1(x) = (\log(1/x))^{\beta}\,(1+r(x))$ and $k_2(x) = (\log(1/x))^{\beta}$. Knowing the corresponding asymptotic formula for the operator A_2, with the kernel k_2, we know this and for the operator A_1 with the kernel k_1.

Remark 2.3. Increasing order of singular values for the operator A is not found, because the upper estimate

$$s_n(A) \leqslant c_2 \frac{\log^{\beta-1} n}{n} \quad (c \text{ is independent on } n).$$

in not obtained. An open problem is to find the exact asymptotic of singular values of the operator $A : L^2(0,1) \to L^2(0,1)$, defined by

$$Af(x) = \int_0^x \log^{\beta} \frac{1}{x-y}\, f(y)\, dy, \quad \beta > 0.$$

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