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REFINEMENTS OF BUZANO'S AND KUREPA'S INEQUALITIES IN INNER PRODUCT SPACES

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Abstract. Refinements of Buzano's and Kurepa's inequalities in inner product spaces and applications for discrete and integral inequalities improving the celebrated Cauchy-Buniakovsky-Schwarz result are given.

1. Introduction

In [3], M.L. Buzano obtained the following extension of the celebrated Schwarz's inequality in a real or complex inner product space $(H; \langle \cdot, \cdot \rangle)$:

$$|\langle a, x\rangle \langle x, b\rangle| \leq \frac{1}{2} [\|a\| \cdot \|b\| + |\langle a, b\rangle|] \|x\|^2, \tag{1.1}$$

for any $a, b, x \in H$. It is clear that for $a = b$, the above inequality becomes the standard Schwarz inequality

$$|\langle a, x\rangle|^2 \leq \|a\|^2 \|x\|^2, \qquad a, x \in H; \tag{1.2}$$

with equality if and only if there exists a scalar $\lambda \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) such that $x = \lambda a$.

It might be useful to observe that, out of (1.1), one may get the following discrete inequality:

$$\left| \sum_{i=1}^{n} p_i a_i \overline{x_i} \sum_{i=1}^{n} p_i x_i \overline{b_i} \right| \leq \frac{1}{2} \left[\left(\sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 \right)^{1/2} + \left| \sum_{i=1}^{n} p_i a_i \overline{b_i} \right| \right] \sum_{i=1}^{n} p_i |x_i|^2, \tag{1.3}$$

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where $p_i \geq 0$, $a_i, x_i, b_i \in \mathbb{C}$, $i \in \{1, \ldots, n\}$.

If one takes in (1.3) $b_i = \overline{a_i}$ for $i \in \{1, \ldots, n\}$, then one obtains

$$\left| \sum_{i=1}^{n} p_i a_i \overline{x_i} \sum_{i=1}^{n} p_i a_i x_i \right| \leq \frac{1}{2} \left[\sum_{i=1}^{n} p_i |a_i|^2 + \left| \sum_{i=1}^{n} p_i a_i^2 \right| \right] \sum_{i=1}^{n} p_i |x_i|^2, \tag{1.4}$$

for any $p_i \geq 0$, $a_i, x_i, b_i \in \mathbb{C}$, $i \in \{1, \ldots, n\}$.

Note that, if x_i, $i \in \{1, \ldots, n\}$ are real numbers, then out of (1.4), we may deduce the de Bruijn refinement of the celebrated Cauchy-Bunyakovsky-Schwarz inequality [2]

$$\left| \sum_{i=1}^{n} p_i x_i z_i \right|^2 \leq \frac{1}{2} \sum_{i=1}^{n} p_i x_i^2 \left[\sum_{i=1}^{n} p_i |z_i|^2 + \left| \sum_{i=1}^{n} p_i z_i^2 \right| \right], \tag{1.5}$$

where $z_i \in \mathbb{C}$, $i \in \{1, \ldots, n\}$. In this way, Buzano's result may be regarded as a generalisation of de Bruijn's inequality.

Similar comments obviously apply for integrals, but, for the sake of brevity we do not mention them here.

The aim of the present paper is to establish some related results as well as a refinement of Buzano's inequality for real or complex inner product spaces. An improvement of Kurepa's inequality for the complexification of a real inner product and the corresponding applications for discrete and integral inequalities are also provided.

2. The Results

The following result may be stated.

Theorem 2.1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. For all $\alpha \in \mathbb{K} \setminus \{0\}$ and $x, a, b \in H$, $\alpha \neq 0$, one has the inequality*

$$\left| \frac{\langle a, x \rangle \langle x, b \rangle}{\|x\|^2} - \frac{\langle a, b \rangle}{\alpha} \right| \leq \frac{\|b\|}{|\alpha| \|x\|} \left[|\alpha - 1|^2 |\langle a, x \rangle|^2 + \|x\|^2 \|a\|^2 - |\langle a, x \rangle|^2 \right]. \tag{2.1}$$

The case of equality holds in (2.1) if and only if there exists a scalar $\lambda \in \mathbb{K}$ so that

$$\alpha \cdot \frac{\langle a, x \rangle}{\|x\|^2} x = a + \lambda b. \tag{2.2}$$

Proof. Using Schwarz's inequality, we have that

$$\left|\left\langle \alpha\cdot\frac{\langle a,x\rangle}{\|x\|^2}x-a,b\right\rangle\right|^2\leq\left\|\alpha\cdot\frac{\langle a,x\rangle}{\|x\|^2}x-a\right\|^2\|b\|^2 \tag{2.3}$$

and since

$$\begin{aligned}\left\|\alpha\cdot\frac{\langle a,x\rangle}{\|x\|^2}x-a\right\|^2 &= |\alpha|^2\frac{|\langle a,x\rangle|^2}{\|x\|^2}-2\frac{|\langle a,x\rangle|^2}{\|x\|^2}\mathrm{Re}\alpha+\|a\|^2\\ &=\frac{|\alpha-1|^2|\langle a,x\rangle|^2+\|x\|^2\|a\|^2-|\langle a,x\rangle|^2}{\|x\|^2}\end{aligned}$$

and

$$\left\langle \alpha\cdot\frac{\langle a,x\rangle}{\|x\|^2}x-a,b\right\rangle=\alpha\left[\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}-\frac{\langle a,b\rangle}{\alpha}\right],$$

hence by (2.1) we deduce the desired inequality (2.1).

The case of equality is obvious from the above considerations related to the Schwarz's inequality (1.2). □

Remark 2.1. Using the continuity property of the modulus, i.e., $||z|-|u||\leq|z-u|$, $z,u\in\mathbb{K}$, we have:

$$\left|\frac{|\langle a,x\rangle\langle x,b\rangle|}{\|x\|^2}-\frac{|\langle a,b\rangle|}{|\alpha|}\right|\leq\left|\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}-\frac{\langle a,b\rangle}{\alpha}\right|. \tag{2.4}$$

Therefore, by (2.1) and (2.4), one may deduce the following double inequality:

$$\begin{aligned}&\frac{1}{|\alpha|}\left[|\langle a,b\rangle|-\frac{\|b\|}{\|x\|}\left[\left(|\alpha-1|^2|\langle x,a\rangle|^2+\|x\|^2\|a\|^2-|\langle a,x\rangle|^2\right)^{1/2}\right]\right]\leq\frac{|\langle a,x\rangle\langle x,b\rangle|}{\|x\|^2}\\ &\qquad\leq\frac{1}{|\alpha|}\left[|\langle a,b\rangle|+\frac{\|b\|}{\|x\|}\right]\left[\left(|\alpha-1|^2|\langle x,a\rangle|^2+\|x\|^2\|a\|^2-|\langle x,a\rangle|^2\right)^{1/2}\right],\end{aligned}$$

for each $\alpha\in\mathbb{K}\setminus\{0\}$, $a,b,x\in H$ and $x\neq0$.

It is obvious that, out of (2.1), we can obtain various particular inequalities. We mention in the following a class of these which is related to Buzano's result (1.1).

Corollary 2.1. *Let $a,b,x\in H$, $x\neq0$ and $\eta\in\mathbb{K}$ with $|\eta|=1$, $\mathrm{Re}\,\eta\neq-1$. Then we have the inequality:*

$$\left|\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}-\frac{\langle a,b\rangle}{1+\eta}\right|\leq\frac{\|a\|\,\|b\|}{\sqrt{2}\sqrt{1+\mathrm{Re}\,\eta}}, \tag{2.5}$$

and, in particular, for $\eta = 1$, the inequality:

$$\left| \frac{\langle a, x\rangle \langle x, b\rangle}{\|x\|^2} - \frac{\langle a, b\rangle}{2} \right| \leq \frac{\|a\| \|b\|}{2}. \tag{2.6}$$

Proof. It follows by Theorem 2.1 on choosing $\alpha = 1 + \eta$ and we omit the details. □

Remark 2.2. Using the continuity property of modulus, we get from (2.5) that:

$$\frac{|\langle a, x\rangle \langle x, b\rangle|}{\|x\|^2} \leq \frac{|\langle a, b\rangle| + \|a\| \|b\|}{\sqrt{2}\sqrt{1 + \operatorname{Re}\eta}}, \qquad |\eta| = 1, \quad \operatorname{Re}\eta \neq -1,$$

which provides, as the best possible inequality, the above result due to Buzano (1.1).

As noted by M. Fujii and F. Kubo in [4], where they provided a simple proof of (1.1) by utilizing orthogonal projection arguments, the case of equality holds in (1.1) if

$$x = \begin{cases} \alpha \left(\dfrac{a}{\|a\|} + \dfrac{\langle a, b\rangle}{|\langle a, b\rangle|} \cdot \dfrac{b}{\|b\|} \right), & \text{when} \quad \langle a, b\rangle \neq 0 \\ \alpha \left(\dfrac{a}{\|a\|} + \beta \cdot \dfrac{b}{\|b\|} \right), & \text{when} \quad \langle a, b\rangle = 0, \end{cases}$$

where $\alpha, \beta \in \mathbb{K}$.

Remark 2.3. If the space is real, then the inequality (2.1) is obviously equivalent to:

$$\frac{\langle a, b\rangle}{\alpha} - \frac{\|b\|}{|\alpha| \|x\|} \left[(\alpha - 1)^2 \langle a, x\rangle^2 + \|x\|^2 \|a\|^2 - \langle a, x\rangle^2 \right]^{1/2} \leq \frac{\langle a, x\rangle \langle x, b\rangle}{\|x\|^2}$$
$$\leq \frac{\langle a, b\rangle}{\alpha} + \frac{\|b\|}{|\alpha| \|x\|} \left[(\alpha - 1)^2 \langle a, x\rangle^2 + \|x\|^2 \|a\|^2 - \langle a, x\rangle^2 \right]^{1/2}$$

for any $\alpha \in \mathbb{R} \setminus \{0\}$ and $a, b, x \in H$, $x \neq 0$.

If in the previous inequalities we take $\alpha = 2$, then we get

$$\frac{1}{2} \left[\langle a, b\rangle - \|a\| \|b\| \right] \|x\|^2 \leq \langle a, x\rangle \langle x, b\rangle \leq \frac{1}{2} \left[\langle a, b\rangle + \|a\| \|y\| \right] \|x\|^2, \tag{2.7}$$

which apparently, as mentioned by T. Precupanu in [7], has been obtained independently of Buzano, by U. Richard in [8].

In [6], Pečarić gave a simple direct proof of (2.7) without mentioning the work of either Buzano or Richard, but tracked down the result, in a particular form, to an earlier paper due to C. Blatter [1].

Obviously, the following refinement of Buzano's result may be stated.

Corollary 2.2. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex inner product space and $a,b,x\in H$. Then*

$$\begin{aligned}(2.8)\qquad |\langle a,x\rangle\langle x,b\rangle| &\le \left|\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\langle a,b\rangle\|x\|^2\right|+\frac{1}{2}|\langle a,b\rangle|\,\|x\|^2\\ &\le \frac{1}{2}\left[\|a\|\,\|b\|+|\langle a,b\rangle|\right]\|x\|^2.\end{aligned}$$

Proof. The first inequality in (2.8) follows by the triangle inequality for the modulus $|\cdot|$. The second inequality is merely (2.6) in which we added the same quantity to both sides. □

Remark 2.4. For $\alpha=1$, we deduce from (2.1) the following inequality:

$$(2.9)\qquad \left|\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}-\langle a,b\rangle\right|\le\frac{\|b\|}{\|x\|}\left[\|x\|^2\|a\|^2-|\langle a,x\rangle|^2\right]^{1/2}$$

for any $a,b,x\in H$ with $x\neq 0$.

If the space is real, then (2.9) is equivalent to

$$\begin{aligned}\langle a,b\rangle-\frac{\|b\|}{\|x\|}\left[\|x\|^2\|a\|^2-|\langle a,x\rangle|^2\right]^{1/2} &\le \frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2}\\ &\le \frac{\|b\|}{\|x\|}\left[\|x\|^2\|a\|^2-|\langle a,x\rangle|^2\right]^{1/2}+\langle a,b\rangle,\end{aligned}$$

which is similar to Richard's inequality (2.7).

3. Applications to Kurepa's Inequality

In 1960, N.G. de Bruijn [2] obtained the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality:

$$(3.1)\qquad \left|\sum_{i=1}^n a_i z_i\right|^2\le\frac{1}{2}\sum_{i=1}^n a_i^2\left[\sum_{i=1}^n|z_i|^2+\left|\sum_{i=1}^n z_i^2\right|\right],$$

provided that a_i are real numbers while z_i are complex for each $i\in\{1,\dots,n\}$.

In an effort to extend this result to inner products, S. Kurepa [5] considered the following setting:

Let H be a real inner product space with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. The *complexification* $H_{\mathbb{C}}$ of H is defined as a complex linear space $H \times H$ of all ordered pairs (x, y) $(x, y \in H)$ endowed with the operations

$$\begin{aligned} (x, y) + (x', y') &:= (x + x', y + y'), && x, x', y, y' \in H; \\ (\sigma + i\tau) \cdot (x, y) &:= (\sigma x - \tau y, \tau x + \sigma y), && x, y \in H \text{ and } \sigma, \tau \in \mathbb{R}. \end{aligned}$$

On $H_{\mathbb{C}}$ one can canonically consider the *scalar product* $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ defined by:

$$\langle z, z' \rangle_{\mathbb{C}} := \langle x, x' \rangle + \langle y, y' \rangle + i \left[\langle y, x' \rangle - \langle x, y' \rangle \right]$$

where $z = (x, y)$, $z' = (x', y') \in H_{\mathbb{C}}$. Obviously,

$$\|z\|_{\mathbb{C}}^2 = \|x\|^2 + \|y\|^2,$$

where $z = (x, y)$.

The conjugate of a vector $z = (x, y) \in H_{\mathbb{C}}$ is defined by $\bar{z} := (x, -y)$.

It is easy to see that the elements of $H_{\mathbb{C}}$ under defined operations behave as formal "complex" combinations $x + iy$ with $x, y \in H$. Because of this, we may write $z = x + iy$ instead of $z = (x, y)$. Thus, $\bar{z} = x - iy$.

In [5], S. Kurepa proved the following generalisation of the de Bruijn result:

Theorem 3.1. *Let $(H; \langle \cdot, \cdot \rangle)$ be a real inner product space and $(H_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$ its complexification. Then for any $a \in H$ and $z \in H_{\mathbb{C}}$, one has the following refinement of Schwarz's inequality*

$$|\langle a, z \rangle_{\mathbb{C}}|^2 \leq \frac{1}{2} \|a\|^2 \left[\|z\|_{\mathbb{C}}^2 + |\langle z, \bar{z} \rangle_{\mathbb{C}}| \right] \leq \|a\|^2 \|z\|_{\mathbb{C}}^2, \tag{3.2}$$

where $\bar{z}$ denotes the conjugate of $z \in H_{\mathbb{C}}$.

As consequences of this general result, Kurepa noted the following integral, respectively, discrete inequality:

Corollary 3.1. *Let (S, Σ, μ) be a positive measure space and let $a, z \in L_2(S, \Sigma, \mu)$, the Hilbert space of complex-valued $2 - \mu-$integrable functions defined on S. If a is a real function and z is a complex function, then*

$$\begin{aligned} &\left| \int_S a(t) z(t) \, d\mu(t) \right|^2 \\ &\leq \frac{1}{2} \int_S a^2(t) \, d\mu(t) \left(\int_S |z(t)|^2 \, d\mu(t) + \left| \int_S z^2(t) \, d\mu(t) \right| \right). \end{aligned} \tag{3.3}$$

Corollary 3.2. *If $a_1, \dots, a_n$ are real numbers, $z_1, \dots, z_n$ are complex numbers and (A_{ij}) is a positive definite real matrix of dimension $n \times n$, then*

$$\left|\sum_{i,j=1}^{n} A_{ij} a_i z_j\right|^2 \leq \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} a_i a_j \left(\sum_{i,j=1}^{n} A_{ij} z_i \overline{z_j} + \left| \sum_{i,j=1}^{n} A_{ij} z_i \overline{z_j} \right| \right). \tag{3.4}$$

The following refinement of Kurepa's result may be stated.

Theorem 3.2. *Let $(H; \langle \cdot, \cdot \rangle)$ be a real inner product space and $(H_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$ its complexification. Then for any $e \in H$ and $w \in H_{\mathbb{C}}$, one has the inequality:*

$$\begin{aligned} |\langle w, e\rangle_{\mathbb{C}}|^2 &\leq \left| \langle w, e\rangle_{\mathbb{C}}^2 - \frac{1}{2} \langle w, \bar{w}\rangle_{\mathbb{C}} \|e\|^2 \right| + \frac{1}{2} |\langle w, \bar{w}\rangle_{\mathbb{C}}| \|e\|^2 \\ &\leq \frac{1}{2} \|e\|^2 \left[\|w\|_{\mathbb{C}}^2 + |\langle w, \bar{w}\rangle_{\mathbb{C}}| \right]. \end{aligned} \tag{3.5}$$

Proof. If we apply Corollary 3.2 for $(H_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$ and $x = e \in H$, $a = w$ and $b = \bar{w}$, then we have

$$\begin{aligned} |\langle w, e\rangle_{\mathbb{C}} \langle e, \bar{w}\rangle_{\mathbb{C}}| &\leq \left| \langle w, e\rangle_{\mathbb{C}} \langle e, \bar{w}\rangle_{\mathbb{C}} - \frac{1}{2} \langle w, \bar{w}\rangle_{\mathbb{C}} \|e\|^2 \right| + \frac{1}{2} |\langle w, \bar{w}\rangle_{\mathbb{C}}| \|e\|^2 \\ &\leq \frac{1}{2} \|e\|^2 \left[\|w\|_{\mathbb{C}} \|\bar{w}\|_{\mathbb{C}} + |\langle w, \bar{w}\rangle_{\mathbb{C}}| \right] \end{aligned} \tag{3.6}$$

Now, if we assume that $w = (x, y) \in H_{\mathbb{C}}$, then, by the definition of $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, we have

$$\begin{aligned} \langle w, e\rangle_{\mathbb{C}} &= \langle (x, y), (e, 0)\rangle_{\mathbb{C}} \\ &= \langle x, e\rangle + \langle y, 0\rangle + i [\langle y, e\rangle - \langle x, 0\rangle] \\ &= \langle e, x\rangle + i \langle e, y\rangle, \end{aligned}$$

$$\begin{aligned} \langle e, \bar{w}\rangle_{\mathbb{C}} &= \langle (e, 0), (x, -y)\rangle_{\mathbb{C}} \\ &= \langle e, x\rangle + \langle 0, -y\rangle + i [\langle 0, x\rangle - \langle e, -y\rangle] \\ &= \langle e, x\rangle + i \langle e, y\rangle = \langle w, e\rangle_{\mathbb{C}} \end{aligned}$$

and

$$\|\bar{w}\|_{\mathbb{C}}^2 = \|x\|^2 + \|y\|^2 = \|w\|_{\mathbb{C}}^2.$$

Therefore, by (3.6), we deduce the desired result (3.5). □

Denote by $\ell_{\rho}^{2}(\mathbb{C})$ the Hilbert space of all complex sequences $z=(z_i)_{i\in\mathbb{N}}$ with the property that for $\rho_i\geq 0$ with $\sum_{i=1}^{+\infty}\rho_i=1$ we have $\sum_{i=1}^{+\infty}\rho_i|z_i|^2<+\infty$. If $a=(a_i)_{i\in\mathbb{N}}$ is a sequence of real numbers such that $a\in\ell_{\rho}^{2}(\mathbb{C})$, then for any $z\in\ell_{\rho}^{2}(\mathbb{C})$ we have the inequality:

$$\begin{aligned}
&\left|\sum_{i=1}^{+\infty}\rho_i a_i z_i\right|^2\\
&\quad\leq\left|\left(\sum_{i=1}^{+\infty}\rho_i a_i z_i\right)^2-\frac{1}{2}\sum_{i=1}^{+\infty}\rho_i a_i^2\sum_{i=1}^{+\infty}\rho_i z_i^2\right|+\frac{1}{2}\sum_{i=1}^{+\infty}\rho_i a_i^2\left|\sum_{i=1}^{+\infty}\rho_i z_i^2\right|\\
&\quad\leq\frac{1}{2}\sum_{i=1}^{+\infty}\rho_i a_i^2\left[\sum_{i=1}^{+\infty}\rho_i|z_i|^2+\left|\sum_{i=1}^{+\infty}\rho_i z_i^2\right|\right].
\end{aligned}$$

Similarly, if by $L_{\rho}^{2}(S,\Sigma,\mu)$ we understand the Hilbert space of all complex-valued functions $f:S\to\mathbb{C}$ with the property that for the $\mu-$measurable function $\rho\geq 0$ with $\int_S\rho(t)\,d\mu(t)=1$ we have

$$\int_S\rho(t)|f(t)|^2\,d\mu(t)<+\infty,$$

then for a real function $a\in L_{\rho}^{2}(S,\Sigma,\mu)$ and any $f\in L_{\rho}^{2}(S,\Sigma,\mu)$, we have the inequalities

$$\begin{aligned}
&\left|\int_S\rho(t)\,a(t)\,f(t)\,d\mu(t)\right|^2\\
&\leq\left|\left(\int_S\rho(t)\,a(t)\,f(t)\,d\mu(t)\right)^2-\frac{1}{2}\int_S\rho(t)\,f^2(t)\,d\mu(t)\int_S\rho(t)\,a^2(t)\,d\mu(t)\right|\\
&\qquad+\frac{1}{2}\left|\int_S\rho(t)\,f^2(t)\,d\mu(t)\right|\int_S\rho(t)\,a^2(t)\,d\mu(t)\\
&\leq\frac{1}{2}\int_S\rho(t)\,a^2(t)\,d\mu(t)\left(\int_S\rho(t)|f(t)|^2\,d\mu(t)+\left|\int_S\rho(t)\,f^2(t)\,d\mu(t)\right|\right).
\end{aligned}$$

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