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# VARIATION OF DOMAINS AND CONCEPT OF MAXIMAL OUTPUT ADMISSIBLE SETS

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**Abstract.** The paper treats the problem of perturbed infinite dimensional discrete-time linear systems given by $x_{i+1}^p = Ax_i^p + B_p u_i$ for every $i \in \mathbb{N}$ and $x_0 \in L^2(\Omega)$, where $B_p$ is a bounded operator representing the perturbation affecting the system. We suppose that just a party $\omega$ of $\Omega$ is controlled. We seek to characterize the set of all variations $\omega_p$ of $\omega$ due to negligent disturbances such that the effect is under a threshold chosen previously. Practical algorithm with simulations are given.

## 1. Introduction

Engineering, biological, economic systems and others are often influenced by some disturbances. These lasts are translated into variables which infiltrate in the mathematical models describing these systems.

In recent years, an extensive studies of the perturbation problems have been elaborated and received considerable attentions. We mention as examples [4], [5], [8], [12], [13], [14], [15] and [16]. This work is a part of same context. Indeed, we consider the following discrete system

$$\begin{cases} x_{i+1} = Ax_i + Bu_i, & i \in \mathbb{N}, \\ u_i = 0, & i \geq I, \\ x_0 \in L^2(\Omega), & \end{cases}$$

the associated output is

$$y_i = Cx_i, \quad i \in \mathbb{N},$$

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where $I$ is a constant in $\mathbb{N}$, $\Omega$ is a nonempty open subset in $\mathbb{R}^n$, $A$ is a bounded operator on $L^2(\Omega)$ and $C$ is a bounded operator from $L^2(\Omega)$ to $\mathbb{R}^q$. The operator $B$ is defined by $B : \mathbb{R} \to L^2(\Omega), v \to v.\mathbb{1}_\omega$, where $\omega$ is a convex compact subset of $\Omega$ which represents the controlled area. $x_i \in L^2(\Omega)$ is the state corresponding to the control $u_i \in U$ of the system at time $i$.

We suppose that $\omega$ sustains a small perturbation and one of the natural questions concerning this perturbation that we address in this work is the following: given a threshold of tolerance $\varepsilon$, under what hypothesis can we characterize all disturbances which are $\varepsilon$-tolerable?

More exactly, we consider the infected discrete linear system

$$\left\{ \begin{array}{l} x^p_{i+1} = Ax^p_i + B_p u_i, \quad i \in \mathbb{N}, \\ x_0 \in L^2(\Omega), \end{array} \right. \tag{1.1}$$

the associated output is

$$y^p_i = Cx^p_i, \quad i \in \mathbb{N},$$

where $x^p_i$ represents the state variable corresponding to the perturbation $\omega_p$ of $\omega$. $B_p$ is the map defined by $B_p : \mathbb{R} \to L^2(\Omega), v \to v.\mathbb{1}_{\omega_p}$.

Our contribution is to determine the set of all disturbances $\omega_p$ that are enough nearby of the domain $\omega$ and for which the corresponding output variables $y^p_i$ remains in a neighborhood of the uninfected output $y_i$ for every $i \in \mathbb{N}$. The fact to suppose that the perturbation $\omega_p$ which affects the domain $\omega$ is "small" leads to consider $d(\omega, \omega_p)$, where $d(\cdot,\cdot)$ designates the Hausdorff distance defined on the compact sets of $\mathbb{R}^n$ and which has as vocation the measure of the maximum distance between two domains or two sets (see, for example, [1] and [3]). Hence, our problem amounts to study of the following set

$$\chi(\alpha,\varepsilon) = \Big\{ \omega_p \in \Delta_\alpha \,:\, \|y^p_i - y_i\| \le \varepsilon,\ i \in \mathbb{N} \Big\}.$$

For all $\alpha > 0$, $\Delta_\alpha$ is the nonempty set defined by

$$\Delta_\alpha = \Big\{ \omega' \subset \Omega \,:\, \omega' \ \text{ convex compact and } \ d(\omega,\omega') \le \alpha \Big\}$$

The paper is organized as follows. Section 2 contains some preliminary results. Sufficient conditions for characterization of the set $\chi(\alpha,\varepsilon)$, including a computational algorithm, are discussed in Section 3. The results are illustrated through a simple example and numerical simulations in Section 4.

We conclude this Section with notations. Let's $E$ be a subset of $\mathbb{N}$. If we denote

$$l^2(E,\mathbb{R}^q) = \Big\{ x = (x_i)_{i\in E} \,:\, x_i \in \mathbb{R}^q, \sum_{i\in E} \|x_i\|^2 < \infty \Big\},$$

$l^2(E, \mathbb{R}^q)$ endowed with the usual addition, the scalar multiplication and the following inner product

$$(x, y)_{l^2(E,\mathbb{R}^q)} = \sum_{i \in E} (x_i, y_i)$$

is a Hilbert space. The corresponding norm is

$$\|x\|^2_{l^2(E,\mathbb{R}^q)} = \sum_{i \in E} \|x_i\|^2.$$

The set of all compacts in $\mathbb{R}^n$ is denoted $K(\mathbb{R}^n)$. $\operatorname{Int}\chi(\alpha, \varepsilon)$ denotes the interior of $\chi(\alpha, \varepsilon)$ and $\dot{\omega}$ is the frontier of $\omega$.

## 2. Basic Results

In order to express the set $\chi(\alpha, \varepsilon)$ in simple terms, we rewrite the solution of the difference equation (1.1) thus

$$x_i^p = A^i x_0 + \sum_{j=0}^{i-1} A^{i-j-1} B_p u_j, \quad i \in \mathbb{N}^\star.$$

Then, for every $i \geq 1$ we have

$$y_i^p - y_i = C x_i^p - C x_i = \sum_{j=0}^{i-1} C A^{i-j-1} \zeta_j^p,$$

where $\zeta_j^p = (B_p - B) u_j$, $j \in \mathbb{N}$.

So, the set of all disturbances $\omega_p \in \Delta_\alpha$ which are $\varepsilon$-tolerable is formally given by

$$\chi(\alpha, \varepsilon) = \left\{ \omega_p \in \Delta_\alpha \,:\, \left\| \sum_{j=0}^{i-1} C A^{i-j-1} \zeta_j^p \right\| \leq \varepsilon,\ i \in \mathbb{N}^\star \right\},$$

or yet $\chi(\alpha, \varepsilon) = \chi_1(\alpha, \varepsilon) \cap \chi_2(\alpha, \varepsilon)$, with

$$\chi_1(\alpha, \varepsilon) = \left\{ \omega_p \in \Delta_\alpha \,:\, \left\| \sum_{j=0}^{i-1} C A^{i-j-1} \zeta_j^p \right\| \leq \varepsilon,\ i \in \{1, \ldots, I\} \right\},$$

and

$$\chi_2(\alpha, \varepsilon) = \left\{ \omega_p \in \Delta_\alpha \,:\, \left\| \sum_{j=0}^{i-1} C A^{i-j-1} \zeta_j^p \right\| \leq \varepsilon,\ i \in \{I+1, \ldots\} \right\}.$$

**Remark 2.1.** As we know that $\zeta_i^p = 0$ for all $i \in \{I, I+1, \ldots\}$, we have

$$\chi_2(\alpha,\varepsilon) = \Big\{\omega_p \in \Delta_\alpha : \|CA^k G_\alpha(\omega_p)\| \le \varepsilon,\ k \in \mathbb{N}\Big\},$$

where $G_\alpha$ is the map defined by

$$\begin{aligned} G_\alpha : \qquad \Delta_\alpha &\longrightarrow L^2(\Omega), \\ \omega' &\longrightarrow G_\alpha(\omega') = \sum_{j=0}^{I-1} A^{I-j}(B'-B)u_j, \end{aligned}$$

with $B'$ the bounded operator defined by

$$\begin{aligned} B' : \qquad \mathbb{R} &\longrightarrow L^2(\Omega) \\ v &\longrightarrow B'(v) \ = \ v.\mathbb{1}_{\omega'}. \end{aligned}$$

It is obvious that the set $\chi(\alpha,\varepsilon)$ contains $\omega$ and so it is nonempty. Moreover, we can prove that $\chi(\alpha,\varepsilon)$ is not limited to a singleton and contains a neighborhood of $\omega$; then we present the following result

**Proposition 2.1.** *If* $\|A\| < 1$ *then* $\omega \in \operatorname{Int}\chi(\alpha,\varepsilon)$.

The proof of this result necessitates two lemmas. To demonstrate the second, we will need the following relation established by [10]

$$\text{mes}(\omega + \varrho B_0) = \text{mes}(\omega) + \sum_{r=1}^{n} \frac{\varrho^r}{r} \int_{\dot\omega} \kappa_{r-1}(\dot\omega) d\sigma, \tag{2.1}$$

where $\omega$ is a convex compact subset of $\Omega$, $B_0$ is the open unit ball in $\mathbb{R}^n$ and $\varrho$ is supposed to be enough small. $\kappa_r(\dot\omega)$ designates the sum of products $r$ with $r$ of principal curvatures of $\dot\omega$ and $d\sigma$ is the surface element.

**Lemma 2.1.** *For two convex compact sets* $\omega_1$ *and* $\omega_2$ *in* $\mathbb{R}^n$*, we have*

$$d(\omega_1, \omega_1 \cap \omega_2) \le d(\omega_1,\omega_2) \quad \textit{and} \quad d(\omega_2, \omega_1 \cap \omega_2) \le d(\omega_1,\omega_2).$$

*Proof.* It suffices to prove the result for the first relation. By definition of the Hausdorff distance, we have [3]

$$d(\omega_1,\omega_2) = \max\left(\max_{x\in\omega_1}\inf_{y\in\omega_2} D(x,y), \max_{y\in\omega_2}\inf_{x\in\omega_1} D(x,y)\right)$$

and

$$d(\omega_1, \omega_1 \cap \omega_2) = \max\left(\max_{x\in\omega_1} \inf_{y\in\omega_1\cap\omega_2} D(x,y), \max_{y\in\omega_1\cap\omega_2} \inf_{x\in\omega_1} D(x,y)\right),$$

where $D(\cdot,\cdot)$ is the euclidian distance on $\mathbb{R}^n$.

Now, $\max\limits_{y\in\omega_1\cap\omega_2} \inf\limits_{x\in\omega_1} D(x,y) = 0$. Then

$$d(\omega_1, \omega_1 \cap \omega_2) = \max_{x\in\omega_1} \inf_{y\in\omega_1\cap\omega_2} D(x,y).$$

Furthermore, for all $x \in \omega_1$ we have $\inf\limits_{y\in\omega_2} D(x,y) \le D(x,y)$, $y \in \omega_1 \cap \omega_2$. Also,

$$\inf_{y\in\omega_2} D(x,y) \le \inf_{y\in\omega_1\cap\omega_2} D(x,y).$$

On the other hand, the fact that $\omega_1$ and $\omega_2$ are compact implies

$$(\exists y_1 \in \omega_1 \cap \omega_2) \inf_{y\in\omega_1\cap\omega_2} D(x,y) = D(x,y_1)$$

and

$$(\exists y_2 \in \omega_2) \inf_{y\in\omega_2} D(x,y) = D(x,y_2).$$

Also, for the same reason

$$(\exists x_1 \in \omega_1) \max_{x\in\omega_1} D(x,y_1) = D(x_1,y_1)$$

and

$$(\exists x_2 \in \omega_1) \max_{x\in\omega_1} D(x,y_2) = D(x_2,y_2).$$

According to the convexity hypothesis, we have $D(x_1,y_1) = D(x_1,y_2)$. Consequently, $D(x_1,y_1) \le D(x_1,y_2) \le D(x_2,y_2)$, which permits to deduce the result. $\square$

**Lemma 2.2.** *The map $G_\alpha$ is continuous for all $\alpha > 0$ enough small.*

*Proof.* The demonstration will be done in the case $n \ge 2$; this one in the case $n = 1$ remains similar (see Remark 2.2).

We seek to prove that

$$(\forall\omega_1 \in \Delta_\alpha)(\forall b > 0)(\exists a > 0)(\forall\omega_2 \in \Delta_\alpha)$$
$$\left[d(\omega_1,\omega_2) < a \Rightarrow \left\|\sum_{j=0}^{I-1} A^{I-j} u_j(\mathbb{1}_{\omega_2} - \mathbb{1}_{\omega_1})\right\| < b\right].$$

Let us remark that

$$\|\mathbb{1}_{\omega_2} - \mathbb{1}_{\omega_1}\|^2 = \operatorname{mes}(\omega_1) + \operatorname{mes}(\omega_2) - 2\operatorname{mes}(\omega_1 \cap \omega_2),$$

or yet

$$\|\mathbb{1}_{\omega_2} - \mathbb{1}_{\omega_1}\|^2 = [\operatorname{mes}(\omega_1) - \operatorname{mes}(\omega_1 \cap \omega_2)] + [\operatorname{mes}(\omega_2) - \operatorname{mes}(\omega_1 \cap \omega_2)].$$

On the other hand, the Hausdorff distance between $\omega_1$ and $\omega_1 \cap \omega_2$ is given by [1]

$$d(\omega_1, \omega_1 \cap \omega_2) = \inf\Big\{\varrho_1 > 0 : \omega_1 \subseteq (\omega_1 \cap \omega_2) + \varrho_1 B_0 \text{ and } (\omega_1 \cap \omega_2) \subseteq \omega_1 + \varrho_1 B_0\Big\}.$$

Also, we define the Hausdorff distance between $\omega_2$ and $\omega_1 \cap \omega_2$ as follows

$$d(\omega_2, \omega_1 \cap \omega_2) = \inf\Big\{\varrho_2 > 0 : \omega_2 \subseteq (\omega_1 \cap \omega_2) + \varrho_2 B_0 \text{ and } (\omega_1 \cap \omega_2) \subseteq \omega_2 + \varrho_2 B_0\Big\}.$$

Then, for all

$$\varrho_1 \in \Big\{\varrho_1 > 0 : \omega_1 \subseteq (\omega_1 \cap \omega_2) + \varrho_1 B_0 \text{ and } (\omega_1 \cap \omega_2) \subseteq \omega_1 + \varrho_1 B_0\Big\}$$

and

$$\varrho_2 \in \Big\{\varrho_2 > 0 : \omega_2 \subseteq (\omega_1 \cap \omega_2) + \varrho_2 B_0 \text{ and } (\omega_1 \cap \omega_2) \subseteq \omega_2 + \varrho_2 B_0\Big\}$$

we have

$$\operatorname{mes}(\omega_1) \leq \operatorname{mes}((\omega_1 \cap \omega_2) + \varrho_1 B_0) \text{ and } \operatorname{mes}(\omega_2) \leq \operatorname{mes}((\omega_1 \cap \omega_2) + \varrho_2 B_0).$$

Since $\alpha$ is enough small, we can applied equality (2.1) for $\varrho_1$ and $\varrho_2$ enough small and we obtain

$$\operatorname{mes}(\omega_1) \leq \operatorname{mes}(\omega_1 \cap \omega_2) + \sum_{r=1}^{n} \frac{\varrho_1^r}{r} \int_{\widehat{\dot{\omega_1 \cap \omega_2}}} \kappa_{r-1}(\widehat{\dot{\omega_1 \cap \omega_2}})\, d\sigma$$

and

$$\operatorname{mes}(\omega_2) \leq \operatorname{mes}(\omega_1 \cap \omega_2) + \sum_{r=1}^{n} \frac{\varrho_2^r}{r} \int_{\widehat{\dot{\omega_1 \cap \omega_2}}} \kappa_{r-1}(\widehat{\dot{\omega_1 \cap \omega_2}})\, d\sigma.$$

Thus,

$$\operatorname{mes}(\omega_1) - \operatorname{mes}(\omega_1 \cap \omega_2) \leq \sum_{r=1}^{n} \frac{\varrho_1^r}{r} \int_{\widehat{\dot{\omega_1 \cap \omega_2}}} |\kappa_{r-1}(\widehat{\dot{\omega_1 \cap \omega_2}})|\, d\sigma$$

and

$$\mathrm{mes}(\omega_2) - \mathrm{mes}(\omega_1 \cap \omega_2) \leq \sum_{r=1}^{n} \frac{\varrho_2^r}{r} \int_{\widehat{\omega_1 \dot{\cap} \omega_2}} |\kappa_{r-1}(\widehat{\omega_1 \dot{\cap} \omega_2})| \, d\sigma,$$

which implies that

$$\mathrm{mes}(\omega_1) - \mathrm{mes}(\omega_1 \cap \omega_2) \leq \sum_{r=1}^{n} \frac{(d(\omega_1, \omega_1 \cap \omega_2))^r}{r} \int_{\widehat{\omega_1 \dot{\cap} \omega_2}} |\kappa_{r-1}(\widehat{\omega_1 \dot{\cap} \omega_2})| \, d\sigma$$

and

$$\mathrm{mes}(\omega_2) - \mathrm{mes}(\omega_1 \cap \omega_2) \leq \sum_{r=1}^{n} \frac{(d(\omega_2, \omega_1 \cap \omega_2))^r}{r} \int_{\widehat{\omega_1 \dot{\cap} \omega_2}} |\kappa_{r-1}(\widehat{\omega_1 \dot{\cap} \omega_2})| \, d\sigma.$$

Consequently,

$$\begin{aligned} &\|\mathbb{1}_{\omega_2} - \mathbb{1}_{\omega_1}\|^2 \\ &\quad \leq \sum_{r=1}^{n} \frac{[(d(\omega_1, \omega_1 \cap \omega_2))^r + (d(\omega_2, \omega_1 \cap \omega_2))^r]}{r} \int_{\widehat{\omega_1 \dot{\cap} \omega_2}} |\kappa_{r-1}(\widehat{\omega_1 \dot{\cap} \omega_2})| \, d\sigma. \end{aligned}$$

Hence,

$$\|\mathbb{1}_{\omega_2} - \mathbb{1}_{\omega_1}\|^2 \leq \sum_{r=1}^{n} \frac{[(d(\omega_1, \omega_1 \cap \omega_2))^r + (d(\omega_2, \omega_1 \cap \omega_2))^r]}{r} \int_{\dot{\omega}_1} |\kappa_{r-1}(\dot{\omega}_1)| \, d\sigma.$$

According to Lemma 2.2, we deduce

$$\|\mathbb{1}_{\omega_2} - \mathbb{1}_{\omega_1}\|^2 \leq 2 \sum_{r=1}^{n} \frac{(d(\omega_1, \omega_2))^r}{r} \int_{\dot{\omega}_1} |\kappa_{r-1}(\dot{\omega}_1)| \, d\sigma.$$

The result can be deduced after remarking that $d(\omega_1, \omega_2) < 1$ for $\alpha$ enough small, and so

$$a = \frac{b \displaystyle\sum_{j=0}^{I-1} \left\|A^{I-j}\right\| \|u_j\|}{2n \displaystyle\sum_{r=1}^{n} \int_{\dot{\omega}_1} |\kappa_{r-1}(\dot{\omega}_1)| \, d\sigma}$$

replies to question. $\square$

**Remark 2.2.** To prove the continuity of $G_\alpha$ for $n = 1$, we write $\omega_1 \cap \omega_2$ as follows $\omega_1 \cap \omega_2 = [a, b]$, where $a$ and $b$ are two reals.

Then $(\omega_1 \cap \omega_2) + \varrho_1 B_0 = [a - \varrho_1, b + \varrho_1]$, and $(\omega_1 \cap \omega_2) + \varrho_2 B_0 = [a - \varrho_2, b + \varrho_2]$, which leads to $\text{mes}((\omega_1 \cap \omega_2) + \varrho_1 B_0) = b - a + 2\varrho_1$ and $\text{mes}((\omega_1 \cap \omega_2) + \varrho_2 B_0) = b - a + 2\varrho_2$, i.e., $\text{mes}((\omega_1 \cap \omega_2) + \varrho_1 B_0) = \text{mes}(\omega_1 \cap \omega_2) + 2\varrho_1$ and $\text{mes}((\omega_1 \cap \omega_2) + \varrho_2 B_0) = \text{mes}(\omega_1 \cap \omega_2) + 2\varrho_2$. Furthermore, $\text{mes}((\omega_1 \cap \omega_2) + \varrho_1 B_0) \le \text{mes}(\omega_1) + 2\varrho_1$ and $\text{mes}((\omega_1 \cap \omega_2) + \varrho_2 B_0) \le \text{mes}(\omega_1) + 2\varrho_2$. We applied then the same technique developed in the case $n \ge 2$ for the rest of the proof.

*Proof of the Proposition* 2.1. If we put by convention $A^i = 0$, $i < 0$, we have

$$\chi_1(\alpha, \varepsilon) = \bigcap_{i=1}^{I} \chi_1^i(\alpha, \varepsilon) \supseteq \bigcap_{i=1}^{I} \hat{\chi}_1^i(\alpha, \varepsilon),$$

where

$$\chi_1^i(\alpha, \varepsilon) = \left\{ \omega_p \in \Delta_\alpha \;:\; \left\| \sum_{j=0}^{I} C A^{i-j-1} \zeta_j^p \right\| \le \varepsilon \right\},$$

and

$$\hat{\chi}_1^i(\alpha, \varepsilon) = \left\{ \omega_p \in \Delta_\alpha \;:\; \left\| \sum_{j=0}^{I} C A^{i-j-1} \zeta_j^p \right\| < \varepsilon \right\}.$$

Moreover, we use the continuity of the map

$$\begin{aligned} f_\alpha : \qquad & \Delta_\alpha \longrightarrow L^2(\Omega), \\ & \omega_p \;\longrightarrow B_p, \end{aligned}$$

and we deduce that the map

$$\omega_p \in \Delta_\alpha \hookrightarrow \sum_{j=0}^{I} C A^{i-j-1} \zeta_j^p$$

is continuous too.

Consequently, for every $i \in \{1, \dots, I\}$, $\hat{\chi}_1^i(\alpha, \varepsilon)$ is an open subset of $K(\mathbb{R}^n)$ and it contains $\omega$. Thus $\omega \in \text{Int}(\hat{\chi}_1(\alpha, \varepsilon))$.

On the other hand the fact that $\|A\| < 1$ implies the existence of a constant $\rho$ such that $\|CA^k x\| \le \rho \|x\|$, for every $x \in L^2(\Omega)$ and $k \in \mathbb{N}$. For every $\omega_p \in \Delta_\alpha$ and every $k \in \mathbb{N}$ we have $\|CA^k G_\alpha(\omega_p)\| \le \rho \|G_\alpha(\omega_p)\|$. Moreover, the continuity of $G_\alpha$ implies that

$$(\forall \gamma > 0)(\exists \eta > 0)\, d(\omega_p, \omega) < \eta \Rightarrow \|G_\alpha(\omega_p)\| \le \frac{\gamma}{\rho}.$$

Thus, for every $\omega_p \in B(\omega, \eta)$ (where $B(\omega, \eta)$ is the ball in $K(\mathbb{R}^n)$ of center $\omega$ and radius $\eta$) and for every $k \in \mathbb{N}$ we have

$$\|CA^k G_\alpha(\omega_p)\| \leq \rho \|G_\alpha(\omega_p)\| \leq \gamma.$$

Hence, $B(\omega, \eta) \subseteq \chi_2(\alpha, \varepsilon)$ and $\omega \in \operatorname{Int} \chi_2(\alpha, \varepsilon)$. $\square$

**Remark 2.3.** Our objective is to characterize the set $\chi(\alpha, \varepsilon)$. For this, we seek to express it by a simpler structure which consists of obtaining it by finite recursive procedures. It's the subject of subsequent sections.

## 3. Sufficient Conditions for Finite Determination of $\chi(\alpha, \varepsilon)$

It is obvious that the set $\chi_1(\alpha, \varepsilon)$ derives from a finite number of inequalities and then a characterization of the set $\chi(\alpha, \varepsilon)$ is equivalent to a characterization of the set $\chi_2(\alpha, \varepsilon)$ which can be rewritten as follows

$$\chi_2(\alpha, \varepsilon) = \Big\{\omega_p \in \Delta_\alpha \, : \, G_\alpha(\omega_p) \in T(\varepsilon)\Big\},$$

where $T(\varepsilon) = \{x \in L^2(\Omega) \, : \, \|CA^i x\| \leq \varepsilon, \; i \in \mathbb{N}\}$. Moreover, if we define for each integer $k$ the set

$$\chi_2^k(\alpha, \varepsilon) = \Big\{\omega_p \in \Delta_\alpha \, : \, G_\alpha(\omega_p) \in T_k(\varepsilon)\Big\},$$

with

$$T_k(\varepsilon) = \Big\{x \in L^2(\Omega) \, : \, \|CA^i x\| \leq \varepsilon, \; i \in \{0, 1, \ldots, k\}\Big\},$$

we remark that the following relations hold

$$\chi_2(\alpha, \varepsilon) = G_\alpha^{-1}(T(\varepsilon)) \quad \text{and} \quad \chi_2^k(\alpha, \varepsilon) = G_\alpha^{-1}(T_k(\varepsilon)).$$

Hence, to characterize the set $\chi_2(\alpha, \varepsilon)$ it suffices to determine the set $T(\varepsilon)$ by easy computational characterization. Hereafter, our basis for describing $T(\varepsilon)$ as desired is to make use of a concept of maximal output admissible sets [6], [7], [10] and [11]. So we consider the following definition.

**Definition 3.1.** The set $T(\varepsilon)$ (resp. $\chi_2(\alpha, \varepsilon)$) is said to be finitely determined if there exists an integer $k$ such that $T(\varepsilon) = T_k(\varepsilon)$ (resp. $\chi_2(\alpha, \varepsilon) = \chi_2^k(\alpha, \varepsilon)$). In this case, we denote $k^\star$ the smallest integer such that $T(\varepsilon) = T_{k^\star}(\varepsilon)$ (resp. $\chi_2(\alpha, \varepsilon) = \chi_2^{k^\star}(\alpha, \varepsilon)$).

Obviously, we have the following relation

$$T(\varepsilon) \subseteq T_{k_2}(\varepsilon) \subseteq T_{k_1}(\varepsilon), \quad k_1, k_2 \in \mathbb{N}, \; k_1 \leq k_2. \tag{3.1}$$

Conditions which imply finite determination of $T(\varepsilon)$ are discussed in the sequel.

**Theorem 3.1.** *$T(\varepsilon)$ is finitely determined if and only if there exists an integer $k$ such that $T_k(\varepsilon) = T_{k+1}(\varepsilon)$.*

*Proof.* If we suppose the existence of an integer $k$ such that $T(\varepsilon) = T_k(\varepsilon)$, then all $x \in T_k(\varepsilon)$ verifies $\|CA^i x\| \leq \varepsilon$ for every $i \in \mathbb{N}$; in particular $\|CA^{k+1}x\| \leq \varepsilon$ and so, $x$ is an element of $T_{k+1}(\varepsilon)$. We apply (3.1) to conclude that $T_k(\varepsilon) = T_{k+1}(\varepsilon)$. From this equality, it is easy confirmed that $x \in T_k(\varepsilon)$ implies that $Ax \in T_k(\varepsilon)$ and recursively $A^j x \in T_k(\varepsilon)$ for each $j \in \mathbb{N}$. This implies that $x$ is an element of $T(\varepsilon)$. The proof is completed by (3.1). $\square$

Using this result, we establish sufficient conditions which assure the finite determination of $T(\varepsilon)$; the main result in this direction is the following theorem. But, at first, we recall a fundamental propriety of the exact-observability for infinite dimensional systems. Thus, we consider the operator $\Lambda$ defined by

$$\begin{aligned} \Lambda: \qquad L^2(\Omega) &\longrightarrow l^2(\mathbb{N}; \mathbb{R}^q), \\ x &\longrightarrow (CA^i x)_{i \in \mathbb{N}}. \end{aligned}$$

**Proposition 3.1.** *The pair $(C, A)$ is exactly observable if and only if there exists a constant $\gamma > 0$ such that*

$$\|x\|_{L^2(\Omega)} \leq \gamma \|\Lambda x\|_{l^2(\mathbb{N};\mathbb{R}^q)}, \quad x \in D(\Lambda),$$

*where $D(\Lambda)$ is the domain of $\Lambda$.*

For each integer $k$ different of zero, we denote by $\Lambda_k$ and $M_k$ the operators

$$\begin{aligned} \Lambda: \qquad L^2(\Omega) &\longrightarrow l^2(\{0, \ldots, k-1\}; \mathbb{R}^q), \\ x &\longrightarrow (CA^i x)_{0 \leq i \leq k-1} \end{aligned}$$

and

$$\begin{aligned} M_k: \qquad L^2(\Omega) &\longrightarrow l^2(\{k, k+1, \ldots\}; \mathbb{R}^q), \\ x &\longrightarrow (CA^i x)_{i \geq k}. \end{aligned}$$

**Theorem 3.2.** *Suppose the following assumptions hold*

(i) *The pair $(C, A)$ is exactly observable;*

(ii) $(\exists k_0 \in \mathbb{N}^\star)\ 0 < \dfrac{1-\gamma^2\|M_{k_0}^\star M_{k_0}\|}{\gamma^2\|\Lambda_{k_0}^\star\|} < \|CA^{k_0}\|$ (*where* $\gamma$ *is given by Proposition* 3.1)*;*

(iii) $A$ *is asymptotically stable.*

*Then* $T(\varepsilon)$ *is finitely determined.*

*Proof.* It is apparent from (i) that for all $k$ integer and in particular for $k_0$ given by (ii) we have

$$\langle \Lambda_{k_0}^\star \Lambda_{k_0} Ax, Ax\rangle + \langle M_{k_0}^\star M_{k_0} Ax, Ax\rangle \geq \frac{1}{\gamma^2}\|Ax\|^2, \quad x \in T_{k_0}(\varepsilon). \tag{3.2}$$

Moreover,

$$\Lambda_{k_0} Ax \in \underbrace{B(0,\varepsilon)\times B(0,\varepsilon)\times\cdots\times B(0,\varepsilon)}_{k_0 \text{ times}}, \quad x \in T_{k_0}(\varepsilon),$$

where $B(0,\varepsilon)$ is the ball in $\mathbb{R}^q$ of center 0 and radius $\varepsilon$.

Consequently,

$$\Lambda_{k_0}^\star \Lambda_{k_0} A(T_{k_0}(\varepsilon)) \subseteq \Lambda_{k_0}^\star(\underbrace{B(0,\varepsilon)\times B(0,\varepsilon)\times\cdots\times B(0,\varepsilon)}_{k_0 \text{ times}})$$

So,

$$\forall x \in T_{k_0}(\varepsilon),\ \exists z \in \underbrace{B(0,\varepsilon)\times B(0,\varepsilon)\times\cdots\times B(0,\varepsilon)}_{k_0 \text{ times}} : \Lambda_{k_0}^\star \Lambda_{k_0} Ax = \Lambda_{k_0}^\star z,$$

or yet

$$\forall x \in T_{k_0}(\varepsilon),\ \exists z \in \underbrace{B(0,\varepsilon)\times\cdots\times B(0,\varepsilon)}_{k_0 \text{ times}} : \langle \Lambda_{k_0}^\star \Lambda_{k_0} Ax, Ax\rangle = \langle \Lambda_{k_0}^\star z, Ax\rangle.$$

Then, the inequality (3.2) becomes

$$\langle \Lambda_{k_0}^\star z, Ax\rangle + \langle M_{k_0}^\star M_{k_0} Ax, Ax\rangle \geq \frac{1}{\gamma^2}\|Ax\|^2, \quad x \in T_{k_0}(\varepsilon),$$

which implies

$$\|\Lambda_{k_0}^\star\|\|z\| + \|M_{k_0}^\star M_{k_0}\|\|Ax\| \geq \frac{1}{\gamma^2}\|Ax\|, \quad x \in T_{k_0}(\varepsilon).$$

That is

$$(1-\gamma^2\|M_{k_0}^\star M_{k_0}\|)\|Ax\| \le \gamma^2\|\Lambda_{k_0}^\star\|\varepsilon, \quad x\in T_{k_0}(\varepsilon).$$

From (ii), it follows that

$$\|Ax\| \le \frac{\varepsilon\,\gamma^2\|\Lambda_{k_0}^\star\|}{1-\gamma^2\|M_{k_0}^\star M_{k_0}\|}, \quad x\in T_{k_0}(\varepsilon).$$

If we denote by $r(k_0)$ the value $\frac{\varepsilon\,\gamma^2\|\Lambda_{k_0}^\star\|}{1-\gamma^2\|M_{k_0}^\star M_{k_0}\|}$, we have

$$AT_i(\varepsilon) \subseteq B(0,r(k_0)), \quad i\ge k_0, \tag{3.3}$$

where $B(0,r(k_0))$ is the ball in $L^2(\Omega)$ of center 0 and radius $r(k_0)$.

On the other hand, the asymptotic stability of A implies the existence of a rank $k_1$ such that

$$\|CA^{k_1}\| < \frac{\varepsilon}{r(k_0)}, \tag{3.4}$$

which implies that $CA^{k_1}(B(0,r(k_0))) \subseteq B(0,\varepsilon)$. If we suppose that $k_1$ is the smallest integer which verifies (3.4), by (ii) we have obligatory $k_1 > k_0$ and from (3.3) we deduce that $AT_{k_1}(\varepsilon)\subseteq B(0,r(k_0))$. Then $CA^{k_1+1}(T_{k_1}(\varepsilon))\subseteq B(0,\varepsilon)$. Hence, $T_{k_1}(\varepsilon)\subseteq T_{k_1+1}(\varepsilon)$. Finally, we use (3.1) to end the proof. □

The hypotheses of Theorem 3.2 are sufficient and not necessary to have the set $T(\varepsilon)$ finitely determined. So, we can think to other sufficient conditions. For this, let's consider the operator $\hat{C}$ defined as

$$\begin{aligned} \hat{C}: \quad & L^2(\Omega) \longrightarrow L^2(\Omega),\\ & x \longrightarrow \hat{C}x = \sum_{j=1}^{q}\alpha_j e_j, \end{aligned}$$

$(e_j)_j$ is a hilbertian basis in $L^2(\Omega)$. For all $j$, $\alpha_j$ is the $j^{th}$ component of the vector $Cx$.

We have

$$\begin{aligned} T(\varepsilon) &= \{x\in L^2(\Omega) : \|CA^ix\| \le \varepsilon,\ i\in\mathbb{N}\}\\ &= \{x\in L^2(\Omega) : CA^ix \in [-\varepsilon,\varepsilon]^q,\ i\in\mathbb{N}\}\\ &= \{x\in L^2(\Omega) : \hat{C}A^ix\in\Sigma,\ i\in\mathbb{N}\}, \end{aligned}$$

where

$$\Sigma = \left\{x\in L^2(\Omega) : x=\sum_{i=1}^{q}x_ie_i,\ x_i\in[-\varepsilon,\varepsilon],\ 1\le i\le q\right\}.$$

Then, we annunciate the following proposition.

**Proposition 3.2.** *Suppose the following hypotheses hold*

(i) $\hat{C}$ *commutes with* $A$.

(ii) $A\Sigma \subset \Sigma$.
*Then,* $T(\varepsilon)$ *is finitely determined. Moreover,* $T(\varepsilon) = T_0(\varepsilon)$.

*Proof.* Let $x \in T_0(\varepsilon)$, then $\hat{C}x \in \Sigma$.

According to (i), we have $\hat{C}Ax = A\hat{C}x$. Now, $A\hat{C}x \in A\Sigma$, while supplementing by (ii), we obtain $T_0(\varepsilon) = T_1(\varepsilon)$. □

Thus, for all system which verifies the hypotheses of Proposition 3.2 it is easy to characterize the set $T(\varepsilon)$. Else, we verify the hypotheses of Theorem 3.2; in this case we seek a procedure to determine an integer $k$ such that $T_k(\varepsilon) \subseteq T_{k+1}(\varepsilon)$. This leads to think to the following recursion

$$T_{k+1}(\varepsilon) = T_k(\varepsilon) \cap \Big\{x \in L^2(\Omega) \,:\, \|CA^{k+1}x\| \leq \varepsilon\Big\}.$$

So the condition $T_{k+1}(\varepsilon) = T_k(\varepsilon)$, which implies the finite determination of the set $T(\varepsilon)$, is verified if and only if we have

$$T_k(\varepsilon) \subseteq \Big\{x \in L^2(\Omega) \,:\, \|CA^{k+1}x\| \leq \varepsilon\Big\},$$

or yet $\|CA^{k+1}x\| \leq \varepsilon$ for all $x \in T_k(\varepsilon)$, i.e.,

$$(\forall x \in T_k(\varepsilon))\ (\forall i \in \{1, \ldots, q\}) \quad |(CA^{k+1}x)_i| - \varepsilon \leq 0,$$

where $(CA^{k+1}x)_i$ is the $i^{\text{th}}$ component of the vector $CA^{k+1}x$.

This means that for every $x \in T_k(\varepsilon)$ and $i \in \{1, \ldots, q\}$ $(CA^{k+1}x)_i - \varepsilon \leq 0$ and $-(CA^{k+1}x)_i - \varepsilon \leq 0$. If we define, for all $i \in \{1, \ldots, 2q\}$, the function $f_i$, $f_i : \mathbb{R}^q \to \mathbb{R}$, by

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix} \mapsto f_i(\mathbf{y}) = \begin{cases} y_i - \varepsilon, & i \in \{1, \ldots, q\}, \\ -y_{i-q} - \varepsilon, & i \in \{q+1, \ldots, 2q\}, \end{cases}$$

the condition $T_{k+1}(\varepsilon) = T_k(\varepsilon)$ is verified if and only if we have

$$\max_{x \in T_k(\varepsilon)} f_i(CA^{k+1}x) \leq 0, \quad i \in \{1, \ldots, 2q\}.$$

Then, to have the numerical procedures for carrying out the finite determination condition of the set $T(\varepsilon)$ we must solve the linear mathematical programming problems

$$
(3.5)\quad \begin{cases} \text{maximize the functional} & J_i(x) = f_i(CA^{k+1}x), \\ \quad\quad\text{subject to} & \\ f_j(CA^l x) \le 0, & j = 1, \dots, 2q,\ l = 0, 1, \dots, k. \end{cases}
$$

The solution is a function which must be chosen in space $L^2(\Omega)$ of infinite dimension. Hence, we purpose to approach problem (3.5) by a sequence of problems in finite dimension (see, for example, [6]). So, we rewrite it in the following way

$$
\begin{cases} \max J_i(x) \\ x \in K \end{cases}
$$

$K$ is the constraints set defined by

$$
K = \{x \in L^2(\Omega) \,:\, f_j(CA^l x) \le 0,\ j = 1, \dots, 2q,\ l = 1, \dots, k\}.
$$

We can verify that $K$ is closed and convex. The Gâteau derivative of $J_i$ exists and $J_i'$ is bounded on all bounded. We are leading to approximate this problem by

$$
\begin{cases} \max\ J_i(x_h) \\ x_h \in K_h \end{cases}
$$

$K_h$ is a closed convex included in space $V_h$ engendered by $(e_i)_{1 \le i \le J}$, where $(e_i)_{i \in \mathbb{N}}$ is a hilbertian basis of $L^2(\Omega)$. $V_h$ has finite dimension $J$ $(h = 1/J)$.

If we put

$$
K_h = \Big\{x_h \in V_h \,:\, f_j(CA^l x_h) \le 0,\ j = 1, \dots, 2q,\ l = 1, \dots, k\Big\}
$$

we have the following hypothesis characterizing the approximation [6]

$$
H_1 : \forall x \in K,\, \exists\, x_h \in K_h \,/\, x_h \longrightarrow x \ \text{ strongly if } \ h \to 0,
$$

$$
H_2 : x_h \in K_h \ \text{ and } \ x_h \longrightarrow x \ \text{ weakly } \Longrightarrow x \in K.
$$

Thus, our problem will be reduced to the resolution of $2q$ linear mathematical programming problems in finite dimension. In this case, much softwares or standard methods can be used for the numerical resolution. Consequently, we introduce the following algorithm.

Algorithm

$$
\left|
\begin{array}{ll}
\text{step 1 :} & \text{let}\, k \leftarrow 0 \\
\text{step 2 :} & \left[
\begin{array}{l}
\text{for } i = 1, \dots, 2q \text{ do} \\
\left\{
\begin{array}{l}
\max J_i(x_h) = f_i(CA^{k+1}x_h) \\
s.t \\
f_j(CA^l x_h) \leq 0;\ j = 1, \dots, 2q,\ l = 0, \dots, k
\end{array}
\right. \\
\text{Set } J_i^* = \max J_i(x_h) \\
\text{end for}
\end{array}
\right. \\
 & \text{if } (J_i^* \leq 0) \text{ for } i = 1, \dots, 2q \\
 & k^* \leftarrow k \text{ and stop.} \\
 & \text{else continue} \\
\text{step 3 :} & k \leftarrow k + 1 \ \text{ and return to step 2.}
\end{array}
\right.
$$

## 4. Example

We take $\Omega = ]0, \pi[$ and we consider the following hilbertian basis of $L^2(]0, \pi[)$

$$e_i(t) = \sqrt{\frac{2}{\pi}} \sin it, \quad i \in \mathbb{N}^\star,\ t \in ]0, \pi[.$$

We define the operator $A$ as

$$
\begin{aligned}
A : \qquad & L^2(]0, \pi[) \to L^2(]0, \pi[), \\
& x = \sum_i x_i e_i \to \sum_i e^{-i^2 \pi^2 t} x_i e_i.
\end{aligned}
$$

We suppose that the information is given by $q$ captors area as follows [2]

$$
\begin{aligned}
C : \qquad & L^2(]0, \pi[) \to \mathbb{R}^q \\
& x \to \mathbf{Cx} = \begin{pmatrix} (g_1, x)_{L^2(D_1)} \\ \vdots \\ (g_q, x)_{L^2(D_q)} \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
g_j : \qquad & \Omega \to \mathbb{R} \\
& t \to g_j(t) = \begin{cases} 1, & \text{if } \ t \in D_j, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

and

$$D_j = [j, j + 1/2], \quad j \in \{1, \dots, q\}.$$

**Remark 4.4.** We suggest that $D_j \subseteq ]0, \pi[$ for all $j \in \{1, \dots, q\}$, which incites to suppose that $q \leq 2$.

We have $C \in \mathcal{L}(L^2(]0, \pi[); \mathbb{R}^q)$ and the adjoint operator $C^\star$ of $C$ is defined by

$$\mathbf{C}^\star \begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix} = \sum_{i=1}^{q} y_i g_i.$$

We remark that the hypotheses of Theorem 3.2 are verified and so the set $T(\varepsilon)$ can be finitely determined; indeed hypothesis (i) and (iii) are easy to see. As for hypothesis (ii), it suffices to choose $k_0 = 1$ and we prove the result for $q = 1$.

Next we generalize the proof for $q = 2$. If we denote by $V_h$ the subset of $L^2(]0, \pi[)$ engendered by $(e_i)_{1 \leq i \leq J}$, where $h = 1/J$, then the restriction of the operator $C$ from $L^2(]0, \pi[)$ to $V_h$, still denoted $C$, is given by the matrix

$$\mathbf{C} = \begin{pmatrix} (g_1, e_1) & \dots & (g_1, e_J) \\ \vdots & & \vdots \\ (g_q, e_1) & \dots & (g_q, e_J) \end{pmatrix}$$

To illustrate the efficiency of the proposed algorithm in the precedent section, we compute the value $k^\star$ for various choices of matrix $C$. All data are summarized in the following table

| | $\varepsilon$ | $k^\star$ |
|---|---|---|
| $q = 1,\ J = 1$ | $10^{-6}$ | 2 |
| $q = 1,\ J = 2$ | $10^{-6}$ | 2 |
| $q = 2,\ J = 2$ | $10^{-6}$ | 2 |
| $q = 2,\ J = 50$ | $10^{-6}$ | 8 |

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