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ORTHONORMAL DECOMPOSITION OF FRACTAL INTERPOLATING FUNCTIONS

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Abstract. In this paper an orthonormal decomposition of affine plane transformation is applied on Iterated Function Systems that define fractal interpolating functions.

1. Introduction

Affine automorphisms seem to be the most elementary dynamic elements in Nature. By iterations of affinity any nonlinearity can be achieved. The simplest example is Horner algorithm which builds up polynomials by repeating affine functions. Of course, this process needs one extra dimension besides the dimension of automorphism domain - dimension of *time* where iterations take place. Many life processes like cells growing and multiplying, tissues forming, neural connections, cortical signals etc., are being product of iterated affinities and can be model and study by using IFS approach [1], [4]. Here, we examine contractive affine transforms of real plane and their application in analyzing fractal functions.

Affine transformation of the real plane $\mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$w : \boldsymbol{x} \mapsto A\boldsymbol{x} + \mathbf{b}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}, \tag{1.1}$$

where $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $a, b, c, d, e, f \in \mathbb{R}$. If $\mathbf{b} = \mathbf{0}$, transformation (1.1) is linear. The set of all affine transformations (1.1) will be denoted by $\mathcal{A}$.

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Regarding composition of mappings, the set $\mathcal{A}$ forms a noncommutative group $(\mathcal{A}, \circ)$.

Linear transformation (1.1) is *orthogonal* if it preserves orthogonality of vectors, i.e. if $\boldsymbol{x}^T \mathbf{y} = 0$ implies $w(\boldsymbol{x})^T w(\mathbf{y}) = 0$. In this case, matrix A is orthogonal, i.e. it satisfies

$$A^T A = \rho^2 \mathbf{I}, \ \ \rho \in \mathbb{R}.$$

If $\rho = 1$, w as well as A are *orthonormal*. The subset $\mathcal{O} \subset \mathcal{A}$ of orthogonal affine transforms forms *orthogonal subgroup* $(\mathcal{O}, \circ)$ of the group $(\mathcal{A}, \circ)$, while $(\mathcal{ON}, \circ)$ is *orthonormal* or *unitary subgroup* of $(\mathcal{O}, \circ)$.

If matrix A of the affine transformation (1.1) is *unimodular*, i.e. if it satisfies $|\det A| = 1$, one gets another subset of the set $\mathcal{A}$, i.e. also a subgroup $(\mathcal{U}, \circ)$ of the group $(\mathcal{A}, \circ)$, which is called *unimodular subgroup.* It is easy to see that $(\mathcal{ON}, \circ) \subset (\mathcal{U}, \circ)$.

In constructive theory of fractal sets an important issue is contraction property of affine mappings (see Section 3). But, it is as well important to distinguish orthogonal part of the mapping from non-orthogonal one. It is important since orthogonal affine mappings preserve angles - crucial geometric elements in constructions like so called initiator-generator ones. On the other hand, non-orthogonal mappings define other geometric parameters like stretch or shear.

So, the set of affine transformations can be expressed as the union $\mathcal{A} = \mathcal{O} \cup \mathcal{O}'$ of orthogonal affine mappings $\mathcal{O}$ and $\mathcal{O}'$ which is complement of $\mathcal{O}$ plus identity mapping. In other words, $\mathcal{O} \cap \mathcal{O}' = \{\boldsymbol{x} \mapsto \mathbf{I}\boldsymbol{x}\}$.

Theorem 1.1. (Orthogonal Decomposition) *The linear transformation $w(\boldsymbol{x}) = A\boldsymbol{x}$, where A is non-orthogonal 2×2 matrix, can be decomposed as*

$$w(\boldsymbol{x}) = w^{\perp}(w^{\Delta}(\boldsymbol{x})) = A^{\perp} A^{\Delta} \boldsymbol{x}, \tag{1.2}$$

where $A^{\perp}$ is orthogonal and A^{Δ} is non-orthogonal or unit matrix.

Proof. The proof is trivial if A is orthogonal, and then $A^{\Delta} = \mathbf{I}$. Otherwise, suppose that $A = A^{\perp} A^{\Delta}$ is non-orthogonal and that $A^{\perp}$ is its orthogonal component. Then $A^{\perp}$ satisfies (eventually multiplied by scalar constant) matrix equation $X^T X = I$, i.e., it coincide (up to the scalar constant) with

$$\begin{pmatrix} p & -q \\ -q & -p \end{pmatrix}, \ \begin{pmatrix} p & -q \\ q & p \end{pmatrix}, \ \begin{pmatrix} p & q \\ -q & p \end{pmatrix}, \ \begin{pmatrix} p & q \\ q & -p \end{pmatrix}, \tag{1.3}$$

where $p^2 + q^2 = 1,\ p, q \in \mathbb{R}$. Matrices (1.3) are unitary matrices, so they satisfy $|\det X| = 1$ and accordingly they are regular and $X^{-1} = X^T$. Since transposition is closed operation over the set (1.3) the same is valid to the inversion operation. Now, if $A^{\Delta} = \mathbf{I}$ then $A = A^{\perp}$ which is a contradiction. If A^{Δ} is non-orthogonal then from $A = A^{\perp}A^{\Delta}$ it follows $A^{\Delta} = A^{\perp}A$, which, in combination with (1.3), yields

$$(A^{\Delta})^T A^{\Delta} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}.$$

Thus, A^{Δ} is non-orthogonal, since if it is, it will be $a^2 + c^2 = 1,\ b^2 + d^2 = 1,\ ab + cd = 0$. The solution of this nonlinear system gives the following configurations for A:

$$\begin{pmatrix} -d & -\sqrt{1-d^2} \\ -\sqrt{1-d^2} & d \end{pmatrix}, \begin{pmatrix} d & \sqrt{1-d^2} \\ -\sqrt{1-d^2} & d \end{pmatrix},$$

$$\begin{pmatrix} -d & \sqrt{1-d^2} \\ \sqrt{1-d^2} & d \end{pmatrix}, \quad \begin{pmatrix} d & -\sqrt{1-d^2} \\ \sqrt{1-d^2} & d \end{pmatrix},$$

which is (up to the sign) identical with (1.3). So, A must be orthogonal which again is a contradiction. $\square$

Decomposition (1.2) can be refined by introducing some important elements of the group $(\mathcal{A}, \circ)$. So, an orthogonal transformation which generally is not unitary is *homogenous scaling* given by the matrix

$$A_s = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix},\ s \neq 0. \tag{1.4}$$

Elements of the orthonormal subgroup $(\mathcal{ON}, \circ)$ are translation, symmetries and rotation. Translation and symmetries (group of symmetries) are defined by the matrix set (1.3) for $q = 0$. The matrix of rotation obtains by setting $p = \cos\theta$ in the second matrix of (1.3)

$$\mathrm{A}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad 0 \leq \theta < 2\pi. \tag{1.5}$$

On the other hand, *non-homogenous scaling* (*stretch*), is a non-orthogonal mapping defined by the matrix

$$A_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix},\ t \neq 0, \pm 1. \tag{1.6}$$

Also, an important element of the affine group, *skew* (*shear*) transformation, given by the matrix

(1.7) $$A_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \ u \in \mathbb{R},$$

belongs to the unimodular group $(\mathcal{U}, \circ)$, and it is not orthonormal.

2. Refined Decomposition of the Group $(\mathcal{A}, \circ)$

Let $\boldsymbol{x} \mapsto A\boldsymbol{x}$ be linear part of transformation (1.1) and therefore a member of $(\mathcal{A}, \circ)$. Then, decomposition (1.2) has refinement according to the following theorem:

Theorem 2.1. *Matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *can be decomposed as*

(2.1) $$A = A_s A_\theta A_t A_u,$$

where matrices A_s, A_θ, A_t *and* A_u *are given by* (1.4), (1.5), (1.6) *and* (1.7) *respectively and decomposition parameters* s, θ, u *and* t *are given by*

(2.2) $$s = \operatorname{sgn} a \sqrt{a^2 + c^2}, \ \theta = \tan^{-1}\Big(\frac{c}{a}\Big), \ u = \frac{ab + cd}{a^2 + c^2}, \ t = \frac{\det A}{a^2 + c^2}.$$

Proof. Multiplying matrices from the right side of (2.1) one gets

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} s\cos\theta & su\cos\theta - st\sin\theta \\ s\sin\theta & st\cos\theta + su\sin\theta \end{pmatrix},$$

which gives the nonlinear system

(2.3) $$\begin{aligned} a &= s\cos\theta, \ c = s\sin\theta, \\ b &= s(u\cos\theta - t\sin\theta), \ d = s(t\cos\theta + u\sin\theta). \end{aligned}$$

From the first row of (2.3) one gets

(2.4) $$a/c = \tan\theta, \ s^2 = a^2 + c^2.$$

By substituting (2.4) in the second row of (2.3), the linear system on $(t,\, u)$ obtains: $-ct + au = b$ and $at + cu = d$ which gives t and u as in (2.2). Now, it yields from (2.4) $\theta = \tan^{-1}(c/a)$, $-\pi/2 < \theta < \pi/2$. Since s may change the sign, the product $A_s A_\theta = sA_\theta$ represents scaled rotation for $\theta \in [0, 2\pi]$

(sometimes called *spiral similarity*) so, $s = \operatorname{sgn} a\,\sqrt{a^2+c^2}$. In this manner, (2.2) obtains as the unique solution for decomposition parameters. □

Besides (2.1) there are other decompositions of A being made by permutations of factors A_θ, A_u and A_t. We found (2.1) is the simplest among them. In [2] decomposition $A = A_s A_t A_u A_\theta$ is mentioned, but it is slightly more complicated than (2.1).

Remark 2.1. Note that Theorem 1.1 can be obtained from Theorem 2.1 by setting $A^\perp = A_s A_\theta$ and $A^\Delta = A_t A_u$. Accordingly,

$$A^\perp = s\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ s \neq 0;\quad A^\Delta = \begin{pmatrix} 1 & u \\ 0 & t \end{pmatrix},\ (t,u) \neq (\pm 1, 0),$$

or

$$A^\perp = s\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},\ s \neq 0;\quad A^\Delta = \begin{pmatrix} 1 & u \\ 0 & t \end{pmatrix},\ (t,u) \neq (\pm 1, 0).$$

3. Contractions in $\mathbb{R}^2$

An important class of fractal sets in $\mathbb{R}^2$ can be roughly defined as invariant sets for collection of affine contractions (1.1) (see [3]). So, it is interesting to examine contractivity conditions for different components in decomposition (2.1). Let $\|\cdot\|$ denotes any vector norm. Then, norm of the matrix A defined by

$$\|A\| = \sup_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\| = \sup_{\boldsymbol{x}} \frac{\|A\boldsymbol{x}\|}{\|\boldsymbol{x}\|},$$

is called matrix norm *inducted* by vector norm $\|\cdot\|$ with properties ([5])

$$\|\lambda A\| = |\lambda|\|A\|,\ \lambda \in \mathbb{R},\quad \|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|,\quad \|A_1 A_2\| \le \|A_1\|\|A_2\|.$$

Let $\lambda_i(\mathrm{M})$ denotes i-th eigenvalue of the matrix M of the type $n \times n$. *Spectral norm*, defined by

$$\|A\| = \max_i \left\{ \sqrt{\lambda_i(AA^T)} \right\} \tag{3.1}$$

is induced by Euclidean vector norm $\|\boldsymbol{x}\| = (\boldsymbol{x}^T\boldsymbol{x})^{1/2}$. The spectral norm of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$\|A\|^2 = \frac{a^2+b^2+c^2+d^2+\sqrt{(a^2+b^2+c^2+d^2)^2 - 4\,(\det A)^2}}{2}. \tag{3.2}$$

Theorem 3.1. *Let* $A^{\Delta} = A_t A_u$*, where* A_t *and* A_u *are given by* (1.6) *and* (1.7)*. Then,*

$$\|\mathrm{A}^{\Delta}\| \geq 1. \tag{3.3}$$

Proof. Since, by (3.2)

$$\|\mathrm{A}^{\Delta}\| = \frac{1}{\sqrt{2}}\sqrt{1 + t^2 + u^2 + \sqrt{(1 - t^2 + u^2)^2 - 4t^2}}, \tag{3.4}$$

and

$$\left(1 + t^2 + u^2\right)^2 - 4t^2 = (1 + t^2 + u^2)^2 + 4t^2u^2 \geq (1 - t^2 + u^2)^2,$$

it follows that

$$1 + t^2 + u^2 + \sqrt{(1 + t^2 + u^2)^2 - 4t^2} \geq 1 + t^2 + u^2 + \sqrt{(1 - t^2 + u^2)^2} \geq 2.$$

□

Remark 3.2. Graph of the function $(t,\ u) \mapsto \|A^{\Delta}\|$ defined by (3.4), shown in Figure 1 (left) illustrates inequality (3.3).

Theorem 3.2. *Affine transformation* $\boldsymbol{x} \mapsto sA_{\theta}A^{\Delta}\boldsymbol{x} + \boldsymbol{b}$ *is a contraction if*

$$|s| < \frac{1}{\|A^{\Delta}\|}, \tag{3.5}$$

where $A^{\Delta} = \begin{pmatrix} 1 & u \\ 0 & t \end{pmatrix}$*,* $t \neq 0$*.*

Proof. Let $B^{\Delta} = A^{\Delta}/\|A^{\Delta}\|$. It follows from Theorem 2.1 that $\|B^{\Delta}\| = 1$ and (2.1) becomes $A = sA_{\theta}A^{\Delta} = s\|A^{\Delta}\|A_{\theta}B^{\Delta}$, whereupon

$$\|A\| \leq |s|\|A^{\Delta}\|\|A_{\theta}\|\|B^{\Delta}\| = |s|\|A^{\Delta}\|.$$

So, the condition $|s| < 1/\|A^{\Delta}\|$ is sufficient for $\boldsymbol{x} \mapsto sA_{\theta}A^{\Delta}\boldsymbol{x} + \boldsymbol{b}$ to be contraction. □

Note that on our notation, by using (3.4) inequality (3.5) becomes

$$s^2 < \frac{2}{1 + t^2 + u^2 + \sqrt{(1 + t^2 + u^2)^2 - 4t^2}}.$$

Here we come to the main result.

Theorem 3.3. (Normalized decomposition) *Any real matrix* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *can be decomposed as* $A = kA_\theta A_\Delta$ *where* A_θ *is rotation matrix* (1.5) *for angle* $\theta = \tan^{-1}(c/a)$ *and*

$$\begin{aligned} k &= \operatorname{sgn} a\,\sqrt{p(a^2+c^2)}, \\ p &= \tfrac{1}{2}\Big(q(a^2+c^2) + \sqrt{q^2(a^2+c^2)^2 - 4\,(\det A)^2}\,\Big), \\ q &= 1 + \frac{(ab+cd)^2}{(\det A)^2} + \frac{(\det A)^2}{(a^2+c^2)^2}, \\ A_\Delta &= \frac{\sqrt{p}}{p(a^2+c^2)}\begin{pmatrix} a^2+c^2 & (\det A)^2 \\ 0 & ab+cd \end{pmatrix}. \end{aligned} \tag{3.6}$$

Moreover, $\|A\| = |k| = \sqrt{p(a^2+c^2)}$.

Proof. By Theorem 2.1 decomposition (2.1) takes place, $A^\Delta = A_t A_u$ and decomposition parameters s, θ, u and t are given by (2.2). Then, in virtue of (3.4) one yields

$$\|A^\Delta\|^2 = \tfrac{1}{2}\Big(q(a^2+c^2) + \sqrt{q^2(a^2+c^2)^2 - 4\,(\det A)^2}\,\Big) = p,$$

and where

$$q = 1 + \frac{(ab+cd)^2}{(\det A)^2} + \frac{(\det A)^2}{(a^2+c^2)^2}.$$

Define normalized matrix $A_\Delta = A^\Delta/\|A^\Delta\| = \sqrt{p}\,A^\Delta/p$, which, after substitutions gets the form (3.6). Obviously, $\|A_\Delta\| = 1$. On the other hand, since the spectral norm (3.1) is unitary invariant (see [6]) it holds $\|A\| = |k|\|A_\theta A_\Delta\| = |k|\|A_\Delta\| = |k|$. □

Example 3.1. Let $A = \begin{pmatrix} 2.1 & 0.8 \\ -1.7 & 1.1 \end{pmatrix}$. Theorem 3.3 gives $k = 2.70308$ (numbers are rounded-off on six decimals) and

$$A_\theta = \begin{pmatrix} 0.777245 & 0.629198 \\ -0.629198 & 0.777245 \end{pmatrix}, \quad A_\Delta = \begin{pmatrix} 0.999547 & -0.0260156 \\ 0 & 0.502512 \end{pmatrix}.$$

It is not difficult to see that A_θ satisfies $A_\theta^T A_\theta = \mathbf{I}_\varepsilon$, where $\mathbf{I}_\varepsilon$ differs from the unit matrix for the amount that decreases as numerical precision increases. In other words A_θ is orthonormal in a given tolerance. The same is with $\|A_\Delta\| = 1$.

Figure 1 (right) illustrates this example as follows: Let Γ be a centered unit circle in $\mathbb{R}^2$. Linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$ maps Γ into an ellipse $A(\Gamma)$. On the other hand, the image of Γ under homothety $\boldsymbol{x} \mapsto k\mathbf{I}\boldsymbol{x}$ is the circle $k\Gamma$ (the biggest circle in Figure 1 (right)). Clearly, $\boldsymbol{x} \mapsto A_\theta\boldsymbol{x}$ maps Γ into itself, while $\boldsymbol{x} \mapsto A_\Delta\boldsymbol{x}$ maps Γ into ellipse with semi-axes being parallel to the coordinate axes.

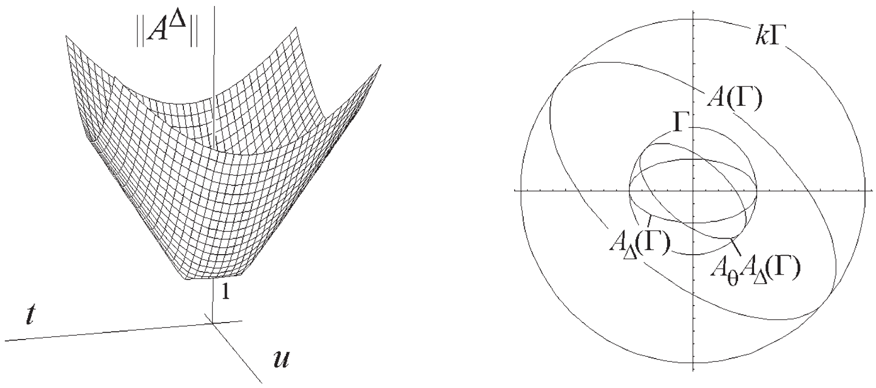

FIG. 1: Illustration of Theorem 3.1 (left) and Theorem 3.3 (right)

4. Applications on Fractal Interpolating Functions

Among many possible applications of Theorem 3.3 we choose issue of interpolating functions.

With the interpolation set of data $Y = \{(x_i, y_i)\}_{i=0}^n$, $(x_{i+1} > x_i)$, and the vector of vertical scaling factors $\mathbf{d} = (d_1, \ldots, d_n)$, one can associate the hyperbolic IFS (see [1]) $\sigma(Y, \mathbf{d}) = \{\mathbb{R}^2; w_1, \ldots, w_n\}$, where each w_i is an affine transformation $\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$w_i(\boldsymbol{x}) = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}, \ \boldsymbol{x} \in \mathbb{R}^2, \ i = 1, \ldots, n. \tag{4.1}$$

where

$$\begin{aligned} a_i &= \frac{\Delta x_{i-1}}{x_n - x_0}, \ c_i = \frac{\Delta y_{i-1}}{x_n - x_0} - d_i \frac{y_n - y_0}{x_n - x_0}, \\ e_i &= x_i - a_i x_n, \ f_i = y_i - c_i x_n - d_i y_n. \end{aligned} \tag{4.2}$$

Attractor of the IFS $\sigma(Y, \mathbf{d})$ is graph of a continuous function passing through data Y called *fractal interpolating function* ([1]). Theorem 3.3 can be applied on analyzing contractions (4.1). Immediate consequence of it is

Corollary 1. *Normalized decomposition of contractive mappings w_i defined by* (4.1) *and* (4.2) $(i = 1, \ldots, n)$ *is given by*

$$k_i = \frac{\sqrt{2}}{2} \alpha_i \beta_i, \ (A_\theta)_i = \frac{1}{\alpha_i} \begin{pmatrix} a_i & -c_i \\ c_i & a_i \end{pmatrix}, \ (A_\Delta)_i = \frac{\sqrt{2}}{\beta_i \alpha_i^2} \begin{pmatrix} \alpha_i^2 & c_i d_i \\ 0 & a_i d_i \end{pmatrix}$$

where

$$\begin{aligned} \alpha_i &= \sqrt{a_i^2 + c_i^2}, \\ \beta_i &= \sqrt{1 + \frac{d_i^2 + \sqrt{a_i^4 + 2a_i^2(c_i^2 - d_i^2) + (c_i^2 + d_i^2)^2}}{a_i^2 + c_i^2}}. \end{aligned}$$

Example 4.1. The interpolating data

$$\{(0,\ 0), (1/3,\ 1/2), (2/3,\ 1/2), (1,\ 1)\}$$

associated with vertical scaling factors $\mathbf{d} = (1/2,\ 0,\ 1/2)$ will give famous *devil staircase* function that is attractor of the IFS with three contractions each associated with one subinterval of the interpolating mesh. The corresponding IFS, given by (4.1) and (4.2) in this case will be

$$\begin{aligned} w_1(\boldsymbol{x}) &= \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \\ w_2(\boldsymbol{x}) &= \begin{pmatrix} 1/3 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix}; \\ w_3(\boldsymbol{x}) &= \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 2/3 \\ 1/2 \end{pmatrix}; \end{aligned} \tag{4.3}$$

By Corollary 1 and with accepting ordered triple notation $\{k_i, (A_\theta)_i, (A_\Delta)_i\}$ for our decomposition, one has

$$\begin{aligned} w_1,\ w_3 &: \left\{ \frac{1}{2}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2/3 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \\ w_2 &: \left\{ \frac{1}{3}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \end{aligned}$$

with translation vectors as in (4.3). The k-parameters show exact value of contractions: $1/2$ for w_1 and w_3, which is not clear from (4.3). Also, the ratio of scaling along x and y axis for w_1 and w_3 is evident from (4.3).

Example 4.2. The setting

$$Y = \{(0,\ 0), (1/2,\ 1/2), (1,\ 1)\}$$

and $\mathbf{d} = (1/2,\ 1/2)$ results in so called *Takagi function* (also *Knopp function* [5]) given by the IFS

$$w_1(\boldsymbol{x}) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad w_2(\boldsymbol{x}) = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

On the other hand, Corollary 1 decompositions give

$$
\begin{aligned}
w_1 &: \left\{\frac{1}{2}\sqrt{\frac{1}{2}(3+\sqrt{5})},\ \frac{\sqrt{2}}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},\ \frac{1}{\sqrt{3+\sqrt{5}}}\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}\right\},\\
w_2 &: \left\{\frac{1}{2}\sqrt{\frac{1}{2}(3+\sqrt{5})},\ \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix},\ \frac{1}{\sqrt{3+\sqrt{5}}}\begin{pmatrix}2 & -1\\ 0 & 1\end{pmatrix}\right\},
\end{aligned}
$$

where the translation vectors remains the same. In this case, contraction factor of both mappings is $\frac{1}{2}\sqrt{\frac{1}{2}(3+\sqrt{5})} \approx 0.809$ which can not be concluded from the IFS. Also, from normalized decomposition of w_1 and w_2, one reveals rotation component for $+\pi/4$ resp. $-\pi/4$ counter-clockwise, the transformation element which is not evident from the IFS. The matrices $(A_\Delta)_1$ and $(A_\Delta)_2$ gives information of stretch ratio, which is twice as big in x-direction, and that the normalized shear in w_1 is $+1$ and in w_2 is -1 which speaks about symmetry of the Takagi function graph.

5. Conclusion

Affine transformation w of $\mathbb{R}^2$ plane can be decomposed into two main sub-transformations, orthogonal and non-orthogonal. The orthogonal one can be further split into (homogeneous) scaling, defined by parameter s and orthonormal transform (parameter θ). The non-orthogonal component consists of non-homogeneous scaling (parameter t) and skew transform (parameter u). By normalization of non-orthogonal component one gets decomposition that comprises three components: Homogeneous scaling that "bears" contractivity of w, one orthonormal and one non-orthogonal component having unit norm. Decomposition parameters s, θ, t, u may help in analyzing Iterative Function Systems for constructing fractal sets, in the sense of dominance of one or two parameters over the others, revealing different symmetries in the IFS and other features. Two examples are given of this analysis, both involving fractal interpolating functions.

REFERENCES

1. M. F. Barnsley: *Fractals Everywhere.* Academic Press, 1988.
2. Y. Fisher (ed.): *Fractal Image Compression. Theory and Applications.* Springer, 1995.
3. J. E. Hutchinson: *Fractals and self similarity.* Indian J. Math. **30** (1981), 713–747.
4. Lj. Kocić: *AIFS – a tool for biomorphic fractal modeling.* Nonlinear Dynam., Psychology Life Sci. **5** (2000), no. 1, 45–63.

5. C. Tricot: *Curves and Fractal Dimension.* Springer-Verlag, 1995.

6. R. S. Varga: *Matrix Iterative Analysis.* Prentice Hall, 1962.

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