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GENERALIZED INVERSES AND SPECIAL TYPE OPERATOR ALGEBRAS

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Abstract. Let $\mathcal{H}$ be a complex Hilbert space. In this work we compute the generalized inverse of a finite rank operator on $\mathcal{H}$ and give necessary and sufficient conditions such that the generalized inverse of the product of two rank-1 operators is the product of the generalized inverses of the corresponding operators in reverse order. We also consider the generalized inverse of products of special type operators. We examine when the generalized inverse of a rank-1 operator in a nest algebra belongs to the nest algebra and give necessary conditions for an operator in a nest algebra with a continuous nest so that its generalized inverse belongs to the nest algebra. Finally, we give equivalent conditions so that an operator with closed range, factors with respect to a closed subalgebra of a von Neumann algebra of operators on $\mathcal{H}$.

1. Introduction

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. It is known that when T is a singular operator with closed range then its unique generalized inverse T^+ (known also as Moore-Penrose inverse) is defined. A lot of work concerning generalized inverses especially in finite dimension has been carried out(e.g. [2]). The generalized inverse operator is a powerful tool for characterization of different classes of operators in finite and in infinite dimension (e.g., normal, hyponormal, EP operators: [3], [5], [4], [9]). In this paper, we study the generalized inverses of finite rank operators, and we provide necessary and sufficient conditions for special type products of

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operators so that the reverse order rule for the Moore-Penrose inverse (see e.g., [1], [4]) is satisfied. Moreover, we examine when the generalized inverse of a rank-1 operator in a nest algebra belongs also to the nest algebra. Furthermore we give necessary conditions for an operator in a nest algebra with a continuous nest so that its generalized inverse belongs also to the nest algebra. Finally, we use the Moore-Penrose inverse for a singular operator T to admit factorizations of T with respect to a closed unital subalgebra of a von Neumann algebra of operators on a complex Hilbert space.

2. Preliminaries and Notation

Let $\mathcal{H}$ be a complex Hilbert space. $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$, and for $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(T)$ denotes the range of T, $\mathcal{N}(T)$ the kernel of T and $\operatorname{Lat} T$ denotes the set of all closed invariant subspaces of T.

The generalized inverse, known as Moore-Penrose inverse, of an operator $T \in \mathcal{B}(\mathcal{H})$ with closed range, is the unique operator satisfying the following conditions:

$$TT^+ = (TT^+)^\star, \qquad T^+T = (T^+T)^\star, \qquad TT^+T = T, \qquad T^+TT^+ = T^+,$$

where $T^\star$ denotes the adjoint operator of T.

It is easy to see that $\mathcal{R}(T^+) = \mathcal{N}(T)^\perp$, TT^+ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T)$ and that T^+T is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(T)^\perp$. It is well known that $\mathcal{R}(T^+) = \mathcal{R}(T^*)$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a finite rank operator if the dimension of its range is finite. The number $n = \dim \mathcal{R}(T)$ is called the rank of T and it is denoted by $r(T)$. For every rank one operator (rank-1) T there are vectors $e, f \in H$ such that $Tx = \langle x, e\rangle f$, for every $x \in H$. The rank-1 operator T is denoted by $e \otimes f$. The adjoint T^* of T is the rank-1 operator $T^* = f \otimes e$. A subset $\mathcal{N}$ of the set of closed subspaces of a Hilbert space $\mathcal{H}$ is called a nest if it is totally ordered by inclusion. The nest $\mathcal{N}$ is called complete if $\{0\}, \mathcal{H} \in \mathcal{N}$ and for any subset $\mathcal{N}_0$ of $\mathcal{N}$ the closed subspaces $\cap\{L : L \in \mathcal{N}_0\}$, $\operatorname{cl}[\cup\{L : l \in \mathcal{N}_0\}]$ belong to $\mathcal{N}$. For $M \in \mathcal{N}$ we denote by M_- the immediate predecessor of M which is the closed subspace $M_- = \cup\{L : L \in \mathcal{N}, L \subset M\}$. A nest $\mathcal{N}$ is called continuous if $M = M_-$ for all $M \in \mathcal{N}$. By $\operatorname{Alg}\mathcal{N}$ we denote the algebra of all bounded linear operators on $\mathcal{H}$ which leave invariant each member of the complete nest $\mathcal{N}$. The algebra $\operatorname{Alg}\mathcal{N}$ is called nest algebra.

3. The Generalized Inverse of a Finite Rank Operator

In this section we compute the generalized inverse of a finite rank operator and give necessary and sufficient conditions such that the generalized inverse of the product of two rank-1 operators is the product of the generalized inverses of the corresponding operators in reverse order. We also consider the generalized inverse of products of special type operators. First we study the case of a rank-1 operator.

Theorem 3.1. *Let $T = e \otimes f$ be a rank-1 operator on a Hilbert space $\mathcal{H}$ and let T^+ be its generalized inverse. Then T^+ is also rank-1 and has the representation $T^+ = f_1 \otimes e$, where $f_1 = \|f\|^{-2}\|e\|^{-2} f$.*

Proof. Let $T = e \otimes f$. Since $\mathcal{R}(T^+) = \mathcal{R}(T^*)$ and T^* is of rank one we have that T^+ is also of rank one and it will have the form $T^+ = f_1 \otimes e$. Thus to determine T^+ it is enough to determine the vector f_1. We have $T^+Te = e$, since T^+T is a projection on $\mathcal{R}(T^*)$. Hence $e = T^+(\langle e, e\rangle f) = \|e\|^2\langle f, f_1\rangle e$ which implies $\langle f, f_1\rangle = \|e\|^{-2}$. Let $f_1 = \alpha f + u$, where $u \perp f$. Then $u \in \mathcal{N}(T^+) = \mathcal{N}(T^*)$ and so $T^+ f_1 = \|f_1\|^2 e = (|\alpha|^2\|f\|^2 + \|u\|^2)e$. Also $T^+ f_1 = T^+(\alpha f + u) = \alpha T^+ f = \alpha\langle f, f_1\rangle e = |\alpha|^2\|f\|^2 e$. Therefore $u = 0$ and $f_1 = \alpha f$. Now using the relation $\langle f, f_1\rangle = \|e\|^{-2}$, we get $\alpha = \|e\|^{-2}\|f\|^{-2}$ and therefore $f_1 = \|e\|^{-2}\|f\|^{-2} f$. □

Corollary 3.1. *Let $T = e \otimes f$ be a rank-1 operator. Then $\|T^+\| = \|T\|^{-1}$.*

Proof. It is evident that

$$\|T^+\| = \|f_1\|\,\|e\| = \frac{1}{\|f\|^2\|e\|^2}\|f\|\,\|e\| = \frac{1}{\|f\|\,\|e\|} = \|T\|^{-1}. \qquad \square$$

Proposition 3.1. *Let $T_1 = e_1 \otimes f_1$ and $T_2 = e_2 \otimes f_2$ be two rank-1 operators. Then $(T_1T_2)^+ = T_2^+T_1^+$ if and only if f_2, e_1 are linearly dependent.*

Proof. It is clear that $T_1T_2 = \langle f_2, e_1\rangle(e_2 \otimes f_1)$ and $(\lambda T)^+ = \frac{1}{\lambda}T^+$ for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and for any operator $T \in \mathcal{B}(H)$ with closed range. By Theorem 3.1 it follows that

$$(T_1T_2)^+ = \frac{f_1 \otimes e_2}{\langle f_2, e_1\rangle\|f_1\|^2\|e_2\|^2} \quad \text{and} \quad T_2^+T_1^+ = \frac{\langle e_1, f_2\rangle f_1 \otimes e_2}{\|e_1\|^2\|f_1\|^2\|e_2\|^2\|f_2\|^2}.$$

Therefore, by simple calculations it is obtained that

$$(T_1T_2)^+ = T_2^+T_1^+ \quad \text{if and only if} \quad |\langle f_2, e_1\rangle| = \|f_2\|^2\|e_1\|^2.$$

The last equality is valid if and only if the vectors f_2, e_1 are linearly dependent (The condition under which the Cauchy-Schwarz inequality becomes an equality). $\square$

The following proposition is a restatement of a part of R. Bouldin's theorem 3.1 [1] (see also [8], [11]). It will be used to prove our next result.

Proposition 3.2. *Let A, B be bounded operators on $\mathcal{H}$ with closed range. Then $(AB)^+ = B^+A^+$ if and only if the following three conditions hold:*

i) *The range of AB is closed,*

ii) *A^+A commutes with BB^*,*

iii) *BB^+ commutes with A^*A.*

Proposition 3.3. *Let $A, T \in \mathcal{B}(\mathcal{H})$ be two operators such that A is invertible and T has closed range. Then*

$$(AT)^+ = T^+A^{-1} \quad \textit{if and only if} \quad \mathcal{R}(T) \in \mathrm{Lat}\,(A^*A).$$

Proof. We will use Proposition 3.2. The first two conditions of Proposition 3.2 are easily verified. In order for the third condition to be valid we must prove that the operator TT^+ commutes with the operator A^*A. Since the operator TT^+ is the projection of $\mathcal{H}$ onto the range of T and A^*A is selfadjoint this is equivalent to $(R)(T) \in \mathrm{Lat}\,(A^*A)$. $\square$

Corollary 3.2. *If $T = e \otimes f$ is a rank-1 operator and $A \in \mathcal{H}$ is invertible then $(AT)^+ = T^+A^{-1}$ if and only if f is an eigenvector of A^*A.*

Let T be a finite rank operator with $r(T) = n$. Then T has the form $T = \sum_{i=1}^n e_i \otimes f_i$, where the vectors $\{e_i : i = 1, 2, \ldots, n\}$ are orthonormal and the vectors $\{f_i : i = 1, 2, \ldots, n\}$ are linearly independent.

Theorem 3.2. *If $T = \sum_{i=1}^n e_i \otimes f_i$ is a rank-n operator then its generalized inverse is also a rank-n operator and it is defined by $T^+x = \sum_{i=1}^n \lambda_i(x)e_i$, where the functions λ_i are the solution of an appropriately defined $n \times n$ linear system.*

Proof. If $T = \sum_{i=1}^n e_i \otimes f_i$ then $\mathcal{R}(T) = \mathrm{cls}\{f_1, f_2, \ldots, f_n\}$ and $\mathcal{R}(T^+) = \mathcal{R}(T^*) = \mathrm{cls}\{e_1, e_2, \ldots, e_n\}$. Therefore for every $x \in \mathcal{H}$ we have $T^+x =$

$\sum_{i=1}^n \lambda_i(x)e_i$. Hence to determine T^+ one must calculate the functions λ_i, $i = 1, 2, \ldots, n$. From $T^*x = T^*TT^+x$, we get

$$\sum_{i=1}^{n}\langle x, f_i\rangle e_i = T^*x = T^*TT^+x = \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_j(x)\langle f_i, f_j\rangle e_i.$$

The last relation leads to the following linear system of n equations in n unknowns

$$\langle x, f_i\rangle = \sum_{j=1}^{n}\lambda_j(x)\langle f_i, f_j\rangle, \;\; i = 1, 2, \ldots, n.$$

The determinant of the system is the Gram determinant of the linear independent vectors $f_1, \ldots, f_n$ and hence the system has a unique solution with unknowns the functions λ_i, $i = 1, 2, \ldots, n$. □

In particular the generalized inverse T^+ of a rank-2 operator

$$T = e_1 \otimes f_1 + e_2 \otimes f_2$$

is the operator

$$T^+ = \frac{1}{D_T}\left\{\left(\|f_2\|^2 f_1 - \langle f_1, f_2\rangle f_2\right)\otimes e_1 + \left(\|f_1\|^2 f_2 - \langle f_2, f_1\rangle f_1\right)\otimes e_2\right\},$$

where $D_T = \|f_1\|^2\|f_2\|^2 - |\langle f_1, f_2\rangle|^2$.

4. Generalized Inverses and Nest Algebras

In the sequel we examine when the generalized inverse of a rank-1 operator in a nest algebra belongs to the nest algebra. We also give necessary conditions for an operator in a nest algebra with a continuous nest so that its generalized inverse belongs to the nest algebra.

Theorem 4.1. *Let* $\mathrm{Alg}\mathcal{N}$ *be a nest algebra and* $T = e \otimes f \in \mathrm{Alg}\mathcal{N}$. *Then* $T^+ \in \mathrm{Alg}\mathcal{N}$ *if and only if there exist a subspace* $M \in \mathcal{N}$, *such that* $M \cap (M_-)^\perp \neq \varnothing$ *and* $f, e \in M \cap (M_-)^\perp$.

Proof. It is well known that $T = e \otimes f \in \mathrm{Alg}\mathcal{N}$ if and only if there exists a subspace $M \in \mathcal{N}$, such that $f \in M$ and $e \in M_-^\perp$(see e.g. [6]). It is obvious that if $f, e \in M \cap (M_-)^\perp$ then $T, T^+ \in \mathrm{Alg}\mathcal{N}$. Conversely let $T^+ = f_1 \otimes e$ and $T^+ \in \mathrm{Alg}\mathcal{N}$. Then there exists a subspace $L \in \mathcal{N}$ such that $e \in L$ and $f_1 \in L_-^\perp$. Since the nest is totally ordered, the subspaces

M, L are comparable. The cases $M \subset L$ or $L \subset M$ lead to a contradiction. For example, if $M \subset L$ then $L_-^\perp \subseteq M^\perp$ and so $f_1 \in M^\perp$. But $f_1 = \alpha f$ and $f \in M$. Therefore $f = 0$ which is a contradiction. Similarly, if $L \subset M$ we get $e = 0$ which is also a contradiction. Hence, we must have $M = L$ and therefore $e, f \in M \cap (M_-)^\perp$. □

Remark 4.1. We pointed out that Theorem 4.1 is valid also in the case where we replace the nest by a commutative subspace lattice (csl). As a consequence of Theorem 4.1, we also have that for a continuous nest, the generalized inverse of every rank-1 operator in the corresponding nest algebra never belongs to the nest algebra.

In the sequel, under certain conditions, we show that $T^+ \in \mathrm{Alg}\mathcal{N}$ where $\mathcal{N}$ is a continuous nest and $T \in \mathrm{Alg}\mathcal{N}$. This generalizes a corresponding result in [7] when T is invertible.

Theorem 4.2. *Let $\mathcal{N}$ be a continuous nest, $\mathrm{Alg}\mathcal{N}$ the corresponding nest algebra and $T \in \mathrm{Alg}\mathcal{N}$ an operator such that $T^+ = A + K$, where $A \in \mathrm{Alg}\mathcal{N}$ and K is a compact operator. If the orthogonal projections T^+T, TT^+ belong to $\mathrm{Alg}\mathcal{N}$ then $T^+ \in \mathrm{Alg}\mathcal{N}$.*

Proof. For simplicity let $P = TT^+$ and $Q = T^+T$. We have

$$T^+ = A + K \Rightarrow TT^+ = TA + TK \Rightarrow TK = P - TA.$$

The compact operator TK belongs to the nest algebra $\mathrm{Alg}\mathcal{N}$, since the operator $P - TA$ belongs to $\mathrm{Alg}\mathcal{N}$. Moreover, since the nest is continuous the operator TK is quasinilpotent and hence the operator $I - TK$ is invertible with $(I - TK)^{-1} = \sum_{n=0}^{\infty} (TK)^n$. Therefore $(I - TK)^{-1} \in \mathrm{Alg}\mathcal{N}$. But

$$\begin{aligned} T^+ &= A + K = Q(A + K) = QA + T^+TK \\ &\Rightarrow T^+(I - TK) = QA \Rightarrow T^+ = QA(I - TK)^{-1}. \end{aligned}$$

Therefore, $T^+ \in \mathrm{Alg}\mathcal{N}$. □

5. A Factorization Result

A well-known problem in operator theory is the following question (for a discussion see [13]): If $\mathcal{F}$ is a unital C^*-algebra and $\mathcal{A} \subseteq \mathcal{F}$ is a unital closed subalgebra then given an invertible element $T \in \mathcal{F}$, when is it possible to write $T^*T = A^*A$, where $A, A^{-1} \in \mathcal{A}$? We consider a version of this problem, using generalized inverses, when $\mathcal{F}$ is a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and $T \in \mathcal{F}$ is an operator (not necessary invertible) with closed range. The techniques we use are based on ideas from [13].

Theorem 5.1. *Suppose $\mathcal{F}$ is a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{A} \subseteq \mathcal{F}$ be a closed unital subalgebra. For an operator $T \in \mathcal{F}$ the following are equivalent:*

1. *There exist an operator $S \in \mathcal{A}$ with closed range $\mathcal{R}(S)$ and a partial isometry $W \in \mathcal{F}$ with initial space $\mathcal{R}(S)$ and final space $\mathcal{R}(T)$ such that $S, S^+ \in \mathcal{A}$ and $T = WS$.*

2. *There exists an operator $S \in \mathcal{A}$ with closed range such that $S^+ \in \mathcal{A}$ and $T^*T = S^*S$.*

Proof. 1. $\Rightarrow$ 2. From $T = WS$ it follows that $T^* = S^*W^*$, as well as $T^*T = S^*W^*WS = S^*S$.

2. $\Rightarrow$ 1. Note that since $\mathcal{F}$ is a von Neumann algebra and $T \in \mathcal{F}$ we have from [12] that $T^+ \in \mathcal{F}$ and from $T^*T = S^*S$, multiplying both sides from the left by T^{+^*}, we get

$$T^{+^*}T^*T = T^{+^*}S^*S \Rightarrow T = T^{+^*}S^*S. \tag{5.1}$$

Let $W = T^{+^*}S^*$. Then $W \in \mathcal{F}$. We will show that W is the appropriate partial isometry. It is $W^*W = ST^+T^{+^*} = S(T^*T)^+S^* = S(S^*S)^+S^* = SS^+S^{+^*}S^* = SS^+$, where SS^+ is the orthogonal projection on the range of S. Similarly, $WW^* = T^{+^*}S^*ST^+ = T^{+^*}T^*TT^+ = TT^+$, where TT^+ is the orthogonal projection on the range of T. Therefore W is a partial isometry with the required initial and final spaces. From equation (5.1) and the definition of W, we have $T = WS$. $\square$

Proposition 5.1. *Suppose $\mathcal{F}$ is a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{A} \subseteq \mathcal{F}$ be a closed unital subalgebra. If for an operator $T \in \mathcal{F}$ there exists an operator $S \in \mathcal{A}$ with closed range such that $S^+ \in \mathcal{A}$ and $T^*T = S^*S$ then there exist an operator $Y \in \mathcal{A}$ with $\mathcal{R}(I - Y)$ closed and $(I-Y)^+ \in \mathcal{A}$ and an operator $Z \in \mathcal{A}^*$ such that $T = TY + T^{+^*}Z$.*

Proof. We have $T^*T = S^*S \Rightarrow T^{+^*}T^*T = T^{+^*}S^*S \Rightarrow T = T^{+^*}S^*S$ and

$$TS^+ = T^{+^*}S^*SS^+ = T^{+^*}S^*. \tag{5.2}$$

Set $Y = I - S^+$ and $Z = S^*$. Then, using (5.2),

$$TY + T^{+^*}Z = T - TS^+ + T^{+^*}S^* = T. \qquad \square$$

Results which are related to this kind of problems could be found also in [10].

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REFERENCES

1. R. Bouldin: *The pseudo-inverse of a product*. SIAM J. Appl. Math. **24** (4) (1973), 489–495.
2. A. Ben-Israel and T. N. E. Grenville: *Generalized Inverses: Theory and Applications*. Springer-Verlag, Berlin, 2002.
3. S. Cheng and Y. Tian: *Two sets of new characterizations for normal and EP matrices*. Linear Algebra Appl. **375** (2003), 181–195.
4. D. S. Djordjević: *Products of EP operatorson Hilbert spaces*. Proc. Amer. Math. Soc. **129** (6) (2001), 1727–1731.
5. D. S. Djordjević: *Characterizations of normal, hyponormal and EP operators*. J. Math. Anal. Appl. (to appear)
6. J. A. Erdős: *Operators of finite rank in nest algebras*. J. London Math. Soc. **43** (1968), 391–397.
7. A. Feintuch and A. Lambert: *Invertibility in nest algebras*. Proc. Amer. Math. Soc. **91** (4) (1984), 573–576.
8. T. N. E. Greville: *Note on the generalized inverse of a matrix product*. SIAM Rev. **8** (1966), 518–521.
9. C. Groetsch: *Generalized Inverses of Linear Operators*. Marcel-Dekker, Inc., New York, 1973.
10. R. E. Harte: *Polar decomposition and the Moore-Penrose inverse*. PanAm. Math. J. **4** (1992), 71–76.
11. S. Izumino: *The product of operators with closed range and an extension of the reverse order law*. Tohoku Math. J. **34** (1982), 43–52.
12. S. Karanasios and D. Pappas: *Approximability of the generalized inverse of an operator*. J. Inst. Math. Comput. Sci. Math. Ser. **19** (1) (2006), 73–78.
13. D. R. Pitts: *Factorization problems for nests: Factorization methods and characterizations of the universal factorization property*. J. Funct. Anal. **79** (1988), 57–90.

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