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SOME MODELS OF CAUSALITY AND WEAK SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH DRIVING SEMIMARTINGALES*

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Abstract. In this paper we consider the stochastic differential equation

$$dX_t = u_t(X)\, dZ_t, \quad X_0 = x,$$

where Z_t is semimartingale and $u_t(X)$ is a predictable functional. We show an equivalence between some models of causality, introduced by Mykland [8] and Petrović [10], and weak uniqueness (for weak solutions of stochastic differential equations with driving semimartingales).

1. Introduction

In the first part of this paper we give various concepts of causality relationship between flow of information (represented by filtrations). Especially, we give a generalization of a causality relationship "**G** is a cause of **J** within **H**" which was first given in [8] and which is based on Granger's definitions of causality (see [2]).

In the second part we give some preliminaries on martingales and consider some kinds of stochastic differential equations and existence of a weak solution to these equations.

In the third part the causality concept is applied to regular solutions of stochastic differential equations with driving semimartingales. Also, the

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equivalence between some models of causality and weakly uniqueness of regular solutions is shown.

2. Some Concepts of Causality

Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space and let

$$\mathbf{F} = \{\mathcal{F}_{\mathbf{t}},\ \mathbf{t} \in \mathbf{T}(\subseteq \mathbf{R})\}$$

be a family of sub-σ-algebras of $\mathcal{F}$. $\mathcal{F}_t$ can be interpreted as the set of events observed up to time t. $\mathcal{F}_{<\infty}$ is the smallest σ-algebra containing all the $\mathcal{F}_t$ (even if $\sup T < +\infty$). So that, we have $\mathcal{F}_{<\infty} = \bigvee_{t \in T} \mathcal{F}_t$.

A filtration $\mathbf{F} = \{\mathcal{F}_{\mathbf{t}},\ \mathbf{t} \in \mathbf{T}\}$ is a nondecreasing family of sub-σ-algebras of $\mathcal{F}$, i.e.such that

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \quad s \leq t.$$

A probabilistic model for a time-dependent system is described by $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space and $\{\mathcal{F}_t,\ t \in T\}$ is a "framework" filtration, i.e.$\mathcal{F}_t$ are all events in the model up to and including time t and $\mathcal{F}_t$ is a subset of $\mathcal{F}$. We suppose that the filtration $\mathcal{F}_t$ satisfy the "usual conditions", which means that $\mathcal{F}_t$ is right continuous and each $\mathcal{F}_t$ is complete.

Analogous notation will be used for filtrations $\mathbf{H} = \{\mathcal{H}_{\mathbf{t}}\}$, $\mathbf{G} = \{\mathcal{G}_{\mathbf{t}}\}$ and $\mathbf{J} = \{\mathcal{J}_{\mathbf{t}}\}$.

Possibly the weakest form of causality can be introduced in the following way.

Definition 2.1. It is said that $\mathbf{H}$ is submitted to $\mathbf{G}$ or that $\mathbf{H}$ is a subfiltration of $\mathbf{G}$ (and written as $\mathbf{H} \subseteq \mathbf{G}$) if $\mathcal{H}_t \subseteq \mathcal{G}_t$ for each t.

It will be said that filtrations $\mathbf{H}$ and $\mathbf{G}$ are equivalent (and written as $\mathbf{H} = \mathbf{G}$) if $\mathbf{H} \subseteq \mathbf{G}$ and $\mathbf{G} \subseteq \mathbf{H}$.

A σ-algebra induced by stochastic process $\mathbf{X} = \{X_t,\ t \in T\}$ is given by $\mathbf{F}^X = \{\mathcal{F}_t^X,\ t \in T\}$, where

$$\mathcal{F}_t^X = \sigma\{X_u,\ u \in T,\ u \leq t\},$$

being the smallest σ-algebra with respect to which the random variables $X_u, u \leq t$ are measurable. The process $\{X_t\}$ is $(\mathcal{F}_t)$-adapted if $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for each t.

On the probability space $(\Omega, \mathcal{F}, P)$ the process $\mathbf{Z} = \{Z_t, t \in T\}$ is a $(\mathcal{F}_t, P)$-martingale if $\{Z_t\}$ is $(\mathcal{F}_t)$-adapted and $Z_s = E(Z_t|\mathcal{F}_s)$ for all $s \geq t$.

Definition 2.2. (compare with [12]) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ arbitrary sub-σ-algebras from $\mathcal{F}$. It is said that $\mathcal{G}$ is splitting for $\mathcal{F}_1$ and $\mathcal{F}_2$ or that $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given $\mathcal{G}$ (and written as $\mathcal{F}_1 \perp \mathcal{F}_2|\mathcal{G}$) if

$$(\forall A_1)\,(A_1 \in \mathcal{F}_1)(\forall A_2)\,(A_2 \in \mathcal{F}_2)\;\; P(A_1A_2|\mathcal{G}) = P(A_1|\mathcal{G})P(A_2|\mathcal{G}).$$

The following results gives an alternative way of defining splitting.

Lemma 2.1. *(see [10]) $\mathcal{F}_1 \perp \mathcal{F}_2|\mathcal{G}$ if and only if $P(\mathcal{F}_i|\mathcal{F}_j \bigvee \mathcal{G}) \subseteq \mathcal{G}$, for $i, j = 1, 2$, $i \neq j$.*

Corollary 2.1. ([10]) *$\mathcal{F}_1 \perp \mathcal{F}_2|\mathcal{G}$ if and only if $\mathcal{F}'_1 \perp \mathcal{F}'_2|\mathcal{G}$ for all $\mathcal{F}'_i \subseteq \mathcal{F}_i \bigvee \mathcal{G}$, $i = 1, 2$.*

The intuitively plausible notion of causality is given in [10]. Let $\mathbf{J}$, $\mathbf{G}$ and $\mathbf{H}$ be arbitrary filtrations. We can say that "$\mathbf{G}$ is a cause of $\mathbf{J}$ within $\mathbf{H}$" if

$$\mathcal{J}_{<\infty} \perp \mathcal{H}_t|\mathcal{G}_t \tag{2.1}$$

because the essence of (2.1) is that all information about $\mathcal{J}_{<\infty}$ that gives $\mathcal{H}_t$ comes via $\mathcal{G}_t$ for arbitrary t; equivalently, $\mathcal{G}_t$ contains all information from the $\mathcal{H}_t$ needed for predicting $\mathcal{J}_{<\infty}$. According to Corollary 2.1, (2.1) is equivalent to $\mathcal{J}_{<\infty} \perp \mathcal{H}_t \bigvee \mathcal{G}_t|\mathcal{G}_t$. The last relation means that condition $\mathbf{G} \subseteq \mathbf{H}$ does not represent essential restriction. Thus, it is natural to introduce the following definition of causality between families of Hilbert spaces.

Definition 2.3. ([10]) It is said that $\mathbf{G}$ is a cause of $\mathbf{J}$ within $\mathbf{H}$ relative to P (and written as $\mathbf{J} \,|\!< \mathbf{G}; \mathbf{H}$;P) if $\mathcal{J}_{<\infty} \subseteq \mathcal{H}_{<\infty}$, $\mathbf{G} \subseteq \mathbf{H}$ and if $\mathcal{J}_{<\infty}$ is conditionally independent of $\mathcal{H}_t$ given $\mathcal{G}_t$ for each t, i.e.$\mathcal{J}_{<\infty} \perp \mathcal{H}_t|\mathcal{G}_t$ for each t, (i.e.$(\forall A \in \mathcal{J}_{<\infty})\, P(A|\mathcal{H}_t) = P(A|\mathcal{G}_t)$).

If there is no doubt about P, we omit "relative to P".

Intuitively, $\mathbf{J} \,|\!< \mathbf{G}; \mathbf{H}$ means that, for arbitrary t, information about $\mathcal{J}_{<\infty}$ provided by $\mathcal{H}_t$ is not "bigger" than that provided by $\mathcal{G}_t$. The meaning of this interpretation will be specified in Lemma 2.2.

A definition, analogous to Definition 2.3 was first given in [8]; however, the definition from [8] contains also the condition $\mathbf{J} \subseteq \mathbf{H}$ (instead of $\mathcal{J}_{<\infty} \subseteq \mathcal{H}_{<\infty}$) which does not have intuitive justification. Since Definition 2.3 is more general than the one given in [8], all results related to causality in the sense of Definition 2.3 will be true and in the sense of the Hilbert space version of the definition from [8], when we add the condition $\mathbf{J} \subseteq \mathbf{H}$ to them.

If $\mathbf{G}$ and $\mathbf{H}$ are such that $\mathbf{G} \mid< \mathbf{G}; \mathbf{H}$, we shall say that $\mathbf{G}$ is its own cause within $\mathbf{H}$ (compare with [8]). It should be mentioned that the notion of subordination (as introduced in [11]) is equivalent to the notion of being one's own cause, as defined here.

If $\mathbf{G}$ and $\mathbf{H}$ are such that $\mathbf{G} \mid< \mathbf{G}; \mathbf{G} \bigvee \mathbf{H}$ (where $\mathbf{G} \bigvee \mathbf{H}$ is a family determined by $(\mathcal{G} \bigvee \mathcal{H})_t = \mathcal{G}_t \bigvee \mathcal{H}_t$), we shall say that $\mathbf{H}$ does not cause $\mathbf{G}$. It is clear that the interpretation of Granger–causality is now that $\mathbf{H}$ does not cause $\mathbf{G}$ if $\mathbf{G} \mid< \mathbf{G}; \mathbf{G} \bigvee \mathbf{H}$ (see [8]). Without difficulty, it can be shown that this term and the term "$\mathbf{H}$ does not anticipate $\mathbf{G}$" (as introduced in [12]) are identical.

We shall give some properties of causality relationship from Definition 2.3 which will be needed later.

Lemma 2.2. (*compare with* [10]) $\mathbf{J} \mid< \mathbf{G}; \mathbf{H}$ *if and only if* $\mathcal{J}_{<\infty} \subseteq \mathcal{H}_{<\infty}$, $\mathbf{G} \subseteq \mathbf{H}$ *and* $P(\mathcal{J}_{<\infty} | \mathcal{H}_t) = P(\mathcal{J}_{<\infty} | \mathcal{G}_t)$ *for each* t.

Lemma 2.3. ([8]) *In the measurable space* $(\Omega, \mathcal{F})$ *let the filtrations* $\mathbf{H} = \{\mathcal{H}_\mathbf{t}\}$, $\mathbf{G} = \{\mathcal{G}_\mathbf{t}\}$ *and* $\mathbf{J} = \{\mathcal{J}_\mathbf{t}\}$ *be given and let* P *and* Q *be probability measures on* $\mathcal{F}$ *satisfying* $Q \ll P$ *with* $\dfrac{dQ}{dP}$ *as* $\mathcal{H}_{<\infty}$*-measurable. Then*

$$\mathbf{J} \mid< \mathbf{G}; \mathbf{H}; P \quad \textit{implies} \quad \mathbf{J} \mid< \mathbf{G}; \mathbf{H}; Q.$$

3. Stochastic Differential Equations With Driving Semimartingales

We consider the following stochastic differential equation

$$\left.\begin{aligned} dX_t &= u_t(X)\, dZ_t \\ X_0 &= x, \end{aligned}\right\} \tag{3.1}$$

where the driving process $\mathbf{Z} = \{Z_t,\, t \in [0, +\infty)\}$ is m-dimensional semimartingale ($Z_0 = 0$) and the coefficient $u_t(X)$ is $n \times m$-dimensional predictable functional (in the sense of [7]).

Jacod and Memin (in [3] and [4]) have studied the existence and uniqueness of solutions of the equation (3.1) by introducing extensions of the given probability space. On that new space they proved the existence of probability measure for which there exist a solution-process X. Such a measure they called the solution measure, but it is also known as a weak solution.

Lebedev (in [6]) generalized the main results from [3] and [4] and proved the existence of weak solution (in the strict sense) for the equation which is more general than equation (3.1), involving random measures. He introduced a slightly different notion of weak solution and established conditions for regularity.

Another approach of the same matter was introduced by Mykland in [9]. Actually, he found conditions in a term of causality for Z_t, under which the equation (3.1) would have a regular weak solution.

For the stochastic differential equation (3.1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X, Z_t)$ is a regular weak solution if:

1. $\mu(A) = P(\mathbf{Z} \in A)$ coincides with a predetermined measure on the function space on which $\mathbf{Z}$ takes values,

2. X_t and Z_t satisfy (3.1),

3. Z_t is its own cause within $\mathbf{F} = \{\mathcal{F}_t\}$ relative to P.

The solution is regular in the sense of [9].

The regular solution is weakly unique [9] if for every regular solution $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$ of the equation (3.1) there is no measure Q on $\mathcal{F}_t^{X,Z}$ so that $(\Omega, \mathcal{F}_{<\infty}^{X,Z}, \mathcal{F}_t^{X,Z}, Q, X_t, Z_t)$ is a regular solution of (3.1).

The object of this paper is to give some conditions for the weak uniqueness (in a sense of [9]) of solution of the equation (3.1).

4. Main Results

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $t \in T = [0, +\infty)$ be a filtered probability space with $\{\mathcal{F}_t\}$ right continuous and complete.

Let H be a set of right continuous modifications of the $(\mathcal{F}_t, P)$-martingales $P(A|\mathcal{F}_t^Z)$

$$H = \{M_t, M_t = P(A|\mathcal{F}_t^Z), A \in \mathcal{F}_{<\infty}^Z\}. \tag{4.1}$$

The following result will be an intermediate step in the proof of next theorem.

Theorem 4.1. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, be a filtered probability system with filtration $\mathbf{F} = \{\mathcal{F}_t\}$. Let $\mathbf{F}^Z = \{\mathcal{F}_t^Z\}$ be a subfiltration of $\{\mathcal{F}_t\}$ (i.e., $\mathcal{F}_t^Z \subseteq \mathcal{F}_t$ for* *each t). Right continuous modifications of the set H are $(\mathcal{F}_t)$-martingales if and only if Z_t is its own cause within $\mathbf{F} = \{\mathcal{F}_t\}$ relative to P, i.e.,*

$$\mathbf{F}^Z \models \mathbf{F}^Z; \mathbf{F}; P$$

holds.

Proof. Let H be of the form (4.1). Then,

$$\forall M_t \in H \qquad E(M_\infty | \mathcal{F}_t) = M_t, t \in T$$

and

$$\forall A \in \mathcal{F}^Z_{<\infty} \qquad E(P(A|\mathcal{F}^Z_{<\infty})|F_t) = M_t, \text{ for all } \; M_t \in H, t \in T.$$

If I_A denotes indicator function of $A \in \mathcal{F}^Z_{<\infty}$, it is $\mathcal{F}^Z_{<\infty}$-measurable, so that we have

$$\forall A \in \mathcal{F}^Z_{<\infty} \qquad P(A|\mathcal{F}_t) = P(A|\mathcal{F}^Z{}_t).$$

Because of $\mathcal{F}^Z_t \subseteq \mathcal{F}_t$, the last relation means that Z_t is its own cause within $\mathbf{F} = \{\mathcal{F}_t\}$ relative to P, i.e.

$$\mathbf{F}^Z \models \mathbf{F}^Z; \mathbf{F}; P$$

holds.

Conversely, let $\mathbf{F}^Z \models \mathbf{F}^Z; \mathbf{F}; P$ holds. We need to prove that

$$M_t = P(A|\mathcal{F}^Z_t), \; A \in \mathcal{F}^Z_{<\infty}$$

is $(\mathcal{F}_t)$-martingale.

According to Lemma 2.2, from $\mathbf{F}^Z \models \mathbf{F}^Z; \mathbf{F}; P$ it follows that

$$\forall A \in \mathcal{F}^Z_{<\infty}, \, P(A|\mathcal{F}_t) = P(A|\mathcal{F}^Z_t).$$

If I_A is the indicator of A, it is bounded and $\mathcal{F}^Z_{<\infty}$-measurable, so that

$$E(E(I_A|\mathcal{F}^Z_{<\infty})|\mathcal{F}_t) = E(I_A|\mathcal{F}^Z_t). \tag{4.2}$$

Since $P(A|\mathcal{F}^Z_{<\infty}) = M_\infty$ we have

$$E(M_\infty|\mathcal{F}_t) = E(I_A|\mathcal{F}^Z_t). \tag{4.3}$$

Now from (4.2) and (4.3) it follows that

$$\forall A \in H_{<\infty} \; M_t = E(M_\infty|\mathcal{F}_t), \, t \in T.$$

So that we proved that the elements of H are $(\mathcal{F}_t)$-martingales. $\square$

The following theorem gives the conditions under which solution of equation (3.1) is weakly unique.

Theorem 4.2. *Every weak solution of stochastic differential equation* (3.1) *is weakly unique if and only if* (X_t, Z_t) *is its own cause within* $\mathbf{F} = \{\mathcal{F}_t\}$ *relative to* P.

Proof. Suppose that a weakly unique solution of equation (3.1) exists. Then, if $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$ and $(\Omega, \mathcal{F}^{X,Z}_{<\infty}, \mathcal{F}^{X,Z}_t, P, X_t, Z_t)$ are two weak solutions, we have

$$\mathbf{F}^Z \ltimes \mathbf{F}^Z; \mathbf{F}; P \quad \text{and} \quad \mathbf{F}^Z \ltimes \mathbf{F}^Z; \mathbf{F}^{X,Z}; Q. \tag{4.4}$$

Since the solutions of (3.1) are weakly unique, we have $P = Q$ on $\mathcal{F}^{X,Z}_{<\infty}$. So that from (4.4) we have

$$\mathbf{F}^Z \ltimes \mathbf{F}^Z; \mathbf{F}^{X,Z}; P. \tag{4.5}$$

If H is of the form (4.1), M_t are right continuous and according to (4.4) and (4.5) they are $(\mathcal{F}^{X,Z}_t, P)$-martingales and $(\mathcal{F}_t, P)$-martingales, so that

$$\forall M_t \in H \qquad E(M_\infty \mid \mathcal{F}^{X,Z}_t) = M_t,\ t \in T,$$

$$\forall A \in \mathcal{F}^Z_{<\infty} \qquad E(P(A \mid \mathcal{F}^Z_{<\infty}) \mid \mathcal{F}^{X,Z}_t) = M_t,\ t \in T.$$

For $\forall A \in \mathcal{F}^Z_{<\infty}$, I_A is the indicator of A, it is bounded and $\mathcal{F}^Z_{<\infty}$-measurable, and

$$E(E(I_A \mid \mathcal{F}^Z_{<\infty}) \mid \mathcal{F}^{X,Z}_t) = P(A \mid \mathcal{F}^Z_t),\ t \in T,$$

$$E(I_A \mid \mathcal{F}^{X,Z}_t) = P(A \mid \mathcal{F}^Z_t),\ t \in T,$$

so that

$$\forall A \in \mathcal{F}^Z_{<\infty} \qquad P(A \mid \mathcal{F}^{X,Z}_t) = P(A \mid \mathcal{F}^Z_t).$$

So that, the elements of H can be represented in form

$$H = \{M_t;\ M_t = P(A \mid \mathcal{F}^{X,Z}_t),\ A \in \mathcal{F}^{X,Z}_{<\infty}\}.$$

According to Theorem 4.1, $\mathbf{F}^Z \ltimes \mathbf{F}^Z; \mathbf{F}; P$ means that the elements of H are $(\mathcal{F}_t)$-martingales, so that

$$\forall M_t \in H \qquad E(M_\infty \mid \mathcal{F}_t) = M_t,\ t \in T,$$

$$\forall A \in \mathcal{F}^{X,Z}_{<\infty} \qquad E(P(A \mid \mathcal{F}^{X,Z}_{<\infty}) \mid \mathcal{F}_t) = P(A \mid \mathcal{F}^{X,Z}_t),\ t \in T.$$

Following the proof as in previous case, we get

$$\forall A \in \mathcal{F}^{X,Z}_{<\infty} \qquad P(A \mid \mathcal{F}_t) = P(A \mid \mathcal{F}^{X,Z}_t).$$

Now, since $\mathcal{F}_t^{X,Z} \subseteq \mathcal{F}_t,\ t \in T$ we get

$$\mathbf{F}^{X,Z} \Join \mathbf{F}^{X,Z}; \mathbf{F}; P.$$

Conversely, let

$$(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$$

be a weak solution of the stochastic differential equation (3.1), than we have

$$\mathbf{F}^{Z} \Join \mathbf{F}^{Z}; \mathbf{F}; P$$

so that

$$\forall A \in \mathcal{F}_{<\infty}^{Z} \qquad P(A \mid \mathcal{F}_t^{Z}) = P(A \mid \mathcal{F}_t).$$

On the other hand, by the assumption of the theorem, (X_t, Z_t) is its own cause within $\mathbf{F} = \{\mathcal{F}_t\}$ relative to P, i.e.$\mathbf{F}^{X,Z} \Join \mathbf{F}^{X,Z}; \mathbf{F}; P$ so that

$$\forall A \in \mathcal{F}_{<\infty}^{X,Z} \qquad P(A \mid \mathcal{F}_t^{X,Z}) = P(A \mid \mathcal{F}_t).$$

Now, it follows that

$$\forall A \in \mathcal{F}_{<\infty}^{Z} \qquad P(A \mid \mathcal{F}_t^{Z}) = P(A \mid \mathcal{F}_t^{X,Z})$$

or

$$\mathbf{F}^{Z} \Join \mathbf{F}^{Z}; \mathbf{F}^{X,Z}; P. \tag{4.6}$$

If, also $(\Omega, \mathcal{F}_{<\infty}^{X,Z}, \mathcal{F}_t^{X,Z}, Q, X_t, Z_t)$ is a weak solution, then it will be

$$\mathbf{F}^{Z} \Join \mathbf{F}^{Z}; \mathbf{F}^{X,Z}; Q, \tag{4.7}$$

or

$$\forall A \in \mathcal{F}_{<\infty}^{Z} \qquad Q(A \mid \mathcal{F}_t^{Z}) = Q(A \mid \mathcal{F}_t^{X,Z}).$$

According to [4] from (4.6) and (4.7) it follows that $Q \sim P$ on $\mathcal{F}_{<\infty}^{X,Z}$ (also $Q \sim P$ on $\mathcal{F}_0$ because of completeness). $\square$

Theorem 4.3. *Weak solution of equation* (3.1) *satisfies that*

$$\mathbf{F}^{X} \Join \mathbf{F}^{X,Z}; \mathbf{F}; P$$

if set H of the form (4.1)*, where M_t are right continuous modifications of martingales, also contains martingales of the form*

$$M_t = P(A \mid \mathcal{F}_t^{X,Z}),\ \forall A \in \mathcal{F}_{\infty}^{X}.$$

Proof. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$ be a weak solution. Then $\mathbf{F}^Z \,|\!\!< \mathbf{F}^Z; \mathbf{F}; P$ holds. It is obvious that the elements of H are martingales relative to filtration $\{\mathcal{F}_t\}$. We supposed that the martingales M_t also have a form $M_t = P(A \mid \mathcal{F}_t^{X,Z}),\ A \in \mathcal{F}_{<\infty}^X$, and they are $(\mathcal{F}_t)$-martingales, so

$$H' = \{M_t;\ M_t = P(A \mid \mathcal{F}_t^{X,Z}),\ A \in \mathcal{F}_{<\infty}^X\},$$

$$\forall M_t \in H' \qquad E(M_\infty \mid \mathcal{F}_t) = M_t,$$

$$\forall A \in \mathcal{F}_{<\infty}^X \qquad E(E(I_A \mid \mathcal{F}_{<\infty}^{X,Z}) \mid \mathcal{F}_t) = M_t,\ t \in T.$$

I_A is obviously $\mathcal{F}_{<\infty}^X$-measurable and $(\mathcal{F}_{<\infty}^X) \subseteq (\mathcal{F}_{<\infty}^{X,Z})$. So that, I_A is $\mathcal{F}_{<\infty}^{X,Z}$-measurable. Now it follows that

$$\forall A \in \mathcal{F}_{<\infty}^X \qquad E(I_A \mid \mathcal{F}_t) = M_t,\ t \in T$$

or

$$\forall A \in \mathcal{F}_{<\infty}^X \qquad P(A \mid \mathcal{F}_t^{X,Z}) = P(A \mid \mathcal{F}_t). \qquad \square$$

REFERENCES

1. R.J. Elliot: *Stochastic Calculus and Applications.* Springer-Verlag, New York, 1982.
2. C.W.J. Granger: *Investigation causal relations by econometric models and cross spectral methods.* Econometrica **37** (1969), 424–438.
3. J. Jacod: *Weak and strong solutions of stochastic differential equations.* Stochastics **3** (1980), 171–191.
4. J. Jacod and J. Memin: *Existence of weak solutions for stochastic differential equation with driving semimartingales.* Stochastics **4** (1981), 317–337.
5. I. Karatzas and S.E. Shreve: *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York, 1988.
6. V.A. Lebedev: *On the existence of weak solutions for stochastic differential equations with driving martingales and random measures.* Stochastics **9** (1981), 37–76.
7. R.S. Liptser and A.N. Shiryaev: *Statistics of Random Processes I.* Springer-Verlag, New York, 1977.
8. P.A. Mykland: *Statistical Causality.* University of Bergen, 1986.
9. P.A. Mykland: *Stable Subspaces Over Regular Solutions of Martingale Problems.* University of Bergen, 1986.

10. Lj. Petrović: *Causality and Markovian reductions and extensions of a family of Hilbert spaces.* J. Math. Systems, Estimat. Control **8** (1998), 12 pp.
11. Yu.A. Rozanov: *Theory of Innovation Processes.* Monographs in Probability Theory and Mathematical Statistics, Izdat. Nauka, Moscow, 1974.
12. Yu.A. Rozanov: *Innovation Processes.* V. H. Winston and Sons, New York, 1977.
13. Yu.A. Rozanov: *Markov Random Fields.* Springer-Verlag, Berlin – New York – Heidelberg, 1982.

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