FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 21 (2006), 23–28

# ON THE APPROXIMATION OF TRIVARIATE FUNCTIONS BY MEANS OF SOME TENSOR-PRODUCT POSITIVE LINEAR OPERATORS

**Ioana Taşcu**

**Abstract.** In this paper we introduce and study a trivariate linear positive operator, depending on some parameters, denoted by $L_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}$, useful for the approximation of functions of three real variables continuous on the unit cub $C_3$. The corresponding approximation formula has the degree of exactness $(1,1,1)$. For the remainder of this formula we give several representations by using the second-order partial derivatives.

## 1. Introduction

It is known that the problems of polynomial approximation for functions of several independent variables are important, but the methods are less developed than in the case of functions of a single variable.

In this paper we will study the problem of approximation of functions of several variables by using a tensor-product of some linear univariate operators of positive type.

For simplicity we will consider the three-dimensional case.

Let $f$ be a function defined on the compact unit 3-cub $C_3$, having the vertices:

$$(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1)$$
$$(1,1,0),\ (1,0,1),\ (0,1,1),\ (1,1,1).$$

Received November 7, 2005.
2000 *Mathematics Subject Classification.* 41A10, 41A36.

Our operators $L_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}$ depending on the nonnegative parameters $\alpha,\beta,\gamma$ are defined by the following formula

$$(L_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}f)(x,y,z)=\sum_{k=0}^{m}\sum_{i=0}^{n}\sum_{j=0}^{r}p_{m,k}^{\langle\alpha\rangle}(x)p_{n,i}^{\langle\beta\rangle}(y)p_{r,j}^{\langle\gamma\rangle}(z)f(x_{m,k},y_{n,i},z_{r,j}),$$

where

$$p_{m,k}^{\langle\alpha\rangle}(x)=\binom{m}{k}\frac{x(x+k\alpha)^{k-1}(1-x)[1-x+(m-k)\alpha]^{m-1-k}}{(1+m\alpha)^{m-1}},$$

$$p_{n,i}^{\langle\beta\rangle}=\binom{n}{i}\frac{y(y+i\beta)^{i-1}(1-y)[1-y+(n-i)\beta]^{n-1-i}}{(1+n\beta)^{n-1}},$$

$$p_{r,j}^{\langle\gamma\rangle}(z)=\binom{r}{j}\frac{z(z+j\gamma)^{j-1}(1-z)[1-z+(r-j)\gamma]^{r-1-j}}{(1+r\gamma)^{r-1}},$$

$\alpha,\beta,\gamma$ being nonnegative parameters, which might depend respectively on $m,n$ and $r$, while the coordinates of the nodes $x_{m,k}$, $y_{n,i}$ and $z_{r,j}$ are certain distinct points of the unit interval $[0,1]$.

## 2. Construction of the Operator

The fundamental polynomials occurring above can be constructed by starting from the combinatorial identity of Abel-Jensen [1], [5], having the form

$$(2.1)\quad (u+v)(u+v+m\alpha)^{m-1}=\sum_{k=0}^{m}\binom{m}{k}u(u+k\alpha)^{k-1}v[v+(m-k)\alpha]^{m-1-k}.$$

By replacing in this identity $u=x$ and $v=1-x$, where $x\in[0,1]$, we obtain the equality

$$(1+m\alpha)^{m-1}=\sum_{k=0}^{m}\binom{m}{k}x(x+k\alpha)^{k-1}(1-x)[1-x+(m-k)\alpha]^{m-1-k}.$$

Given a function $\varphi\in C[0,1]$, we associate to it a linear operators $L_m^{\langle\alpha\rangle}$ which depends on the parameter $\alpha$, satisfying the conditions $0\le\alpha=\alpha(m)\to 0$, as $m$ tends to infinity. This operator is defined by the following formula

$$\left(L_m^{\langle\alpha\rangle}\varphi\right)(x;a,a')=\sum_{k=0}^{m}p_{m,k}^{\langle\alpha\rangle}(x)\varphi\left(\frac{k+a}{m+a'}\right),$$

where $0 \le a \le a'$ and

$$p_{m,k}^{\langle\alpha\rangle}(x) = \frac{1}{(1+m\alpha)^{m-1}}\binom{m}{k}x(x+k\alpha)^{k-1}(1-x).$$

In the case $a = a' = 0$, we can see that the approximating polynomial $L_m^{\langle\alpha\rangle}\varphi$ is interpolatory at both sides of the interval $[0,1]$, for any nonnegative value of the parameter $\alpha$.

On the other hand, we can see that the operator $L_m^{\langle\alpha\rangle}$ reproduces the linear functions. The proof of this property can be seen in the paper [9] by D. D. Stancu and C. Cismaşiu.

In like manner we can construct the univariate operators $L_n^{\langle\beta\rangle}$ and $L_r^{\langle\gamma\rangle}$, corresponding to variables $y$ and $z$, depending on the nonnegative parameters $\beta$ and $\gamma$.

By using the tensor-product of these operators we can get the trivariate operator $L_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}$, applied to the function $f \in C(C_3)$, with the coordinates of the nodes

$$x_{m,k} = \frac{k+a}{m+a'}\,, \quad y_{n,i} = \frac{i+b}{n+b'}\,, \quad z_{r,j} = \frac{j+c}{r+c'}\,,$$

where $0 \le a \le a'$, $0 \le b \le b'$ and $0 \le c \le c'$.

Thus, we obtain for our operator the representation

$$\begin{aligned}&(L_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}f)(x,y,z)\\ &\qquad = \sum_{k=0}^{m}\sum_{i=0}^{n}\sum_{j=0}^{r} p_{m,k}^{\langle\alpha\rangle}(x)p_{n,i}^{\langle\beta\rangle}(y)p_{r,j}^{\langle\gamma\rangle}(z)f\left(\frac{k+a}{m+a'}\,,\frac{i+b}{n+b'}\,,\frac{j+c}{r+c'}\right).\end{aligned}$$

Now we consider the approximation formula

$$f(x,y,z) = \left(L_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}f\right)(x,y,z) + \left(R_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}f\right)(x,y,z). \tag{2.2}$$

Assuming that $a = a' = 0$, $b = b' = 0$ and $c = c' = 0$, this formula has the degree of exactness $(1,1,1)$.

## 3. The Remainder Term

By using a theorem of Peano-Milne-Stancu, given in the paper [8], we are able to give an integral representation for the remainder of the approximation formula (2.2). It has the following form

$$
\begin{aligned}
(R_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}f)(x,y,z) &= \int_0^1 G_1^{\langle\alpha\rangle}(t_1,x)f^{(2,0,0)}(t_1,y,z)dt_1 \\
&+ \int_0^1 G_2^{\langle\beta\rangle}(t_2,y)f^{(0,2,0)}(x,t_2,z)dt_2 + \int_0^1 G_3^{\langle\gamma\rangle}(t_3,z)f^{(0,0,2)}(x,y,t_3)dt_3 \\
&- \int_0^1\int_0^1 G_1^{\langle\alpha\rangle}(t_1,x)G_2^{\langle\beta\rangle}(t_2,y)f^{(2,2,0)}(t_1,t_2,z)dt_1dt_2 \\
&- \int_0^1\int_0^1 G_1^{\langle\alpha\rangle}(t_1,x)G_3^{\langle\gamma\rangle}(t_3,z)f^{(2,0,2)}(t_1,y,t_3)dt_1dt_3 \\
&- \int_0^1\int_0^1 G_2^{\langle\beta\rangle}(t_2,y)G_3^{\langle\gamma\rangle}(t_3,z)f^{(0,2,2)}(x,t_2,t_3)dt_2dt_3 \\
&+ \int_0^1\int_0^1\int_0^1 G_1^{\langle\alpha\rangle}(t_1,x)G_2^{\langle\beta\rangle}(t_2,y)G_3^{\langle\gamma\rangle}(t_3,z)f^{(2,2,2)}(t_1,t_2,t_3)dt_1dt_2dt_3,
\end{aligned}
$$

where we have

$$
\begin{aligned}
G_1^{\langle\alpha\rangle}(t_1,x) &= (R_m^{\langle\alpha\rangle}\varphi_x)(t_1),\ \varphi_x(t_1) = (x-t_1)_+ = \frac{x-t_1+|x-t_1|}{2}, \\
G_2^{\langle\beta\rangle}(t_2,y) &= (R_n^{\langle\beta\rangle}\varphi_y)(t_2),\ \varphi_y(t_2) = (y-t_2)_+ = \frac{y-t_2+|y-t_2|}{2}, \\
G_3^{\langle\gamma\rangle}(t_3,z) &= (R_r^{\langle\gamma\rangle}\varphi_z)(t_3),\ \varphi_z(t_3) = (z-t_3)_+ = \frac{z-t_3+|z-t_3|}{2}.
\end{aligned}
$$

We have used above for the partial derivatives of order $(p,q,s)$, in the point $(\overline{x},\overline{y},\overline{z})$, the following notation

$$
f^{(p,q,s)}(\overline{x},\overline{y},\overline{z}) = \left.\frac{\partial^{p+q+s}f(x,y,z)}{\partial x^p\,\partial y^q\,\partial z^s}\right|_{(x=\overline{x},\,y=\overline{y},\,z=\overline{z})}.
$$

Assuming that $x,y,z$ are fixed points in the interval $[0,1]$ and that $\alpha\geq 0$, $\beta\geq 0$, $\gamma\geq 0$, it follows that we have $G_1^{\langle\alpha\rangle}(t_1,x)\leq 0$, $G_2^{\langle\beta\rangle}(t_2,y)\leq 0$ and $G_3^{\langle\gamma\rangle}(t_3,z)\leq 0$ on this interval.

If we have now apply the mean value theorem to the integrals occurring above, we obtain

$$\int_0^1 G_1^{\langle\alpha\rangle}(t_1,x)dt_1 = \frac{1}{2}(R_m^{\langle\alpha\rangle}e_{2,0,0})(x),$$

$$\int_0^1 G_2^{\langle\beta\rangle}(t_2,y)dt_2 = \frac{1}{2}(R_n^{\langle\beta\rangle}e_{0,2,0}))(y),$$

$$\int_0^1 G_3^{\langle\gamma\rangle}(t_3,z)dt_3 = \frac{1}{2}(R_r^{\langle\gamma\rangle}e_{0,0,2})(z).$$

In the case $\alpha = \beta = \gamma = 0$ we are able to conclude that we can give for the corresponding remainder an expression, in terms of second-order partial derivatives of the function $f$, having the following form:

$$(3.1)\quad (R_{m,n,r}f)(x,y,z) = \frac{x(1-x)}{2m} f^{(2,0,0)}(\xi,y,z) + \frac{y(1-y)}{2n} f^{(0,2,0)}(x,\eta,z)$$
$$+\frac{z(1-z)}{2r} f^{(0,0,2)}(x,y,\zeta) - \frac{x(1-x)y(1-y)}{4mn} f^{(2,2,0)}(\xi,\eta,z)$$
$$-\frac{x(1-x)z(1-z)}{4mr} f^{(2,0,2)}(\xi,y,\zeta) - \frac{y(1-y)z(1-z)}{4nr} f^{(0,2,2)}(x,\eta,\zeta)$$
$$-\frac{x(1-x)y(1-y)z(1-z)}{8mnr} f^{(2,2,2)}(\xi,\eta,\zeta),$$

which corresponds to the approximation of the function $f \in C^{2,2,2}(C_3)$ by using the Bernstein operators $B_{m,n,s}$.

Since the formula (2.2) has the degree of exactness $(1,1,1)$ and the remainder does not vanish if $f$ is a convex function of the first order, with respect to the variables $x$, $y$ and $z$, if we apply a criterion of T. Popoviciu [6] we can find that this remainder is of simple form and it can be represented by means of the second-order partial divided differences of the functions $f$ on some distinct points from the unit cub $C_3$. If we further assume that this function has continuous second-order partial derivatives, then we can arrive at the formula (3.1).

Ending this paper we want to mention that by using an extension given by A. Hurwitz [4] to the identity (2.1) of Abel-Jensen, D. D. Stancu has given in

[10] a generalization of the Cheney-Sharma second operator, from the paper [3], by constructing a multiparameter Bernstein type operator depending on a set of parameters $\beta_1, \beta_2, \dots, \beta_m$, which are nonnegative numbers. Such a bivariate extension was given in our recent paper [11].

## REFERENCES


1. N. H. Abel: *Démonstration d'une expression de laquelle la formule du binome est un cas particulier.* Oeuvres Completes, nouvelle édition (1981), **1**, 102–103.
2. L. Beutel and H. H. Gonska: *Quantitative inheritance properties for simultaneous approximation by tensor product operators.* International Conference on Numerical Algorithms, Vol. I (Marrakesh, 2001). Numer. Algorithms **33** (2003), no. 1-4, 83–92.
3. E. W. Cheney and A. Sharma: *On a generalization of Bernstein polynomials.* Riv. Univ. Parma (2) **5** (1964), 77–84.
4. A. Hurwitz: *Über Abel's Verallgemeinerung der binomische Formel*, Acta Mathematica **26** (1902), 199–203.
5. J. L.W.V. Jensen: *Sur une identité d'Abel et sur d'autres formules analogues.* Acta Mathematica **26** (1902), 307–318.
6. T. Popoviciu: *Sur le reste dans certaines formules linéaires d'approximation de l'analyse.* Mathematica (Cluj) 1 (**24**) (1959), 59–142.
7. D. D. Stancu: *Evaluation of the remainder term in approximation formulas by Bernstein polynomials.* Math. Comp. **17** (1963), 270–278.
8. D. D. Stancu: *The remainder of certain linear approximation formulas in two variables.* J. SIAM Numer. Anal. B, **1** (1964), 137–163.
9. D. D. Stancu, C. Cismaşiu: *On a approximating linear positive operator of Cheney-Sharma.* Rev. Anal. Numér. Theor. Approx. **26** (1997), 221–227.
10. D. D. Stancu: *Use of an identity of A. Hurwitz for construction of a linear positive operator of approximation.* Rev. Anal. Numér. Theor. Approx. **31** (2002), 115–118.
11. I. Taşcu: *Approximation of bivariate functions by means of a polynomial operator of Stancu-Hurwitz type.* Facta Univ. Ser. Math. Inform. **20** (2005), 33–39.



North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania

itascu@yahoo.com