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SOME MODELS OF CAUSALITY AND STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION*

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Abstract. In this paper we consider some concepts of causality between filtrations and between stochastic processes. Then, we consider a generalization of a causality relationship "G is a cause of E within H" which was first given by Mykland [3] and which is based on Granger's definition of causality [1]. Than we apply this concept on weak solutions of stochastic differential equations driven by fractional Brownian motions.

1. Introduction

In the first part of this paper we give various concepts of causality relationship between flows of information (represented by filtrations). Especially, we consider connections between a generalized causality relationship "$\mathbf{G}$ is a cause of $\mathbf{E}$ within $\mathbf{H}$" which was given in [6] (which is based on Granger's definitions of causality) and some known relationships.

In the second part we give some preliminaries on fractional calculus. Then we consider some kinds of stochastic differential equations driven by fractional Brownian motions and existence of a weak solution to these equations.

In the third part the causality concepts are applied to different kinds of stochastic differential equations driven by fractional Brownian motion.

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2. Preliminary Notions and Definitions

We first give precise definitions of all terms used.

Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space and let $\mathbf{F} = (\mathcal{F}_\mathbf{t})$, $t \in \mathbb{R}$, be a family of σ-subalgebras of $\mathcal{F}$. $\mathcal{F}_t$ can be interpreted as the set of events observed up to time t. We define $\mathcal{F}_{<\infty}$ by $\mathcal{F}_{<\infty} = \bigvee_{t \in R} \mathcal{F}_t$.

A filtration $\mathbf{F} = (\mathcal{F}_\mathbf{t})$, $t \in \mathbb{R}$, is a nondecreasing family of σ-subalgebras of $\mathcal{F}$, i.e. such that

$$\mathcal{F}_s \subseteq \mathcal{F}_t, s \leq t.$$

Analogous notation will be used for filtrations $\mathbf{G} = (\mathcal{G}_\mathbf{t})$, $\mathbf{H} = (\mathcal{H}_\mathbf{t})$ and $\mathbf{J} = (\mathcal{J}_\mathbf{t})$, $t \in \mathbb{R}$.

Possibly the weakest form of causality can be introduced in the following way.

Definition 2.1. It is said that $\mathbf{H}$ is submitted to $\mathbf{G}$ or that $\mathbf{H}$ is a subfiltration of $\mathbf{G}$ (and written as $\mathbf{H} \subseteq \mathbf{G}$) if $\mathcal{H}_t \subseteq \mathcal{G}_t$ for each t.

It will be said that filtrations $\mathbf{H}$ and $\mathbf{G}$ are equivalent (and written as $\mathbf{H} = \mathbf{G}$) if $\mathbf{H} \subseteq \mathbf{G}$ and $\mathbf{G} \subseteq \mathbf{H}$.

A σ-algebra induced by stochastic process $X = (X_t), t \in T$, is given by $\mathbf{F}^X = (\mathcal{F}_t^X), t \in T$, where

$$\mathcal{F}_t^X = \sigma\{X_u, u \in T, u \leq t\},$$

being the smallest σ-algebra with respect to which the random variables $X_u, u \leq t$ are measurable. Process (X_t) is $(\mathcal{F}_t)$-adapted if $(\mathcal{F}_t^X) \subseteq (\mathcal{F}_t)$.

Definition 2.2. (compare with [8]) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ arbitraty σ-subalgebras from $\mathcal{F}$. It is said that $\mathcal{G}$ is splitting for $\mathcal{F}_1$ and $\mathcal{F}_2$ or that $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given $\mathcal{G}$ (and written as $\mathcal{F}_1 \perp \mathcal{F}_2 | \mathcal{G}$) if

$$(\forall A_1)\,(A_1 \in \mathcal{F}_1)(\forall A_2)\,(A_2 \in \mathcal{F}_2)\ \ P(A_1 A_2|\mathcal{G}) = P(A_1|\mathcal{G})P(A_2|\mathcal{G}).$$

The following result gives an alternative way of defining splitting.

Lemma 2.1. *(see [6]) $\mathcal{F}_1 \perp \mathcal{F}_2|\mathcal{G}$ if and only if $E(\mathcal{F}_i|\mathcal{F}_j \bigvee \mathcal{G}) \subseteq \mathcal{G}$, for $i, j = 1, 2$, $i \neq j$.*

Corollary 2.1. *$\mathcal{F}_1 \perp \mathcal{F}_2|\mathcal{G}$ if and only if $F_1' \perp \mathcal{F}_2'|\mathcal{G}$ for all $\mathcal{F}_i' \subseteq \mathcal{F}_i \bigvee \mathcal{G}$, $i = 1, 2$.*

The following result will be needed later.

Lemma 2.2. *(see [6])* $\mathcal{F}_1 \perp \mathcal{F}_2|\mathcal{G}$ *if and only if* $P(A_1|\mathcal{F}_2 \bigvee \mathcal{G}) = P(A_1|\mathcal{G})$ *for all* $A_1 \in \mathcal{F}_1$.

In [6] the intuitively plausible notion of causality is given. Let **G**, **H** and **J** be arbitrary filtrations. We can say that "**G** is a cause of **J** within **H**" if

$$\mathcal{J}_{<\infty} \perp \mathcal{H}_t|\mathcal{G}_t \tag{2.1}$$

because the essence of (2.1) is that all information about $\mathcal{J}_{<\infty}$ that gives $\mathcal{H}_t$ comes via $\mathcal{G}_t$ for arbitrary t; equivalently, $\mathcal{G}_t$ contains all the information from the $\mathcal{H}_t$ needed for predicting $\mathcal{J}_{<\infty}$. According to Corollary 2.1, (2.1) is equivalent to $\mathcal{J}_{<\infty} \perp \mathcal{H}_t \bigvee \mathcal{G}_t|\mathcal{G}_t$. The last relation means that condition $\mathbf{G} \subseteq \mathbf{H}$ does not represent essential restriction. Thus, it is natural to introduce the following definition of causality between filtrations.

Definition 2.4. ([6]) It is said that **G** is a cause of **J** within **H** relative to P (and written as $\mathbf{J} \mid\!< \mathbf{G}; \mathbf{H}; P$) if $\mathcal{J}_{<\infty} \subseteq \mathcal{H}_{<\infty}$, $\mathbf{G} \subseteq \mathbf{H}$ and if $\mathcal{J}_{<\infty}$ is conditionally independent of $\mathcal{H}_t$ given $\mathcal{G}_t$ for each t, i.e. $\mathcal{J}_{<\infty} \perp \mathcal{H}_t|\mathcal{G}_t$ for each t, (i.e. $\forall A \in \mathcal{J}_{<\infty}$, $P(A|\mathcal{H}_t) = P(A|\mathcal{G}_t)$).

If there is no doubt about P, we omit "relative to P".

Intuitively, $\mathbf{J} \mid\!< \mathbf{G}; \mathbf{H}$ means that, for arbitrary t, information about $\mathcal{J}_{<\infty}$ provided by $\mathcal{H}_t$ is not "bigger" than that provided by $\mathcal{G}_t$. The meaning of this interpretation will be specified in Lemma 2.4.

A definition, similar to Definition 2.4 was given in [3]; however, the definition from [3] contains also the condition $\mathbf{J} \subseteq \mathbf{H}$ (instead of $\mathcal{J}_{<\infty} \subseteq \mathcal{H}_{<\infty}$) which does not have intuitive justification. Since Definition 2.4 is more general than the definition given in [3], all results related to causality in the sense of Definition 2.4 will be true and in the sense of the definition from [3], when we add the condition $\mathbf{J} \subseteq \mathbf{H}$ to them.

If **G** and **H** are such that $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{H}$, we shall say that **G** is its own cause within **H** (compare with [3]). It should be mentioned that the notion of subordination (as introduced in [7]) is equivalent to the notion of being one's own cause, as defined here.

If **G** and **H** are such that $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{G} \bigvee \mathbf{H}$ (where $\mathbf{G} \bigvee \mathbf{H}$ is a family determined by $(\mathcal{G} \bigvee \mathcal{H})_t = \mathcal{G}_t \bigvee \mathcal{H}_t$), we shall say that **H** does not cause **G**. It is clear that the interpretation of Granger–causality is now that **H** does not cause **G** if $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{G} \bigvee \mathbf{H}$ (see [3]). Without difficulty, it can be shown that this term and the term '"**H** does not anticipate **G**" (as introduced in [8]) are identical.

We shall give some properties of causality relationship from Definition 2.4 which will be needed later.

Lemma 2.3. *From* $\mathcal{J}_{<\infty} \subseteq \mathcal{G}_{<\infty}$ *and* $\mathbf{G} \,|\!\!< \mathbf{G}; \mathbf{H}$ *it follows* $\mathbf{J} \,|\!\!< \mathbf{G}; \mathbf{H}$.

Lemma 2.4. (*compare with* [6]) $\mathbf{J} \,|\!\!< \mathbf{G}; \mathbf{H}$ *if and only if* $\mathcal{J}_{<\infty} \subseteq \mathcal{H}_{<\infty}$, $\mathbf{G} \subseteq \mathbf{H}$ *and* $E(\mathcal{J}_{<\infty}|\mathcal{H}_t) = E(\mathcal{J}_{<\infty}|\mathcal{G}_t)$ *for each* t.

Lemma 2.5. (*compare with* [6]) *From* $\mathbf{J} \,|\!\!< \mathbf{G}; \mathbf{H}$ *and* $\mathbf{J} \subseteq \mathbf{H}$ *it follows* $\mathbf{J} \subseteq \mathbf{G}$.

Lemma 2.6. *From* $\mathbf{G} \,|\!\!< \mathbf{G}; \mathbf{H}$ *and* $\mathbf{H} \,|\!\!< \mathbf{H}; \mathbf{J}$ *it follows* $\mathbf{G} \,|\!\!< \mathbf{G}; \mathbf{J}$.

Lemma 2.7. ([3]) *In the measurable space* $(\Omega, \mathcal{F})$ *let the filtrations* $\mathbf{H} = (\mathcal{H}_\mathbf{t})$, $\mathbf{G} = (\mathcal{G}_\mathbf{t})$ *and* $\mathbf{J} = (\mathcal{J}_\mathbf{t})$ *be given and let* P *and* $\tilde{P}$ *be probability measures on* $\mathcal{F}$ *satisfying* $\tilde{P} \ll P$ *with* $\frac{d\tilde{P}}{dP}$ *as* $\mathcal{H}_{<\infty}$*-measurable. Then*

$$\mathbf{J} \,|\!\!< \mathbf{G}; \mathbf{H}; P \quad \textit{implies} \quad \mathbf{J} \,|\!\!< \mathbf{G}; \mathbf{H}; \tilde{P}.$$

3. Stochastic Differential Equations Given by Fractional Brownian Motion

A probabilistic model for a time-dependent system is described by $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_t), t \in T$ is a "framework" filtration, i.e. $\mathcal{F}_t$ are all events in the model up to and including time t and $\mathcal{F}_t$ is a subset of $\mathcal{F}$.

For $f \in L^1([a, b])$ and $\alpha > 0$ the left fractional Riemann-Liouville integral of f of order α on (a, b) is given at almost all x by (see [5])

$$I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy,$$

where Γ denotes the Euler function.

Let

$$B^H = \{B^H_t, \, t \in [0, T]\}$$

be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. That is, B^H is a centered Gaussian process with covariance

$$R_H(t,s) = E(B_t^H B_s^H) = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}.$$

For each $t \in [0,T]$ we denote by $\mathcal{F}_t^{B^H}$ the σ-algebra generated by random variables B_s^H, $s \in [0,t]$ and the sets of probability zero. So, $\mathbf{F}^{B^H} = (\mathcal{F}_t^{B^H})$, $t \in [0,T]$ is the filtration of fractional Brownian motion

$$B^H = \{B_t^H,\ t \in [0,T]\}.$$

If $H = \frac{1}{2}$ the process B^H is standard Brownian motions.

Consider the stochastic differential equation (see [5])

$$X_t = x + B_t^H + \int_0^t b(s, X_s)ds,\ 0 \le t \le T \tag{3.1}$$

where b is Borel function on $[0,T] \times \mathbb{R}$.

By weak solutions of equation (3.1) we mean a couple of adapted continuous processes (B^H, X) on filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, such that:

(i) B^H is $\mathcal{F}_t$-fractional Brownian motion,

(ii) X and B^H satisfy equation (3.1).

The following result is a version of the Girsanov theorem for the fractional Brownian motion.

Theorem 3.1. ([5, Theorem 2]) *Consider the shifted process*

$$\tilde{B}_t^H = B_t^H + \int_0^t u_s ds$$

defined by the process $u = \{u_t,\ t \in [0,T]\}$ with integrable trajectories. Assume that

(i) $\int_0^{\cdot} u_s ds \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$, almost surely.

(ii) $E(\xi_T) = 1$, where

$$\xi_T = \exp\left(-\int_0^T \left(K_H^{-1}\int_0^{\cdot} u_s ds\right)(s)dW_s - \frac{1}{2}\int_0^T \left(K_H^{-1}\int_0^{\cdot} u_s ds\right)^2 (s)ds\right).$$

Than the shifted process $\tilde{B}^H$ is an $\mathcal{F}_t^{B^H}$-fractional Brownian motions with Hurst parameter H under new probability $\tilde{P}$ defined by $\frac{d\tilde{P}}{dP} = \xi_T$.

The following theorems give conditions for existence and uniqueness of a weak solution to equation (3.1).

Theorem 3.2. ([5, Theorem 3]) *Suppose that* $b(t,x)$ *satisfies the linear growth condition*

$$|b(t,x)| \leq C(1+|x|)$$

if $H < \frac{1}{2}$ *(singular case) or the Hölder continuity condition of order* $1 > \alpha > 1 - \frac{1}{2H}$ *in* x *and of order* $\gamma > H - \frac{1}{2}$ *in time*

$$|b(t,x) - b(s,y)| \leq C(|x-y|^{\alpha} + |t-s|^{\gamma})$$

if $H > \frac{1}{2}$ *(regular case). Than equation*

$$X_t = x + B_t^H + \int_0^t b(s, X_s)ds, \ 0 \leq t \leq T$$

has a week solution.

Theorem 3.3. ([5, Theorem 4]) *Suppose that* $b(t,x)$ *satisfies the assumptions of Theorem 3.2. Then two weak solutions must have the same distribution.*

Theorem 3.4. ([5, Theorem 5]) *Suppose that* $b(t,x)$ *satisfies the assumptions of Theorem 3.2. Then two weak solutions defined on the same filtered space must coincide almost surely.*

4. Main Results

Now we apply preeciding results on weak solution on stochastic differential equations driven by fractional Brownian motion.

When $H \notin \left\{\frac{1}{2}, 1\right\}$ fractional Brownian motion is neither a Markov process nor a semimartingale, but property that it is its own cause is inherited from standard Brownian motion. Precisely, it is easy to prove the following result that we need later.

Theorem 4.1. *Fractional Brownian motion* $B^H = (B_t^H)$ *on filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *is its own cause within* $\mathbf{F} = (\mathcal{F}_t), t \in T$*, relative to probability* P*.*

The proof follows by Theorem 3.4 from [4] in which was prowed that exists a martingale and, moreover, standard Brownian motion which generates the same filtration as fractional Brownian motion.

Theorem 4.2. *Weak solution of equation*

$$X_t = x + B_t^H + \int_0^t b(s, X_s)ds,\ 0 \leq t \leq T,$$

is its own cause within $\mathbf{F}^{B^H} = (\mathcal{F}_t^{B^H}), t \in T$ *relative to probability* $\tilde{P}$ *defined in Theorem 3.1.*

Proof. Under probability $\tilde{P}$ process

$$\tilde{B}_t^H = B_t^H - \int_0^t b(s, B_s^H + x)ds$$

is an $\mathcal{F}_t^{B^H}$-fractional Brownian motion (Girsanov transform), and couple $(B^H, \tilde{B}^H)$ is a weak solution of given equation on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t^{B^H}, \tilde{P})$. According to Theorem 3.1 we have

$$\mathbf{F}^{\tilde{B}^H} \mid\!< \mathbf{F}^{\tilde{B}^H}; \mathbf{F}^{B^H}; \tilde{P}.$$

The proof is completed. $\square$

Theorem 4.3. *Weak solution of equation*

$$X_t = x + B_t^H + \int_0^t b(s, X_s)ds,\ 0 \leq t \leq T,$$

is its own cause within $\mathbf{F} = (\mathcal{F}_t), t \in T$ *relative to probability* P.

Proof. First we have $\tilde{P} \sim P$, because of $\frac{d\tilde{P}}{dP} = \xi_T$ and $E\xi_T = 1$ [2, Theorem 7.1]. According to Theorem 4.2 and Lemma 2.7, it follows

$$\mathbf{F}^{\tilde{B}^H} \mid\!< \mathbf{F}^{\tilde{B}^H}; \mathbf{F}^{B^H}; P.$$

Now, since $\mathbf{F}^{\tilde{B}^H} \subseteq \mathbf{F}^{B^H}$ and $\mathbf{F}^{B^H} \mid\!< \mathbf{F}^{B^H}; \mathbf{F}; P$ (it follows from Theorem 4.2), from Lemma 1.6 it follows

$$\mathbf{F}^{\tilde{B}^H} \mid\!< \mathbf{F}^{\tilde{B}^H}; \mathbf{F}; P.$$

The proof is completed. $\square$

Consider now, in the case $H > \frac{1}{2}$, stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_s)dB_s^H + \int_0^t b(s, X_s)ds, \ \ 0 \le t \le T,$$

where σ is a Hölder continuous function of order $\delta > \frac{1}{H} - 1$ such that $|\sigma(z)| \ge c > 0$.

Now, we can give following theorem which consider weak solution of this type equations.

Theorem 4.4. *Weak solution of equation*

$$X_t = x + \int_0^t \sigma(X_s)dB_s^H + \int_0^t b(s, X_s)ds, \ \ 0 \le t \le T, \tag{4.1}$$

is its own cause within $\mathbf{F} = (\mathcal{F}_t), t \in T$ *relative to probability* P.

Proof. Stochastic integral that appears in (4.1) exists pathwise. We refer to [9] for definition of this pathwise integral using fractional calculus.
Set

$$F(x) = \int_0^x \frac{1}{\sigma(z)}dz.$$

Since

$$F'(x) = \frac{1}{\sigma(x)}$$

and

$$\left| \frac{1}{\sigma(x)} - \frac{1}{\sigma(y)} \right| \le \frac{1}{c^2} \left| \sigma(y) - \sigma(x) \right| \le \frac{1}{c^2} \left| y - x \right|^\delta$$

we can use the change-of-variables formula for the fractional Brownian motions [9, Theorem 4.3.1] and obtain

$$Y_t = F(x) + B_t^H + \int_0^t \frac{b(s, F^{-1}(Y_s))}{\sigma(F^{-1}(Y_s))}ds. \tag{4.2}$$

Now, process X is a solution of equation (4.1) if and only if the process $Y_t = F(X_t)$ is solution of equation (4.2). First, we prove existence of a weak solution to equation (4.2). Since for the function $\dfrac{b(s, F^{-1}(Y_s))}{\sigma(F^{-1}(Y_s))}$ holds:

$$\left| \frac{b(s, F^{-1}(Y_s))}{\sigma(F^{-1}(Y_s)} - \frac{b(t, F^{-1}(Y_t))}{\sigma(F^{-1}(Y_t)} \right| \le \frac{1}{c} \left| b(s, F^{-1}(Y_s)) - b(t, F^{-1}(Y_t)) \right|$$

$$\leq \frac{1}{c} \cdot K(|s-t|^{\alpha} + \left|F^{-1}(Y_s) - F^{-1}(Y_t)\right|^{\gamma}),$$

from Theorem 3.2 it follows that weak solution $Y = (Y_t)$, $t \in [0,T]$ of equation (4.2) exists. Moreover, from Theorem 4.3 it follows that process Y is its own cause within $(\mathcal{F}_t)$ relative to probability P, that is,

$$\mathbf{F}^Y \,|\!\!< \mathbf{F}^Y; \mathbf{F}; P.$$

Also, because σ is a Hölder continuous function of order $\delta > \frac{1}{H} - 1$ such that $|\sigma(z)| \geq c > 0$ we have

$$\left|F'(x)\right| = \left|\frac{1}{\sigma(x)}\right| \leq \frac{1}{c} \quad \text{and} \quad F'(x) \neq 0,$$

so it follows that function $F(x)$ is increasing or decreasing. If $F(x)$ is increasing we have

$$(\forall c \in \mathbb{R}) \{\omega : X_t \geq c\} = \{\omega : F(X_t) \geq F(c)\} = \{\omega : Y_t \geq F(c)\},$$

that is,

$$\mathbf{F}^X \subseteq \mathbf{F}^Y$$

and, also,

$$(\forall c \in \mathbb{R}) \{\omega : Y_t \geq c\} = \{\omega : F(X_t) \geq c\} = \left\{\omega : X_t \geq F^{-1}(c)\right\},$$

that is,

$$\mathbf{F}^Y \subseteq \mathbf{F}^X.$$

So, we proved that processes X and Y generate the same filtration. Now from $\mathbf{F}^Y \,|\!\!< \mathbf{F}^Y; \mathbf{F}; P$ it follows that process X is its own cause within $\mathbf{F} = (\mathcal{F}_t), t \in T$ relative to probability P, i.e.

$$\mathbf{F}^X \,|\!\!< \mathbf{F}^X; \mathbf{F}; P.$$

For the case when $F(x)$ is decreasing proof is similar. $\square$

Remark 4.1. According to Theorem 3.3 and Theorem 3.4, all weak solutions considered in Theorem 4.2, Theorem 4.3 and Theorem 4.4 are unique.

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