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ON MAXIMUM OF THE MODULUS OF KERNELS IN GAUSS-TURÁN QUADRATURES WITH CHEBYSHEV WEIGHTS: THE CASES S = 1,2*

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Abstract. We study the kernels $K_{n,s}(z)$ in the remainder terms $R_{n,s}(f)$ of Gauss-Turán quadrature formulae for analytic functions on elliptical contours with foci at ± 1, when the weight ω is Chebyshev weight function of the first and third kind. It is shown that the modulus of the kernel attains its maximum on the real axis $\forall n \geq n_0,\ n_0 = n_0(\rho, s)$ in the case $s = 1$. Analogous results can be performed in the case $s = 2$.

1. Introduction

We consider the Gauss-Turán quadrature formula with multiple nodes

$$\int_{-1}^{1} f(t)\omega(t)dt = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,s}(f) \ (n \in \mathbb{N};\ s \in \mathbb{N}_0), \tag{1.1}$$

where ω is nonnegative and integrable function on interval $(-1, 1)$, which is exact for all algebraic polynomials of degree at most $2(s + 1)n - 1$. The nodes τ_ν in (1.1) must be zeros of the s-orthogonal polynomials with respect to the weight function $\omega(t)$. The s-orthogonal polynomials $\pi_n = \pi_{n,s}$ with respect to the weight function $\omega(t)$ are polynomials which satisfy the following orthogonality conditions

$$\int_{-1}^{1} \pi_n(t)^{2s+1} t^k \omega(t)dt = 0\,, \qquad k = 0, 1, \ldots, n - 1.$$

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Numerically stable methods for constructing nodes τ_ν and coefficients $A_{i,\nu}$ can be found in [1, 4, 6]. For more details on quadrature formulae with multiple nodes see [2] and [3].

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and let D be its interior. If integrand f is analytic on D and continuous on $\overline{D}$, then the remainder term $R_{n,s}$ in (1.1) admits the contour integral representation (see, for instance, [5] and reference therein)

$$R_{n,s}(f) = \frac{1}{2\pi i} \oint_\Gamma K_{n,s}(z) f(z) dz. \tag{1.2}$$

The kernel is given by

$$K_{n,s}(z) = \frac{\rho_{n,s}(z)}{[\pi_{n,s}(z)]^{2s+1}}\,, \qquad z \notin [-1,1],$$

where

$$\rho_{n,s}(z) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t} \omega(t) dt.$$

The modulus of the kernel is symmetric with respect to real axis, i.e., $|K_{n,s}(\overline{z})| = |K_{n,s}(z)|$. If the weight function in (1.1) is even the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(-\overline{z})| = |K_{n,s}(z)|$ (see [5, Lemma 2.1.]).

The integral representation (1.2) leads to the error estimate

$$|R_{n,s}| \le \frac{l(\Gamma)}{2\pi} \left(\max_{z\in\Gamma} K_{n,s}(z) \right) \left(\max_{z\in\Gamma} (f(z) \right),$$

where $l(\Gamma)$ denotes the length of the contour Γ. First maximum depends only on the quadrature rule (i.e., on ω) and not on f.

2. The Maximum Modulus of the Kernel on Confocal Ellipses

In this section we take as contour Γ an ellipse $\mathcal{E}_\rho$ with foci at points ± 1 and a sum of semiaxes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} : \; z = \frac{1}{2} \left(\rho e^{i\theta} + \rho^{-1} e^{-i\theta} \right), \; 0 \le \theta \le 2\pi \right\}.$$

When $\rho \to 1$ the ellipse shrinks to the interval $[-1,1]$, while with increasing ρ it becomes more and more circle-like.

We study the magnitude of $|K_{n,s}(z)|$ on the contour $\mathcal{E}_\rho$ for the generalized Chebyshev weight functions of first and third kind, respectively, (cf. [5])

$$\omega_1(t) = (1-t^2)^{-1/2} \quad \text{and} \quad \omega_3(t) = \frac{(1+t)^{1/2+s}}{(1-t)^{1/2}}.$$

2.1. The weight function $\omega_1(t) = (1-t^2)^{-1/2}$. Explicit representation of the kernel $K_{n,s}^{(1)}(z)$ on the ellipse $\mathcal{E}_\rho$ for the weight function $\omega_1(t)$ was given by Milovanović and Spalević in [5], as well

$$\left|K_{n,s}^{(1)}(z)\right| = \frac{2^{1-s}\pi}{\rho^n} \frac{\left|Z_{n,s}^{(1)}(\rho e^{i\theta})\right|}{(a_2-\cos 2\theta)^{1/2}(a_{2n}+\cos 2n\theta)^{1/2+s}}\ ,\ z \in \mathcal{E}_\rho, \tag{2.1}$$

where

$$a_j = a_j(\rho) = \frac{1}{2}\left(\rho^j + \rho^{-j}\right), \quad j \in \mathbb{N}, \tag{2.2}$$

and

$$Z_{n,s}^{(1)}\left(\rho e^{i\theta}\right) = \sum_{k=0}^{s} \binom{2s+1}{s+k+1} \left(\rho e^{i\theta}\right)^{-2nk}. \tag{2.3}$$

The weight function $\omega_1(t)$ is even, so we can take $\theta \in [0, \pi/2]$.

Using the representation (2.1) Milovanović and Spalević stated the following conjecture:

Conjecture 2.1. *For each fixed $\rho > 1$ and $s \in \mathbb{N}_0$ there exists $n_0 = n_0(\rho, s)$ such that*

$$\max_{z\in\mathcal{E}_\rho} \left|K_{n,s}^{(1)}(z)\right| = K_{n,s}^{(1)}\left(\frac{1}{2}(\rho+\rho^{-1})\right)$$

for each $n \geq n_0$.

Theorem 2.1. *Conjecture* 2.1 *holds for $s = 1$.*

Proof. Because (2.1) it is sufficiently to prove

$$\begin{aligned} &(9+6\rho^{-2n}\cos 2n\theta + \rho^{-4n})(a_2-1)(a_{2n}+1)^3 \\ &\quad \leq (9+6\rho^{-2n}+\rho^{-4n})(a_2-\cos 2\theta)(a_{2n}+\cos 2n\theta)^3, \end{aligned} \tag{2.4}$$

for sufficiently large n ($n \geq n_0(\rho)$) and $\theta \in (0, \pi/2]$, where a_j are given by (2.2). Introducing half-angles, this is equivalent to

$$\begin{aligned} &[(3+\rho^{-2n})^2 - 12\rho^{-2n}\sin^2 n\theta](a_2-1)(a_{2n}+1)^3 \\ &\quad \leq (3+\rho^{-2n})^2[(a_2-1)+2\sin^2\theta][(a_{2n}+1)^3 - 6a_{2n}^2\sin^2 n\theta \\ &\qquad -12a_{2n}\sin^2 n\theta\cos^2 n\theta - 6\sin^2 n\theta + 12\sin^4 n\theta - 8\sin^6 n\theta]. \end{aligned}$$

Now, it is sufficiently to prove

$$
\begin{aligned}
(2.5)\qquad & (a_{2n}+1)^3 - \frac{\sin^2 n\theta}{\sin^2 n\theta}(a_2-1)(3a_{2n}^2+6a_{2n}\cos^2 n\theta \\
& +3-6\sin^2 n\theta+4\sin^4 n\theta)-2\sin^2 n\theta(3a_{2n}^2 \\
& +6a_{2n}\cos^2 n\theta+3-6\sin^2 n\theta+4\sin^4 n\theta)\ge 0,
\end{aligned}
$$

if $n \ge n_0(\rho)$ and $\theta \in (0,\pi/2]$. Since

$$\left|\frac{\sin n\theta}{\sin\theta}\right| \le n,\ \ (a_2-1)>0,$$

and

$$(\forall n\in\mathbb{N})\quad 3a_{2n}^2+6a_{2n}\cos^2 n\theta+3-6\sin^2 n\theta+4\sin^4 n\theta\ge 0,$$

the left-hand side of (2.5) is larger or equal to

$$(a_{2n}+1)^3-n^2(a_2-1)(3a_{2n}^2+6a_{2n}+7)-2(3a_{2n}^2+6a_{2n}+7):=F(n).$$

Using (2.2) we get

$$
\begin{aligned}
F(n)=\frac{1}{8}\big[&\rho^{6n}-(3An^2+6)\rho^{4n}-(12An^2+33)\rho^{2n} \\
&-(34An^2+116)-(12An^2+33)\rho^{-2n}-(3An^2+6)\rho^{-4n}+\rho^{-6n}\big],
\end{aligned}
$$

where $A=(a_2-1)=(\rho-\rho^{-1})^2=$ const. Since $F(n)$ is continuous on $\mathbb{R}$ and $\lim_{n\to+\infty}=+\infty$, it follows that $F(n)>0$, for all $n>t$, where t is the largest zero of $F(n)$. For n_0 we can take $[t]+1$. □

We can use the function $F(n)$ from the proof to estimate n_0. Numerical values of $[t]+1$ (t is the largest zero of F) for some values of ρ are presented in Table 1. The least possible values of n_0 are also presented. We can see that the least possible n_0 is estimated by $[t]+1$ very well.

Table 1

ρ	$[t]+1$	the l.p. n_0	ρ	$[t]+1$	the l.p. n_0
1.01	207	165	1.2	12	10
1.02	104	83	1.3	8	7
1.03	70	56	1.4	7	6
1.04	53	42	1.5	6	5
1.05	43	34	1.6	5	4
1.06	36	29	1.7	4	4
1.07	31	25	1.8	4	4
1.08	27	22	1.9	4	3
1.09	24	20	2	4	3
1.1	22	18	2.5	3	3

Analogous results can be derived in the case $s = 2$, in a similar way. But when s increases the derivation becomes drastically complex.

2.2. The weight function $\omega_3(t) = (1+t)^{1/2+s}(1-t)^{-1/2}$. Explicit representation of the kernel $K_{n,s}^{(3)}(z)$ on the ellipse $\mathcal{E}_\rho$ for the generalized Chebyshev weight function of third kind $\omega_3(t)$ was given by Milovanović and Spalević in [5], as well

$$\left|K_{n,s}^{(3)}(z)\right| = \frac{2^{1-s}\pi}{\rho^{n+1/2}} \frac{(a_1+\cos\theta)\left|Z_{n,s}^{(3)}(\rho e^{i\theta})\right|}{(a_2-\cos 2\theta)^{1/2}(a_{2n+1}+\cos{(2n+1)\theta})^{1/2+s}}, \tag{2.6}$$

where

$$Z_{n,s}^{(3)}\left(\rho e^{i\theta}\right) = \sum_{k=0}^{s} \binom{2s+1}{s+k+1} \left(\rho e^{i\theta}\right)^{-(2n+1)k}. \tag{2.7}$$

Using representation (2.6) in [5] was been stated the following conjecture:

Conjecture 2. For each fixed $\rho > 1$ and $s \in \mathbb{N}_0$ there exists $n_0 = n_0(\rho, s)$ such that

$$\max_{z\in\mathcal{E}_\rho} \left|K_{n,s}^{(3)}(z)\right| = K_{n,s}^{(3)}\left(\frac{1}{2}(\rho+\rho^{-1})\right)$$

for each $n \geq n_0$.

Theorem 2.2. *The conjecture* 2 *holds for* $s = 1$.

Proof. Because (2.6) it is sufficiently to prove

$$\begin{aligned}(9+6\rho^{-2n-1}&\cos{(2n+1)\theta}+\rho^{-4n-2})(a_2-1)(a_{2n+1}+1)^3\\ &\leq (9+6\rho^{-2n-1}+\rho^{-4n-2})(a_2-\cos 2\theta)(a_{2n+1}+\cos{(2n+1)\theta})^3,\end{aligned} \tag{2.8}$$

for enough large n ($n \geq n_0(\rho)$) and $\theta \in (0, \pi]$, where a_j are given by (2.2). Introducing the new variable k with $n = (2k-1)/2$ inequality (2.8) becomes inequality (2.4), which holds $\forall k,\ k > t$, where t is the largest zero of the function $F(k)$ from the proof of Theorem 2.1. Furthermore, we can conclude that inequality (2.8) holds for every n, such that $n > (2t-1)/2$. For n_0 we can take $[(2t-1)/2] + 1$. □

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