FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 20 (2005), 75–101

SOME GENERALIZATIONS OF THE FIRST FREDHOLM THEOREM TO HAMMERSTEIN EQUATIONS AND THE NUMBER OF SOLUTIONS

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Abstract. We prove some generalizations of the first Fredholm theorem for Hammerstein operator equations in Banach spaces and study the number of their solutions using a projection like method.The linear part is assumed to be either selfadjoint or nonseladjoint while the nonlinearities are such that the corresponding map is (pseudo) A-proper. In particular, the nonlinearities can be either of monotone type, or of type (S_+), or condensing, or the sum of such maps.

1. Introduction

In this paper, we shall prove some generalizations of the first Fredholm theorem to Hammerstein operator equations of the form

$$x - KFx = f, \tag{1.1}$$

where K is linear and F is a nonlinear map. We shall consider (1.1) in a general setting between two Banach spaces. To that end, we shall use two approaches. One is based on applying the Brouwer degree theory directly to the finite dimensional approximations of the map $I-KF$ in conjunction with the (pseudo) A-proper mapping approach. The other one is based on splitting first the map K as a product of two suitable maps and then using again this degree theory. The linear part K is assumed to be either selfadjoint or nonselfadjoint. In the second case, we assume that K is either positive in the sense of Krasnoselskii, P-positive (i.e., angle-bounded) or that it is

Received March 12, 2005.

2000 *Mathematics Subject Classification.* Primary 47H15, 35L70, 35L75; Secondary 35J40.

P-quasi-positive, that is that its selfadjoint part has at most a finite number of negative eigenvalues of finite multiplicity. The nonlinear part is assumed to be such that either $I - KF$ is (pseudo) A-proper or that the corresponding map in an equivalent reformulation of Eq. (1.1) is a k-ball contractive or a quasimonotone perturbation of a strongly monotone map and is therefore A-proper. Applications of the abstract theory to Hammerstein integral equations will be given elsewhere.

We begin by proving some generalized first Fredholm theorems for general (pseudo) A-proper maps and their uniform limits. In the case of A-proper maps, we also establish the number of solutions of these equations. Then we use them to establish various results on the number of solutions of Eq. (1.1) assuming different conditions on the nonlinearity F that imply a-priori estimates on the solution set. Unlike earlier studies, we also study Eq. (1.1) with nonlinearities that are the sum of a strongly monotone and k-ball condensing maps. This work is a continuation of our study of these equations in [19, 23]. There is an extensive literature on Hammerstein equations and we refer to the books [9, 10, 29] as well as to [1, 3, 7, 6, 25, 26, 27, 28]. In particular, for the unique (approximation) solvability of these equations we refer to [29, 2, 19, 23].

2. Generalizations of the First Fredholm Theorem and the Number of Solutions

We begin with some definitions.

Let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of Banach spaces X and Y respectively such that $\dim X_n = \dim Y_n$ for each n and

$$\operatorname{dist}(x, X_n) \to 0 \quad \text{as} \quad n \to +\infty \quad \text{for each} \quad x \in X.$$

Let $P_n : X \to Y_n$ and $Q_n : Y \to Y_n$ be linear projections onto X_n and Y_n respectively such that $P_n x \to x$ for each $x \in X$ and $\delta = \max ||Q_n|| < +\infty$. Then $\Gamma = \{X_n, P_n; Y_n, Q_n\}$ is a projection scheme for (X, Y).

A map $T : D \subset X \to Y$ is said to be *approximation-proper* (*A-proper* for short) with respect to Γ if

(i) $Q_n T : D \cap X_n \to Y_n$ is continuous for each n and

(ii) whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $||Q_{n_k} T x_{n_k} - Q_{n_k} f|| \to 0$ for some $f \in Y$, then a subsequence $x_{n_{k(i)}} \to x$ and $Tx = f$.

T is said to be *pseudo A-proper* w.r.t. Γ if in (ii) above we do not require that a subsequence of $\{x_{n_k}\}$ converges to x for which $Tx = f$. If f is given in advance, we say that T is (pseudo) A-proper at f.

For the developments of the (pseudo) A-proper mapping theory and applications to differential equations, we refer to [15]–[22] and [26, 27]. To demonstrate the generality and the unifying nature of the (pseudo) A-proper mapping theory, we state now a number of examples of A-proper and pseudo A-proper maps.

To look at ϕ-condensing maps, we recall that the *set measure of noncompactness* of a bounded set $D \subset X$ is defined as

$$\gamma(D) = \inf\{d > 0 : D \text{ has a finite covering by sets of diameter less than } d\}.$$

The *ball-measure of noncompactness* of D is defined as

$$\chi(D) = \inf\left\{ r > 0 \,\middle|\, D \subset \bigcup_{i=1}^{n} B(x_i, r),\ x_i \in X,\ n \in \mathbb{N} \right\}.$$

Let ϕ denote either the set or the ball-measure of noncompactness. Then a map $N : D \subset X \to X$ is said to be $k - \phi$ *contractive* (ϕ*-condensing*) if $\phi(N(Q)) \leq k\phi(Q)$ (respectively $\phi(N(Q)) < \phi(Q)$) whenever $Q \subset D$ (with $\phi(Q) \neq 0$).

Recall that $N : X \to Y$ is K-monotone for some $K : X \to Y^*$ if

$$(Nx - Ny, K(x - y)) \geq 0 \ \text{ for all } \ x, y \in X.$$

It is said to be generalized pseudo-K-monotone (of type (KM)) if whenever $x_n \rightharpoonup x$ and $\limsup(Nx_n, K(x_n - x)) \leq 0$ then $(Nx_n, K(x_n - x)) \to 0$ and $Nx_n \rightharpoonup Nx$ (then $Nx_n \rightharpoonup Nx$). Recall that N is said to be of type (KS_+) if $x_n \rightharpoonup x$ and $\limsup(Nx_n, K(x_n - x)) \leq 0$ imply that $x_n \to x$. If $x_n \rightharpoonup x$ implies that $\limsup(Nx_n , K(x_n - x)) \geq 0$, N is said to be K-quasimonotone. If $Y = X^*$ and K is the identity map, then these maps are called monotone, generalized pseudo monotone, of type (M) and (S_+) respectively. If $Y = X$ and $K = J$ the duality map, then J-monotone maps are called accretive. It is known that bounded monotone maps are of type (M). We say that N is demicontinuous if $x_n \to x$ in X implies that $Nx_n \rightharpoonup Nx$. It is well known that $I - N$ is A-proper if N is ball-condensing and that K-monotone like maps are pseudo A-proper under some conditions on N and K. Moreover, their perturbations by Fredholm or hyperbolic like maps are A-proper or pseudo A-proper. (see [15]–[17], [20]–[22]). In [11] we have shown that ball-condensing perturbations of stable A-proper maps are also A-proper. In particular, a ball-condensing perturbation of a c-strongly K- monotone map for a suitable $K : X \to Y^*$, i.e., $(Tx - Ty, K(x - y)) \geq c||x - y||^2$ for all $x, y \in X$, is an A-proper map.

3. General Equations

We begin with some extensions of the first Fredholm theorem to general equations of the form $Tx = f$, where T is a nonlinear (pseudo) A-proper map.

We say that a map $T : X \to Y$ satisfies condition $(+)$ if whenever $Tx_n \to f$ in Y then $\{x_n\}$ is bounded in X. It satisfies condition $(*)$ if whenever $Tx_n \to f$ in Y then $Tx = f$ for some $f \in Y$. T is locally injective at $x_0 \in X$ if there is a neighborhood $U(x_0)$ of x_0 such that T is injective on $U(x_0)$. T is locally injective on X if it is locally injective at each point $x_0 \in X$. A continuous map $T : X \to Y$ is said to be locally invertible at $x_0 \in X$ if there are a neighborhood $U(x_0)$ and a neighborhood $U(T(x_0))$ of $T(x_0)$ such that T is a homeomorphism of $U(x_0)$ onto $U(T(x_0))$. It is locally invertible on X if it is locally invertible at each point $x_0 \in X$.

Let Σ be the set of all points $x \in X$ where T is not locally invertible and let $\operatorname{card} T^{-1}(\{f\})$ be the cardinal number of the set $T^{-1}(\{f\})$.

We need the following basic theorem on the number of solutions of nonlinear equations for A-proper maps (see [21]).

Theorem 3.1. *Let $T : X \to Y$ be a continuous A-proper map that satisfies condition $(+)$. Then*
(a) *The set $T^{-1}(\{f\})$ is compact (possibly empty) for each $f \in Y$.*
(b) *The range $R(T)$ of T is closed and connected.*
(c) *Σ and $T(\Sigma)$ are closed subsets of X and Y, respectively, and $T(X \setminus \Sigma)$ is open in Y.*
(d) *$\operatorname{card} T^{-1}(\{f\})$ is constant and finite (it may be 0) on each connected component of the open set $Y \setminus T(\Sigma)$.*

Theorem 3.2. *Let $A, T : X \to Y$ be nonlinear maps such that*
(i) *A is odd on $X \setminus B(0, R)$ for some $R > 0$, and there are an $n_0 \geq 1$ and a function $c : R^+ \to R^+$ such that $c(r) \to +\infty$ as $r \to +\infty$ and*

$$||Q_n Ax|| \geq c(||x||) \quad \text{for all } x \in X_n \setminus B(0, R),\ n \geq n_0$$

(ii) *T is asymptotically close to A, i.e.,*

$$|T - A| = \limsup_{||x|| \to +\infty} ||Tx - Ax||/c(||x||) < 1/\delta,$$

where $\delta = \max ||Q_n||$.

Then

(a) *If T is A-proper w.r.t. Γ, then Eq. $Tx = f$ is approximation solvable for each $f \in Y$. Moreover, if*

$$\Sigma = \{x \in X \mid T \text{ is not invertible at } x\},$$

and T is continuous, then $(T)^{-1}(\{f\})$ is compact for each $f \in Y$ and the cardinal number card $(T)^{-1}(\{f\}))$ *is constant, finite and positive on each connected component of the set $Y \setminus T(\Sigma)$.*

(b) *If T is pseudo A-proper w.r.t. Γ, then $T(X) = Y$.*

(c) *If $T + \mu G$ is A-proper w.r.t. Γ for each $\mu \in (0, \mu_0)$ with some small μ_0 and a bounded map $G : X \to Y$, and T satisfies condition $(*)$, then $T(X) = Y$.*

Proof. The solvability of Eq. $Tx = f$ has been proven in [12, 14] using the finite dimensional antipodes theorem of Borsuk. Here we shall give a more direct proof. Let $\epsilon > 0$ be such that $|T - A| + 2\epsilon < 1/\delta$. Then there is $\beta > 0$ such that $|T - A| + 2\epsilon < (1 - \beta)/\delta$. Let $r \geq R$ be such that $c(r) \geq \max\{1, 2\delta||f||/\beta\}$ and $||Tx - Ax|| \leq (|T - A| + \epsilon)c(||x||)$ for all $||x|| \geq r$. Define a homotopy

$$H(t, x) = (1 - t)Ax + tTx - tf \quad \text{for } t \in [0, 1] \text{ and } x \in B(0, r).$$

We claim that $Q_nH(t, x) \neq 0$ for all $t \in [0, 1]$ and $x \in X_n \cap \partial B(0, r)$ for all $n \geq n_0$. If not, then there would exist infinitely many $x_n \in X_n \cap \partial B(0, r)$ and $t_n \in [0, 1]$ such that $Q_nH(t_n, x_n) = 0$ for these $n's$. Then

$$\begin{aligned} c(||x_n||) &\leq ||Q_nAx_n|| = t||Q_n(T - A)x_n - Q_nf|| \\ &\leq \delta[||Tx_n - Ax_n|| + ||f||] \leq \delta[(|T - A| + \epsilon)c(||x_n||) + ||f||] \\ &< \delta[(1 - \beta)c(||x_|\,|)/\delta + ||f||] = (1 - \beta)c(||x_n||) + \delta||f||. \end{aligned}$$

But, $\delta||f|| \leq \beta c(||x_n||)/2$ and therefore

$$c(||x_n||) \leq (1 - \beta)c(||x_n||) + \beta c(||x_n||)/2 = (1 - \beta/2)c(||x_n||).$$

Dividing by $c(||x_n||)$, we get a contradiction $1 \leq 1 - \beta/2$. Hence, our claim is valid. Thus, $\deg(Q_nT, X_n \cap B(0, r), Q_nf) = \deg(Q_nA, X_n \cap B(0, r), 0) \neq 0$ for all large n. Thus, by the pseudo A-properness of T, the equation $Tx = f$ is solvable.

Next, we shall show that T satisfies condition $(+)$. Indeed, let $Tx_n \to f$ in Y as $n \to +\infty$. Then there is a constant $M > 0$ such that $||Tx_n|| \leq M$ for all n. As above, select $\epsilon > 0$ and $\beta > 0$ such that $|T - A| + 2\epsilon < (1-\beta)/\delta$. Suppose that $||x_n|| \to +\infty$ and, as above, let $r \geq R$ be large such that $c(r) \geq \max\{1, 2\delta M/\beta\}$ and $||Tx - Ax|| \leq (|T-A|+\epsilon)c(||x||)$ for all $||x|| \geq r$. Then for $||x_n|| \geq r$, we get

$$\begin{aligned} c(||x_n||) &\leq ||Q_n A x_n|| \leq ||Q_n(Ax_n - Tx_n) + Q_n T x_n|| \\ &\leq \delta(||Ax_n - Tx_n|| + M) \leq \delta(|T-A|c(||x_n||) + M) \\ &< (1-\beta)c(||x_n||) + \delta M \leq (1-\beta)c(||x_n||) + \beta c(||x_n||)/2. \end{aligned}$$

This leads to a contradiction as above. Hence, T satisfies condition (+). Now, the theorem follows from Theorem 3.1. The proofs of (b)–(c) can be found in [12]–[14]. □

The following result gives some conditions for (i) in Theorem 3.2 to hold.

Lemma 3.1. *Let $A : X \to Y$ be A-proper w.r.t. Γ and k-positively homogeneous outside of some ball in X, i.e., $A(\alpha x) = \alpha^k Ax$ for all $||x|| \geq R$, all $\alpha \geq 1$ and some $k \geq 1$. Suppose that there is an $M > 0$ such that if $Ax = 0$, then $||x|| \leq M$. Then there are $c > 0$ and $n_0 \geq 1$ such that for $r \geq M+1$ and each $n \geq n_0$*

$$||Q_n Ax|| \geq c||x||^k \quad \textit{for all } \; x \in X_n \setminus B(0,r).$$

Proof. If this is not the case, then there are $x_{n_j} \in X_{n_j} \setminus B(0,r)$ such that

$$||Q_{n_j} A x_{n_j}|| \leq 1/j ||x_{n_j}||^k \quad \text{for all } \; j \geq 1.$$

Set $v_{n_j} = r x_{n_j}/||x_{n_j}||$. Then $||v_{n_j}|| = r$, $x_{n_j} = r^{-1}||x_{n_j}||v_{n_j}$, and

$$Q_{n_j} A x_{n_j} = Q_{n_j}\left(r^{-1}||x_{n_j}||v_{n_j}\right) = r^{-k}||x_{n_j}||^k Q_{n_j} A v_{n_j}$$

since $r^{-1}||x_{n_j}|| \geq 1$. Hence,

$$||Q_{n_j} A x_{n_j}|| = r^{-k}||x_{n_j}^k||\, ||Q_{n_j} A v_{n_j}|| \leq 1/j ||x_{n_j}||^k$$

and therefore

$$||Q_{n_j} A v_{n_j}|| \leq r^k/j \to 0 \;\text{ as }\; j \to +\infty.$$

Since A is A-proper (at 0), we may assume that $x_{n_j} \to v$ and $Av = 0$ with $||v|| = r$. This contradicts the choice of r. Hence, the lemma is valid. □

Remark 3.1. The condition that $||x|| \leq M$ if $Tx = 0$ is implied in particular if $x = 0$ when $Tx = 0$. This particular condition has been used in all earlier generalizations of the first Fredholm theorem.

In view of Lemma 3.1, we have the following generalized first Fredholm theorem.

Theorem 3.3. *Let $A : X \to Y$ be A-proper w.r.t. Γ, odd and k-positively homogeneous outside of some ball in X for some $k \geq 1$ and $T : X \to Y$ be such that*

(i) *There is an $M > 0$ such that if $Ax = 0$, then $||x|| \leq M$.*

(ii) *T is asymptotically close to A, i.e.,*

$$|T - A| = \limsup_{||x|| \to +\infty} ||Tx - Ax||/c||x||^k < 1/\delta$$

where c is as in Lemma 3.1 and $\delta = \max ||Q_n||$.

Then

(a) *Eq. $Tx = f$ is approximation solvable for each $f \in Y$ if T is A-proper w.r.t. Γ. Moreover, if*

$$\Sigma = \{x \in X \mid T \textit{ is not invertible at } x\}$$

and T is continuous, then $(T)^{-1}(\{f\})$ is compact for each $f \in Y$ and the cardinal number $\operatorname{card}(T)^{-1}(\{f\}))$ is constant, finite and positive on each connected component of the set $Y \setminus T(\Sigma)$.

(b) *Eq. $Tx = f$ is solvable for each $f \in Y$ if T is pseudo A-proper w.r.t. Γ.*

(c) *If $T + \mu G$ is A-proper w.r.t. Γ for each $\mu \in (0, \mu_0)$ with some small μ_0 and a bounded map $G : X \to Y$, and T satisfies condition $(*)$, then $T(X) = Y$.*

Proof. By Lemma 3.1, there are $c > 0$ and $n_0 \geq 1$ such that for $r \geq M + 1$ and each $n \geq n_0$

$$||Q_n Ax|| \geq c||x||^k \quad \text{for all } \; x \in X_n \setminus B(0, r).$$

Hence, the theorem follows by Theorem 3.2. $\square$

4. Applications to Hammerstein Equations

In this section, we shall apply Theorem 3.3 to Hammerstein equations.

Theorem 4.1. *Let $K : X^* \to X$ be linear and $F = F_1 + F_2 : X \to X^*$ be nonlinear such that $I - KF_1$ is an A-proper map w.r.t. Γ and*

(i) *F_1 is odd and positively homogeneous outside some ball in X.*

(ii) *There is an $M > 0$ such that if $x - KF_1x = 0$, then $||x|| \le M$.*

(iii) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Then

(a) *Eq.* (1.1) *is approximation solvable w.r.t. Γ for each $f \in X$ if $I - KF$ is A-proper w.r.t. Γ. Moreover, if*

$$\Sigma = \{x \in X \mid I - KF \ \textit{ is not invertible at } \ x\}$$

and $I - KF$ is continuous, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$ and the cardinal number card $(I - KF)^{-1}(\{f\}))$ *is constant, finite and positive on each connected component of the set $X \setminus (I - KF)(\Sigma)$.*

(b) *Eq.* (1.1) *is solvable for each $f \in Y$ if $I - KF$ is pseudo A-proper w.r.t. Γ.*

(c) *If $I - KF + \mu G$ is A-proper w.r.t. Γ for each $\mu \in (0, \mu_0)$ with some small μ_0 and a bounded map $G : X \to X$, and $I - KF$ satisfies condition $(*)$, then $(I - KF)(X) = X$.*

Proof. Set $A = I - KF_1$ and $T = A - KF_2$. Then T is asymptotically close to A and

$$|T - A| = |KF_2| = \limsup_{||x|| \to +\infty} ||KF_2x||/||x|| < |F| \, ||K||.$$

Hence, A and T satisfy all the conditions of Theorem 3.3. $\square$

Next, we shall look at some special classes of nonlinearities.

Corollary 4.1. *Let $K : X^* \to X$ be linear and $F = F_1 + F_2 : X \to X^*$ be nonlinear such that*

(i) *F_1 is odd and positively homogeneous outside some ball in X.*

(ii) *There is an $M > 0$ such that if $x - KF_1x = 0$, then $||x|| \leq M$.*

(iii) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x||\to+\infty} ||F_2x||/||x||$$

is sufficiently small.

Assume that either K is compact and F_1 and F_2 are demicontinuous, or K is continuous and F and F_1 are k-ball contractive and k_1-ball contractive, respectively with $k||K|| < 1$ and $k_1||K|| < 1$. Then Eq. (1.1) *is approximation solvable w.r.t. Γ for each $f \in X$. Moreover, if*

$$\Sigma = \{x \in X \mid I - KF \ \ \textit{is not invertible at} \ \ x\}$$

and $I - KF$ is continuous, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$ and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\}))$ *is constant, finite and positive on each connected component of the set $X \setminus (I - KF)(\Sigma)$.*

Proof. Set $A = I - KF_1$ and $T = A - KF_2$. Then $I - KF_1$ and $I - KF$ are A-proper maps w.r.t. Γ and Theorem 4.1 applies. $\square$

Next, we shall discuss other sets of conditions on K and F that imply the A-properness of an operator in an equivalent formulation of our equation. Recall that a map K acting in a Hilbert space H is called positive in the sense of Krasnoselski if there exists a number $\mu > 0$ for which

$$(Kx, Kx) \leq \mu(Kx, x), \ \ x \in H.$$

The infimum of all such numbers μ is called the positivity constant of K and is denoted by $\mu(K)$. The simplest example of a positive map is provided by any bounded selfadjoint positive definite map K on H. Then $\mu(K) = ||K||$ for such maps. A compact normal map K in a Hilbert space is positive on H if and only if (cf. [8]) the number

$$\left[\inf_{\substack{\lambda\in\sigma(K)\\ \lambda\neq 0}} \operatorname{Re}\left(\lambda^{-1}\right)\right]^{-1}$$

is well defined and positive. In that case, it is equal to $\mu(K)$.

Let X be a reflexive embeddable Banach space, that is, there is a Hilbert space H such that $X \subset H \subset X^*$ so that $\langle y, x\rangle = (y, x)$ for each $y \in H, x \in X$, where $\langle , \rangle$ is the duality pairing of X and X^*. Let $K : X^* \to X$ be a positive semidefinite bounded selfadjoint map in the sense that $\langle Kx, y\rangle = \langle x, Ky\rangle$ for all $x, y \in X^*$. Then the positive semidefinite square root $K_H^{1/2}$ of the restriction K_H of K to H can be extended to a bounded linear map $T : X^* \to H$ such that $K = T^*T$, where the adjoint map $T^* = K_H^{1/2}$ of T is a bounded map from H to X (see [29]).

We shall look at the following equivalent formulation of Eq.(1.1)

$$y - TFCy = h, \quad h \in H. \tag{4.1}$$

We need the following lemma (cf. [29, 24]).

Lemma 4.1. *Equations* (1.1) *and* (4.1) *are equivalent with f restricted to $C(H)$; each solution y of* (4.1) *determines a solution $x = Cy$ of* (1.1) *and each solution x of* (1.1) *with $f \in C(H)$ determines a solution $y = TFx + h$ of* (4.1) *with $f = Ch$ and $x = Cy$. Moreover, the map*

$$C : S(h) = (I - TFC)^{-1}(\{h\}) \to S = (I - KF)^{-1}(\{Ch\})$$

is bijective.

Proof. Let y_1 and y_2 be distinct solutions of (4.1). Applying C to $y_i - TFCy_i = h$ and using the fact that $K = CT$, we get that $x_1 = Cy_1$ and $x_2 = Cy_2$ are solutions of (1.1). They are distinct since

$$\begin{aligned} 0 < ||y_1 - y_2||^2 &= (TFCy_1 - TFCy_2, y_1 - y_2) \\ &= (FCy_1 - FCy_2, C(y_1 - y_2)) = (Fx_1 - Fx_2, x_1 - x_2). \end{aligned}$$

Conversely, let $f \in C(H)$ and x_1 and x_2 be distinct solutions of (1.1). Let $f = Ch$ for some $h \in H$. Set $y_i = TFx_i + h$. Then $Cy_i = CTFx_i + h = KFx_i + f$ and so $x_i = Cy_i$. Hence, $y_i = TFCy_i + h$, i.e., y_i are solutions of (4.1). They are distinct since $y_1 = y_2$ implies that $x_1 = Cy_1 = Cy_2 = x_2$. These arguments show that $C : S(h) \to S$ is a bijection. □

Theorem 4.2. *Let $K : X^* \to X$ be linear and $F = F_1 + F_2 : X \to X^*$ be nonlinear such that $I - TF_1C$ is an A-proper map w.r.t. Γ and*

(i) *F_1 is odd and positively homogeneous outside some ball in X.*

(ii) *There is an $M > 0$ such that if $x - TF_1Cx = 0$, then $||x|| \le M$.*

(iii) *The quasinorm of* F_2

$$|F_2| = \limsup_{||x||\to+\infty} ||F_2x||/||x||$$

is sufficiently small.

Then

(a) *If* $I - KF$ *is A-proper w.r.t.* Γ, *Eq.* (1.1) *is approximation solvable in* X *for each* $f \in C(H) \subset X$ *w.r.t. a projection scheme* $\Gamma = \{X_n, P_n\}$ *for* X, $\delta = \max ||P_n|| = 1$. *Moreover, if*

$$\Sigma_H = \{h \in H \mid I - TFC \ \textit{ is not locally invertible at } \ h\},$$

then $(I-KF)^{-1}(\{f\})$ *is compact for each* $f \in C(H)$, *and the cardinal number* $\operatorname{card}(I-KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set* $H \setminus (I - TFC)(\Sigma_H)$ *intersected by* $C(H)$.

(b) *Eq.* (1.1) *is solvable for each* $f \in C(H)$ *if* $I - KF$ *is pseudo A-proper w.r.t.* Γ.

(c) *If* $I - KF + \mu G$ *is A-proper w.r.t.* Γ *for each* $\mu \in (0, \mu_0)$ *with some small* μ_0 *and a bounded map* $G : X \to X$, *and* $I - KF$ *satisfies condition* $(*)$, *then Eq.* $x - KFx = f$ *is solvable for each* $f \in C(H)$.

Proof. Set $A = I - TF_1C$ and $B = A - TFC$. Then A and B satisfy all the conditions of Theorem 3.3. Hence, we have that the equation $y - TFCy = h$ is solvable for each $h \in H$, $S(h) = (I - TFC)^{-1}(\{h\}) \neq \varnothing$ and compact, and $\operatorname{card} S(h)$ is constant and finite on each connected component of the open set $H \setminus (I - TFC)(\Sigma_H)$, where

$$\Sigma_H = \{h \in H \mid I - TFC \ \text{ is not locally invertible at } \ h\}.$$

Next, applying C to $y - TFCy = h$ and using the fact that $K = CT$, we get that $x - KFx = f$ with $x = Cy \in X$. By Lemma 4.1, we get that

$$\operatorname{card} S = (I - KF)^{-1}(\{Ch\}) = \operatorname{card} S(h).$$

Hence, $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite and positive on each connected component of $H \setminus (I - TFC)(\Sigma_H)$ intersected by $C(H)$. $\square$

Corollary 4.2. *Let* X *be a reflexive embeddable Banach space* ($X \subset H \subset X^*$), $K : X^* \to X$ *be a positive semidefinite bounded selfadjoint map, and* $C = K_H^{1/2}$, *where* K_H *is the restriction of* K *to* H, $\mu(K) = ||C||^2$ *and* T :

$X^* \to H$ be a bounded linear extension of $K_H^{1/2}$. Let $F = F_1 + F_2 : X \to X^*$ be a nonlinear map such that

(i) *F_1 is odd and positively homogeneous outside some ball in X.*

(ii) *There is an $M > 0$ such that if $x - TF_1Cx = 0$, then $||x|| \leq M$.*

(iii) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Assume that TFC and TF_1C are k-ball contractive and k_1-ball contractive, respectively with $k < 1$ and $k_1 < 1$. Then Eq. (1.1) *is approximation solvable in X for each $f \in C(H) \subset X$ w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max ||P_n|| = 1$. Moreover, if*

$$\Sigma_H = \{h \in H \mid I - TFC \text{ is not locally invertible at } h\},$$

then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in C(H)$, and the cardinal number card $(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set $H \setminus (I - TFC)(\Sigma_H)$ intersected by $C(H)$.*

Proof. Set $A = I - TF_1C$ and $B = A - TFC$. Then A and B are A-proper w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max ||P_n|| = 1$. Hence, Theorem 4.2 applies. □

Corollary 4.3. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a positive semidefinite bounded selfadjoint map, and $C = K_H^{1/2}$, where K_H is the restriction of K to H, $\mu(K) = ||C||^2$ and $T : X^* \to H$ be a bounded linear extension of $K_H^{1/2}$. Let $F = F_1 + F_2 : X \to X^*$ be a nonlinear map, such that either $-F_2$ is quasimonotone or TF_2C is k ball contractive with $k < 1 - c\mu(K)$ where c is the smallest number such that*

(i) *$(F_1x - F_1y, x - y) \leq c||x - y||^2$ for all $x, y \in X$.*

(ii) *F_1 is odd and positively homogeneous outside some ball in X.*

(iii) *There is an $M > 0$ such that if $x - TF_1Cx = 0$, then $||x|| \leq M$.*

(iv) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Then Eq. (1.1) *is approximation solvable in* X *for each* $f \in C(H) \subset X$ *w.r.t. a projection scheme* $\Gamma = \{X_n, P_n\}$ *for* X, $\delta = \max ||P_n|| = 1$. *Moreover, if*

$$\Sigma_H = \{h \in H \mid I - TFC \ \ \textit{is not locally invertible at} \ \ h\},$$

then $(I - KF)^{-1}(\{f\})$ *is compact for each* $f \in C(H)$, *and the cardinal number* $\operatorname{card}(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set* $H \setminus (I - TFC)(\Sigma_H)$ *intersected by* $C(H)$.

Proof. We claim that the map $I - TF_1C : H \to H$ is $1 - c\mu(K)$-strongly monotone. Indeed, for $x, y \in H$, we have

$$\begin{aligned}(x - TF_1Cx - y + TF_1Cy, x - y) &= ||x - y||^2 - (TF_1Cx - TF_1Cy, x - y) \\ &= ||x - y||^2 - (F_1Cx - F_1Cy, Cx - Cy) \\ &\geq (1 - c\mu(K))||x - y||^2.\end{aligned}$$

Hence, $I - TF_1C$ is A-proper w.r.t. $\Gamma = \{H_n, P_n\}$ for H. Since TF_2C is k-ball condensing with $k < 1 - c\mu(K)$, we see that $I - tTFC$ is A-proper w.r.t. Γ (cf. [11]). If $-F_2$ is quasimonotone, then $I - TFC$ is of type (S_+) as a sum of a strongly monotone map and a quasimonotone map. Hence, Theorem 4.2 applies. □

Remark 4.1. Corollary is also valid if we assume that TF_1C is k-ball contractive and F_2 satisfies condition (i).

Corollary 4.4. *Let* X *be a reflexive embeddable Banach space* ($X \subset H \subset X^*$), $K : X^* \to X$ *be a positive semidefinite bounded selfadjoint map, and* $C = K_H^{1/2}$, *where* K_H *is the restriction of* K *to* H, $\mu(K) = ||C||^2$ *and* $T : X^* \to H$ *be a bounded linear extension of* $K_H^{1/2}$. *Let* $F = F_1 + F_2 : X \to X^*$ *be a nonlinear map, such that* $-F_1$ *and* $-F_2$ *are quasibounded and either pseudomonotone, or bounded generalized pseudomonotone, or quasimonotone and*

(i) F_1 *is odd and positively homogeneous outside some ball in* X.

(ii) *There is an* $M > 0$ *such that if* $x - TF_1Cx = 0$, *then* $||x|| \leq M$.

(iii) *The quasinorm of* F_2

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Then Eq. (1.1) *is approximation solvable in* X *for each* $f \in C(H) \subset X$ *w.r.t. a projection scheme* $\Gamma = \{X_n, P_n\}$ *for* X, $\delta = \max ||P_n|| = 1$. *Moreover, if*

$$\Sigma_H = \{h \in H \mid I - TFC \text{ is not locally invertible at } h\},$$

then $(I - KF)^{-1}(\{f\})$ *is compact for each* $f \in C(H)$*, and the cardinal number* $\operatorname{card}(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set* $H \setminus (I - TFC)(\Sigma_H)$ *intersected by* $C(H)$.

Proof. Set $A = I - TF_1C$ and $B = A - TF_2C$. We shall show that A and B are of type (S_+) in either case. Suppose first that F_1 is quasimonotone. Let $x_n \rightharpoonup x$ and $\limsup(Ax_n, x_n - x) \le 0$. Since

$$(Ax_n, x_n - x) = (x_n, x_n - x) - (F_1Cx_n, Cx_n - Cx)$$

and $Cx_n \rightharpoonup Cx$, the quasimonotonicity of $-F_1$ implies that

$$\limsup(x_n, x_n - x) \le 0.$$

Since $||x_n - x||^2 = (x_n, x_n - x) + (x, x - x_n)$, we get that $x_n \to x$. Hence, A is of type (S_+).

Next, we shall show that B is also of type (S_+). Let $x_n \rightharpoonup x$ and $\limsup(Bx_n, x_n - x) \le 0$. Since

$$(Bx_n, x_n - x) = (Ax_n, x_n - x) - (F_2Cx_n, Cx_n - Cx)$$

and $Cx_n \rightharpoonup Cx$, the quasimonotonicity of $-F_2$ implies that

$$\limsup(Ax_n, x_n - x) \le 0.$$

Hence, by the (S_+) property of A, we get that $x_n \to x$ and so B is of type (S_+).

Next, we shall show that a bounded generalized pseudomonotone map $-F_1$ is quasimonotone. If not, then for some $x_n \rightharpoonup x$ we have that

$$\limsup(-F_1x_n, x_n - x) < 0.$$

Since $-F_1$ is bounded, we may assume that $-F_1x_n \rightharpoonup y$. By the generalized pseudomonotonicity of $-F_1$, it follows that $\lim(-F_1x_n, x_n - x) = 0$, a contradiction. Hence, $-F_1$ is quasimonotone. Similarly, we get that $-F_1$ is quasimonotone if it is pseudomonotone (see [5]). Thus, A and B are of type (S_+) in all cases and are therefore A-proper w.r.t. Γ. Hence, the theorem follows from Theorem 4.2. $\square$

Next, let us look at the case when K is not selfadjoint. We begin by describing the setting of the problem. Let X be an embeddable Banach space, $X \subset H \subset X^*$. Let $K : X^* \to X$ be a linear map and K_H be the restriction of K to H such that $K_H : H \to H$. Let $A = (K + K^*)/2$ denote the selfadjoint part of K and $B = (K - K^*)/2$ be the skew-adjoint part of K. Assume that A is positive definite. Under our assumptions on K, both A and B map X^* into X. We know that A can be represented in the form $A = CC^*$, where $C = A^{1/2}$ is the square root of A, $C : H \to X$, and the adjoint map $C^* : X^* \to H$.

As in [2] and [23], we say that K is P-positive if $C^{-1}K(C^*)^{-1}$ exists and is bounded in H. It is S-positive if $K(C^*)^{-1}$ exists and is bounded in H. Clearly, the P-positivity implies the S-positivity but not conversely. It is easy to see that K is P-positive if and only if $C^{-1}B(C^*)^{-1}$ is bounded in H, and is S-positive if and only if $B(C^*)^{-1}$ is bounded in H. Moreover, K is P-positive if and only if K is angle-bounded, i.e.,

$$|(Kx, y) - (y, Kx)| \leq a(Kx, x)^{1/2}(Ky, y)^{1/2}, \;\; x, y \in H.$$

Denote by M and N the closure of the maps $C^{-1}K(C^*)^{-1}$ and $K(C^*)^{-1}$, respectively, in H. Note that M and N are defined on the closure (in H) of the range of $C = A^{1/2}$ and suppose that their domains coincide with H. We require that the following decompositions hold

$$K = CMC^*, \; K = NC^*.$$

Note that K, M and N are related as: $N = CM, N^* = M^*C^*$ and we have $(Mx, x) = ||x||^2$ for all $x \in H$. Hence, both M and M^* have trivial nullspaces. Denote by $\mu(K) = ||N||^2$, which is the positivity constant of K in the sense of Krasnoselski.

Let $F : X \to X^*$ be a nonlinear map and consider the Hammerstein equation

$$x - KFx = f. \tag{4.2}$$

For $f \in N(H)$, let $h \in H$ be a solution of

$$M^*h - N^*FNh = M^*k, \tag{4.3}$$

where $f = Nk$ for some $k \in H$. Then $M^*(h - C^*FNh - k) = 0$ since $N = CM$ and $N^* = M^*C^*$. Hence, $h = C^*FNh + k$ since M^* is injective and therefore

$$Nh = NC^*FNh + Nk = KFNh + f$$

since $K = NC^*$. Thus $x = Nh$ is a solution of (4.2). So the solvability of (4.2) is reduced to the solvability of (4.3). Actually these two equations are equivalent. Namely, we have (cf. [24])

Lemma 4.2. *Equations* (4.2) *and* (4.3) *are equivalent with f restricted to $N(H)$; each solution h of* (4.3) *determines a solution $x = Nh$ of* (4.2) *and each solution x of* (4.2) *with $f \in N(H)$ determines a solution $h = C^*Fx + k$ of* (4.3) *with $f = Nk$ and $x = Nh$. Moreover, the map*

$$N : S(M^*k) = (M^* - N^*FN)^{-1}(M^*k) \to S = (I - KF)^{-1}(Nk)$$

is bijective.

Proof. Let h_1 and h_2 be distinct solutions of (4.3). We have seen above that $x_1 = Nh_1$ and $x_2 = Nh_2$ are solutions of (4.2). They are distinct since

$$\begin{aligned} 0 < ||h_1 - h_2||^2 &= (M(h_1 - h_2), h_1 - h_2) = (N^*FNh_1 - N^*FNh_2, h_1 - h_2) \\ &= (FNh_1 - FNh_2, N(h_1 - h_2)) = (Fx_1 - Fx_2, x_1 - x_2). \end{aligned}$$

Conversely, let $f \in N(H)$ and x_1 and x_2 be distinct solutions of (4.2). Let $f = Nk$ for some $k \in H$. Set $h_i = C^*Fx_i + k$. Then $Nh_i = NC^*Fx_i + Nk = KFx_i + f$ and so $x_i = Nh_i$. Hence,

$$M^*h_i = M^*C^*FNh_i + M^*k = N^*FNh_i + M^*k,$$

i.e., h_i are solutions of (4.3). They are distinct since $h_1 = h_2$ implies that $x_1 = Nh_1 = Nh_2 = x_2$. These arguments show that $N : S(M^*k) \to S$ is bijective. □

Theorem 4.3. *Let $K : X^* \to X$ be linear P-positive and $F = F_1 + F_2 : X \to X^*$ be nonlinear such that $M^* + N^*F_1N$ is an A-proper map w.r.t. Γ and*

(i) *F_1 is odd and positively homogeneous outside some ball in X.*

(ii) *There is an $M_1 > 0$ such that if $M^* - N^*F_1Nx = 0$, then $||x|| \leq M_1$.*

(iii) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Then

(a) *If* $M^* - N^*FN$ *is A-proper w.r.t.* Γ, *Eq.* (1.1) *is approximation solvable in* X *for each* $f \in N(H) \subset X$ *w.r.t. a projection scheme* $\Gamma = \{X_n, P_n\}$ *for* X, $\delta = \max ||P_n|| = 1$. *Moreover, if*

$$\Sigma_H = \{h \in H \mid M^* - N^*FN \quad \textit{is not locally invertible at} \ \ h\},$$

then $(I - KF)^{-1}(\{f\})$ *is compact for each* $f \in C(H)$, *and the cardinal number* $\operatorname{card}(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set* $H \setminus (M^* - N^*FN)(\Sigma_H)$ *intersected by* $N(H)$.

(b) *Eq.* (1.1) *is solvable for each* $f \in N(H)$ *if* $M^* - N^*FN$ *is pseudo A-proper w.r.t.* Γ.

(c) *If* $M^* - N^*FN + \mu G$ *is A-proper w.r.t.* Γ *for each* $\mu \in (0, \mu_0)$ *with some small* μ_0 *and a bounded map* $G : X \to X$, *and* $M^* - N^*KN$ *satisfies condition* $(*)$, *then Eq.* $x - KFx = f$ *is solvable for each* $f \in N(H)$.

Proof. Set $A = M^* - N^*F_1N$ and $B = A - N^*FN$. Then A and B satisfy all the conditions of Theorem 3.3. Hence, we have that $M^*h - N^*FNh = M^*k$ for some $h \in H$ by Theorem 3.3. As before, we get that

$$Nh = NC^*FNh + Nk = KFNh + f$$

since $K = NC^*$. Thus, $x - KFx = f$ with $x = Nh \in X$. Next, we have that $Y = N(H)$ is a Banach subspace of X and $I - KF : Y \to Y$, since $N : H \to X$ is continuous and therefore it is closed. Moreover, $S(M^*k)$ is nonempty and compact, and $\operatorname{card} S(M^*k)$ is constant and finite on each connected component of the open set $H \setminus (M^* - N^*FN)(\Sigma_H)$ by Theorem 3.1 By Lemma 4.2, we get $\operatorname{card} S = (I - KF)^{-1}(f) = \operatorname{card} S(M^*k)$ with $f = Nk$. Hence, $\operatorname{card}(I - KF)^{-1}(f)$ is constant and finite on each connected component of $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$. □

Corollary 4.5. *Let* X *be a reflexive embeddable Banach space* ($X \subset H \subset X^*$), $K : X^* \to X$ *be a linear P-positive map and* $F = F_1 + F_2 : X \to X^*$ *be a nonlinear map such that*

(i) F_1 *is odd and positively homogeneous outside some ball in* X.

(ii) *There is an* $M_1 > 0$ *such that if* $x - TF_1Cx = 0$, *then* $||x|| \leq M_1$.

(iii) *The quasinorm of* F_2

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Assume that N^*FN and N^*F_1N are k-ball contractive and k_1-ball contractive, respectively with $k < 1$ and $k_1 < 1$. Then Eq. (1.1) *is approximation solvable in X for each $f \in N(H) \subset X$ w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max ||P_n|| = 1$. Moreover, if*

$$\Sigma_H = \{h \in H \mid M^* - N^*FN \ \ \textit{is not locally invertible at} \ \ h\}$$

then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in C(H)$, and the cardinal number card $(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.*

Proof. Set $A = M^* - N^*F_1N$ and $B = A - N^*FN$. Then A and B are A-proper w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max ||P_n|| = 1$. Hence, Theorem 4.3 applies. □

Corollary 4.6. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a linear P-positive map and $F = F_1 + F_2 : X \to X^*$ be a nonlinear map, such that either $-F_2$ is quasomonotone or N^*F_2N is k ball contractive with $k < 1 - c\mu(K)$ where c is the smallest number such that*
(i) $(F_1x - F_1y, x - y) \le c||x - y||^2$ *for all* $x, y \in X$*;*
(ii) F_1 *is odd and positively homogeneous outside some ball in X;*
(iii) *There is an $M_1 > 0$ such that if $M^*x - N^*F_1Nx = 0$, then $||x|| \le M_1$.*
(iv) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Then Eq. (1.1) *is approximation solvable in X for each $f \in N(H) \subset X$ w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max ||P_n|| = 1$. Moreover, if*

$$\Sigma_H = \{h \in H \mid M^* - N^*FN \ \ \textit{is not locally invertible at} \ \ h\}$$

then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in N(H)$, and the cardinal number card $(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.*

Proof. We claim that the map $M^* - N^*F_1N : H \to H$ is $1 - c\mu(K)$-strongly monotone. Indeed, for $x, y \in H$, we have

$$\begin{aligned}(M^*(x - y) - (N^*(F_1Nx - F_1Ny), x - y)& \\ &= ||x - y||^2 - (F_1Nx - F_1Ny, Nx - Ny) \\ &\ge (1 - c\mu(K))||x - y||^2.\end{aligned}$$

Since N^*F_2N is k-ball contraction, $M^* - N^*FN$ is A-proper w.r.t. Γ (cf. [11]). As before, $M^* - N^*FN$ is A-proper if $-F_2$ is quasimonotone.

Next, let $f \in N(H) \subset X$, $f = Nk$, be fixed. The A-properness of $M^* - N^*FN$ imply that $M^*h - N^*FNh = M^*k$ for some $h \in H$ by Theorem 3.1. As before, we get that $Nh = NC^*FNh + Nk = KFNh + f$ since $K = NC^*$. Thus, $x - KFx = f$ with $x = Nh \in X$. Next, we have that $Y = N(H)$ is a Banach subspace of X and $I - KF : Y \to Y$, since $N : H \to X$ is continuous and therefore it is closed. Moreover, $S(M^*k)$ is nonempty and compact, and $\operatorname{card} S(M^*k)$ is constant and finite on each connected component of the open set $H \setminus (M^* - N^*FN)(\Sigma_H)$ by Theorem 3.1. By Lemma 4.2, we get $\operatorname{card} S = (I - KF)^{-1}(f) = \operatorname{card} S(M^*k)$ with $f = Nk$. Hence, $\operatorname{card}(I - KF)^{-1}(f)$ is constant and finite on each connected component of $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$. $\square$

Corollary 4.7. *Let $K : X^* \to X$ be linear P-positive and $F = F_1 + F_2 : X \to X^*$ be nonlinear such that $-F_1$ and $-F_2$ are quasibounded and either pseudomonotone, or bounded generalized pseudomonotone, or quasimonotone and*

(i) *F_1 is odd and positively homogeneous outside some ball in X.*

(ii) *There is an $M_1 > 0$ such that if $M^* - N^*F_1Nx = 0$, then $||x|| \leq M_1$.*

(iii) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small. Then Eq. (1.1) *is approximation solvable in X for each $f \in N(H) \subset X$ w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max ||P_n|| = 1$. Moreover, if*

$$\Sigma_H = \{h \in H \mid M^* - N^*FN \;\; \textit{is not locally invertible at} \;\; h\},$$

then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in C(H)$, and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.

Proof. Set $A = M^* - N^*F_1N$ and $B = A - M^*F_2N$. We shall show that A and B are of type (S_+) in either case. Suppose first that F_1 is quasimonotone. Let $x_n \rightharpoonup x$ and $\limsup(Ax_n, x_n - x) \leq 0$. Since

$$(Ax_n, x_n - x) = (M^*x_n, x_n - x) - (F_1Nx_n, Nx_n - Nx)$$

and $Nx_n \rightharpoonup Nx$, the quasimonotonicity of $-F_1$ implies that

$$\limsup(M^* x_n, x_n - x) \le 0.$$

Since $||x_n - x||^2 = (M^* x_n, x_n - x) + (M^* x, x - x_n)$, we get that $x_n \to x$. Hence, A is of type (S_+).

Next, we shall show that B is also of type (S_+). Let $x_n \rightharpoonup x$ and $\limsup(Bx_n, x_n - x) \le 0$. Since

$$(Bx_n, x_n - x) = (Ax_n, x_n - x) - (F_2 N x_n, N x_n - Nx)$$

and $Nx_n \rightharpoonup Nx$, the quasimonotonicity of $-F_2$ implies that

$$\limsup(Ax_n, x_n - x) \le 0.$$

Hence, by the (S_+) property of A, we get that $x_n \to x$ and so B is of type (S_+).

Next, we shall show that a bounded generalized pseudomonotone map $-F_1$ is quasimonotone. If not, then for some $x_n \rightharpoonup x$ we have that

$$\limsup(-F_1 x_n, x_n - x) < 0.$$

Since $-F_1$ is bounded, we may assume that $-F_1 x_n \rightharpoonup y$. By the generalized pseudomonotonicity of $-F_1$, it follows that $\lim(-F_1 x_n, x_n - x) = 0$, a contradiction. Hence, $-F_1$ is quasimonotone. Similarly, we get that $-F_1$ is quasimonotone if it is pseudomonotone (see [5]). Thus, A and B are of type (S_+) in all cases and are therefore A-proper w.r.t. Γ. Hence, the theorem follows from Theorem 4.3. □

Next, we shall look at the case when the selfadjoint part A of K is not positive definite. Suppose that A is quasi-positive definite in H, i.e., it has at most a finite number of negative eigenvalues of finite multiplicity. Let U be the subspace spanned by the eigenvectors of A corresponding to these negative eigenvalues of A and $P : H \to U$ be the orthogonal projection onto U. Then P commutes with A, but not necessarily with B, and acts both in X and X^*. Then the operator $|A| = (I - 2P)A$ is easily seen to be positive definite. Hence,we have the decomposition $|A| = DD^*$, where $D = |A|^{1/2} : H \to X$ and $D^* : X^* \to H$.

Following [2] and [23], we call the map K P-quasi-positive if the map $D^{-1}K(D^*)^{-1}$ exists and is bounded in H, and S-quasi-positive if the map $K(D^*)^{-1}$ exists and is bounded in H. Let M and N denote the closure in H

of the the bounded maps $D^{-1}K(D^*)^{-1}$ and $K(D^*)^{-1}$ respectively. Assume that they are both defined on the whole space H. We assume that we have the following decompositions

$$K = DMD^*, \quad K = ND^*.$$

Then we have $N = DM$, $N^* = M^*D^*$, and $\langle Mh, h\rangle = ||h||^2 - 2||Ph||^2$ for all $h \in H$. Define the number

$$\nu(K) = \sup\Big\{\nu \ : \ \nu > 0, \ ||Nh|| \geq (\nu)^{1/2}||Ph||, \ h \in H\Big\}.$$

Note that for a selfadjoint map K, $\nu(K)$ is the absolute value of the largest negative eigenvalue of K. We need (cf. [24])

Lemma 4.3. *Equations* (4.2) *and* (4.3) *are equivalent with f restricted to $N(H)$; each solution h of* (4.3) *determines a solution $x = Nh$ of* (4.2) *and each solution x of* (4.2) *with $f \in N(H)$ determines a solution $h = D^*Fx + k$ of* (4.3) *with $f = Nk$ and $x = Nh$. Moreover, the map $N : S(M^*k) \to S = (I - KF)^{-1}(Nk)$ is bijective.*

Proof. Let h_1 and h_2 be distinct solutions of (4.3). Since $N = DM$ and and $K = ND^*$, we get as before that $x_1 = Nh_1$ and $x_2 = Nh_2$ are solutions of (4.2). They are distinct since

$$\begin{aligned} 0 \neq ||h_1 - h_2||^2 - 2||P(h_1 - h_2)||^2 &= (M(h_1 - h_2), h_1 - h_2) \\ &= (N^*FNh_1 - N^*FNh_2, h_1 - h_2) = (FNh_1 - FNh_2, N(h_1 - h_2)) \\ &= (Fx_1 - Fx_2, x_1 - x_2). \end{aligned}$$

Conversely, let $f \in N(H)$ and x_1 and x_2 be distinct solutions of (4.2). Let $f = Nk$ for some $k \in H$. Set $h_i = D^*Fx_i + k$. Then

$$Nh_i = ND^*Fx_i + Nk = KFx_i + f$$

and so $x_i = Nh_i$. Hence,

$$M^*h_i = M^*D^*FNh_i + M^*k = N^*FNh_i + M^*k,$$

i.e., h_i are solutions of (4.3). They are distinct since $h_1 = h_2$ implies that $x_1 = Nh_1 = Nh_2 = x_2$. These arguments show that $N : S(M^*k) \to S$ is bijective. $\square$

We have the following result when K is P-quasi-positive.

Theorem 4.4. *Let $K : X^* \to X$ be linear P-quasi-positive and $F = F_1 + F_2 : X \to X^*$ be nonlinear such that $M^* - N^* F_1 N$ is an A-proper map w.r.t. Γ and*

(i) *F_1 is odd and positively homogeneous outside some ball in X.*

(ii) *There is an $M_1 > 0$ such that if $M^* - N^* F_1 N x = 0$, then $||x|| \le M_1$.*

(iii) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2 x|| / ||x||$$

is sufficiently small.

Then

(a) *If $M^* - N^* F N$ is A-proper w.r.t. Γ, Eq.* (1.1) *is approximation solvable in X for each $f \in N(H) \subset X$ w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max ||P_n|| = 1$. Moreover, if*

$$\Sigma_H = \{h \in H \mid M^* - N^* F N \ \text{ is not locally invertible at } \ h\},$$

then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in C(H)$, and the cardinal number card $(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^* F N)(\Sigma_H)$ intersected by $N(H)$.*

(b) *Eq.* (1.1) *is solvable for each $f \in N(H)$ if $M^* - N^* F N$ is pseudo A-proper w.r.t. Γ.*

(c) *If $M^* - N^* F N + \mu G$ is A-proper w.r.t. Γ for each $\mu \in (0, \mu_0)$ with some small μ_0 and a bounded map $G : X \to X$, and $M^* - N^* K N$ satisfies condition $(*)$, then Eq. $x - KFx = f$ is solvable for each $f \in N(H)$.*

Proof. Set $A = M^* - N^* F_1 N$ and $B = A - N^* F N$. Then A and B satisfy all conditions of Theorem 3.3. Hence, we have that $M^* h - N^* F N h = M^* k$ for some $h \in H$ by Theorem 3.3. As before, we get that

$$Nh = NC^* F N h + Nk = KFNh + f$$

since $K = NC^*$. Thus, $x - KFx = f$ with $x = Nh \in X$. Next, we have that $Y = N(H)$ is a Banach subspace of X and $I - KF : Y \to Y$, since $N : H \to X$ is continuous and therefore it is closed. Moreover, $S(M^* k)$ is nonempty and compact, and card $S(M^* k)$ is constant and finite on each connected component of the open set $H \setminus (M^* - N^* F N)(\Sigma_H)$ by Theorem 3.1. By Lemma 4.3, we get

$$\operatorname{card} S = (I - KF)^{-1}(f) = \operatorname{card} S(M^* k) \ \text{ with } \ f = Nk.$$

Hence, card $(I - KF)^{-1}(f)$ is constant and finite on each connected component of $H \setminus (M^* - N^* F N)(\Sigma_H)$ intersected by $N(H)$. $\square$

Corollary 4.8. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a linear P-quasi-positive map and $F = F_1 + F_2 : X \to X^*$ be a nonlinear map such that*

(i) *F_1 is odd and positively homogeneous outside some ball in X.*

(ii) *There is an $M_1 > 0$ such that if $x - TF_1Cx = 0$, then $||x|| \le M_1$.*

(iii) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Assume that N^*FN and N^*F_1N are k-ball contractive and k_1-ball contractive, respectively with $k < 1$ and $k_1 < 1$. Then Eq. (1.1) is approximation solvable in X for each $f \in N(H) \subset X$ w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max ||P_n|| = 1$. Moreover, if

$$\Sigma_H = \{h \in H \mid M^* - N^*FN \ \ \text{is not locally invertible at} \ \ h\},$$

then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in C(H)$, and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.

Proof. Set $A = M^* - N^*F_1N$ and $B = A - N^*FN$. Then A and B are A-proper w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max ||P_n|| = 1$. Hence, Theorem 4.4 applies. □

Corollary 4.9. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a linear continuous P-quasi-positive map with $c\nu(K) < -1$. and $F = F_1 + F_2 : X \to X^*$ be a nonlinear map, such that either $-F_2$ is quasomonotone or N^*F_2N is k ball contractive with $k < -(1 + c\nu(K))$ where c is the smallest number such that*

(i) *$(F_1x - F_1y, x - y) \le c||x - y||^2$ for all $x, y \in X$*

(i) *F_1 is odd and positively homogeneous outside some ball in X*

(iii) *There is an $M_1 > 0$ such that if $M^*x - N^*F_1Nx = 0$, then $||x|| \le M_1$.*

(iv) *The quasinorm of F_2*

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Then Eq. (1.1) *is approximation solvable in* X *for each* $f \in N(H) \subset X$ *w.r.t. a projection scheme* $\Gamma = \{X_n, P_n\}$ *for* X, $\delta = \max ||P_n|| = 1$. *Moreover, if*

$$\Sigma_H = \{h \in H \mid M^* - N^*FN \text{ is not locally invertible at } h\}$$

then $(I - KF)^{-1}(\{f\})$ *is compact for each* $f \in N(H)$, *and the cardinal number* $\operatorname{card}(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set* $H \setminus (M^* - N^*FN)(\Sigma_H)$ *intersected by* $N(H)$.

Proof. We claim that the map $M^* - P - N^*F_1N : H \to H$ is $-(1 + c\nu(K)$-strongly monotone. Indeed, for $x, y \in H$, we have

$$\begin{aligned}(M^*(x-y) &- (N^*(F_1Nx - F_1Ny), x - y)\\ &= ||x-y||^2 - 2||P(x-y)||^2 - (F_1Nx - F_1Ny, Nx - Ny)\\ &\geq ||x-y||^2 - 2||P(x-y)||^2 - c||Nx - Ny||^2)\\ &\geq ||x-y||^2 - 2||P(x-y)||^2 - c\nu(K)||P(x-y)||^2)\\ &\geq -(1 + c\nu(K))||x-y||^2.\end{aligned}$$

Since N^*F_2N is k-ball contraction, $M^* - N^*FN$ is A-proper w.r.t. Γ (cf. [11]). As before, we get that $M^* - N^*FN$ is A-proper when $-F_2$ is quasimonotone. Hence, Theorem 4.4 applies. □

Corollary 4.10. *Let* $K : X^* \to X$ *be linear P-quasi-positive and* $F = F_1 + F_2 : X \to X^*$ *be nonlinear such that* $-F_1$ *and* $-F_2$ *are quasibounded and either pseudomonotone, or bounded generalized pseudomonotone, or quasimonotone and*

(i) F_1 *is odd and positively homogeneous outside some ball in* X.

(ii) *There is an* $M_1 > 0$ *such that if* $M^* - N^*F_1Nx = 0$, *then* $||x|| \leq M_1$.

(iii) *The quasinorm of* F_2

$$|F_2| = \limsup_{||x|| \to +\infty} ||F_2x||/||x||$$

is sufficiently small.

Then Eq. (1.1) *is approximation solvable in* X *for each* $f \in N(H) \subset X$ *w.r.t. a projection scheme* $\Gamma = \{X_n, P_n\}$ *for* X, $\delta = \max ||P_n|| = 1$. *Moreover, if*

$$\Sigma_H = \{h \in H \mid M^* - N^*FN \text{ is not locally invertible at } h\},$$

then $(I - KF)^{-1}(\{f\})$ *is compact for each* $f \in C(H)$*, and the cardinal number* $\operatorname{card}(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set* $H \setminus (M^* - N^*FN)(\Sigma_H)$ *intersected by* $N(H)$*.*

Proof. Set $A = M^* - N^*F_1N$ and $B = A - N^*F_2N$. We shall show that $A + 2P$ and $B + 2P$ are of type (S_+) in either case. Suppose first that F_1 is quasimonotone. Let $x_n \rightharpoonup x$ and $\limsup(Ax_n + 2Px_n, x_n - x) \leq 0$. Since

$$(Ax_n + 2Px_n, x_n - x) = (M^*x_n, x_n - x) - (F_1Nx_n, Nx_n - Nx)$$

and $Nx_n \rightharpoonup Nx$, the quasimonotonicity of $-F_1$ implies that

$$\limsup(M^*x_n, x_n - x) \leq 0.$$

Since $||x_n - x||^2 = (M^*x_n, x_n - x) + (M^*x, x - x_n)$, we get that $x_n \to x$. Hence, $A + 2P$ is of type (S_+).

Next, we shall show that $B + 2P$ is also of type (S_+). Let $x_n \rightharpoonup x$ and $\limsup(Bx_n + 2Px_n, x_n - x) \leq 0$. Since

$$(Bx_n + 2Px_n, x_n - x) = (Ax_n + 2Px_n, x_n - x) - (F_2Nx_n, Nx_n - Nx)$$

and $Nx_n \rightharpoonup Nx$, the quasimonotonicity of $-F_2$ implies that

$$\limsup(Ax_n + 2Px_n, x_n - x) \leq 0.$$

Hence, by the (S_+) property of $A + 2P$, we get that $x_n \to x$ and so $B + 2P$ is of type (S_+).

Next, as we have shown above, a bounded generalized pseudomonotone map $-F_1$ is quasimonotone, as is a pseudomonotone map. Thus, $A + 2P$ and $B + 2P$ are of type (S_+) in all cases and are therefore A-proper w.r.t. Γ. Since P is compact, we have that A and B are also A-proper w.r.t. Γ. Hence, the theorem follows from Theorem 4.4. $\square$

REFERENCES

1. H. Amann: *Existence theorems for equations of Hammerstein type.* Applicable Anal. **1** (1972), 385–397.
2. J.A. Appell, E. de Pascale, P.P. Zabrejko: *On the unique solvability of Hammerstein integral equations with non-symmetric kernels.* In Progress in Nonlinear Differential Equations and Their Applications, vol. **40** (2000), 27–34.

3. F.E. Browder: *Nonlinear functional analysis and nonlinear integral equations of Hammerstein and Urisohn type.* In Zarantonello, E. [ed.] (1971), 425–500.
4. C. Dolph: *Nonlinear integral equations of Hammerstein type.* Trans. Amer. Math. Soc. **66** (1949), 289–307.
5. P.M. Fitzpatrick: *Surjectivity results for nonlinear mappings from a Banach space to its dual.* Math. Ann. **204** (1973), 177–188.
6. P. Hess: *On nonlinear mappings of monotone type homotopic to odd operators.* J. Funct. Anal. **11** (1972), 138–167.
7. R.I. Kachurovskii: *On the Fredholm theory for nonlinear operator equations.* Dokl. Akad. Nauk SSSR **192** (1970), 969–972.
8. A.M. Krasnosel'skii: *Asymptotics of Nonlinearities and Operator Equations.* Birkhauser, 1995.
9. M.A. Krasnosel'skii: *Topological Methods in the Theory of Nonlinear Integral Equations.* MacMillan, N.Y., 1964.
10. M.A. Krasnoselskii and P.O. Zabreiko: *Geometrical Methods of Nonlinear Analysis.* Springer-Verlag, 1984.
11. P.S. Milojević: *A generalization of the Leray-Schauder theorem and surjectivity results for multivalued A-proper and pseudo A-proper mappings.* Nonlinear Anal., TMA **1** (1977), 263–276.
12. P.S. Milojević: *Some generalizations of the first Fredholm theorem to multivalued A-proper mappings with applications to nonlinear elliptic equations.* J. Math. Anal. Appl. **65** (1978), 468–502.
13. P.S. Milojević: *Fredholm alternatives and surjectivity results for multivalued A-proper and condensing mappings with applications to nonlinear integral and differential ewquations.* Czech. Math. J. **30 (105)** (1980), 387–417.
14. P.S. Milojević: *Fredholm theory and semilinear equations without resonance involving noncompact perturbations, I, II. Applications* Publications de l'Institut Math. **42** (1987), 71–82 and 83–95.
15. P.S. Milojević: *Solvability of semilinear operator equations and applications to semilinear hyperbolic equations.* In: Nonlinear Functoinal Analysis, (P.S. Milojević, ed.) vol. **121** Marcel Dekker, 1989, pp. 95–178.
16. P.S. Milojević: *Solvability of operator equations and periodic solutions of semilinear hyperbolic equations.* Publ. Inst. Math. **48** (1990), 133–168.
17. P.S. Milojević: *Approximation-solvability of nonlinear equations and applications.* In: Fourier Analysis (W. Bray, P.S. Milojević, C.V. Stanojević, eds.), Lect. Notes Pure Appl. Math., Marcel Dekker Inc., N.Y. vol. **157** (1994), 311–373.
18. P.S. Milojević: *On the dimension and the index of the solution set of nonlinear equations.* Trans. Amer. Math. Soc. **347** (1995), 835–856.

19. P.S. Milojević: *Approximation solvability of Hammerstein equations.* Publ. Inst. Math. **58** (1995), 71–84.

20. P.S. Milojević: *Approximation-Solvability of Semilinear Equations and Applications.* In Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A.G. Kartsatos, ed.), Lect. Notes Pure Appl. Math., Marcel Dekker Inc., NY. vol. **178** (1996), 149–208,

21. P.S. Milojević: *Existence and the number of solutions of nonresonant semilinear equations and applications to boundary value problems.* Math. Computer Modelling **32** (2000), 1395–1416.

22. P.S. Milojević: *Continuation theory for A-proper mappings and their uniform limits and nonlinear perturbations of Fredholm mappings.* SIMAA **4** (2002), 311–359.

23. P.S. Milojević: *On the unique approximation solvability of Hammerstein equations.* Electronic J. Diff. Equations **103** (2002), 1–15.

24. P.S. Milojević: *Solvability and the number of solutions of Hammerstein equations.* Electronic J. Diff. Equations **54** (2004), 1–25.

25. J. Necas: *Fredholm alternative for nonlinear operators and applications to partial differential equations and integral equations.* Cas. Pest. Mat. **9** (1972), 15–21.

26. W.V. Petryshyn: *Fredholm alternative for nonlinear A-proper mappings with applications to nonlinear elliptic boundary value problems.* J. Funct. Anal. **18** (1975), 288–317.

27. W.V. Petryshyn: *On the approximation - solvability of equations involving A-proper and pseudo A-proper mappings.* Bull. Amer. Math. Soc. **81** (1975), 223–312.

28. S.I. Pohozaev: *On the solvability of nonlinear equations involving odd operators.* Funktional Anal. i Prilozenia, **1(3)** (1967), 66–73.

29. M.M. Vainberg: *Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations.* Holsted Press, NY., 1973.

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