FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 20 (2005), 57–64

ON THE BEST APPROXIMATION IN SMOOTH AND UNIFORMLY CONVEX REAL BANACH SPACE

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Abstract. Let X be a smooth and uniformly convex real Banach space and g functional defined as (2). The best approximation, ax, the vector y with vectors from $[x] = \operatorname{span}\{x\}$ $(P_{[x]}y = ax)$ is characterized with the equation $g(y - ax, x) = 0$. In certain Banach space, this equation, is possibly resolve for a. The above idea applied for $P_M y$, where M is a $n-$dimensional subspace of X.

Let X be a real normed space and $S(X)$ unit sphere in X. It is well known that the functional

$$g(x,y) := \frac{\|x\|}{2}\left(\lim_{t\to -0}\frac{\|x+ty\|-\|x\|}{t} + \lim_{t\to +0}\frac{\|x+ty\|-\|x\|}{t}\right) \tag{1}$$

always exists on X^2. If X is smooth then (1) reduces to

$$g(x,y) = \|x\| \lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \qquad (x, y \in X). \tag{2}$$

In this case the functional g is linear in second argument and it has the properties:

$$g(\alpha x, y) = \alpha g(x,y) \ \ (\alpha \in \mathbb{R}), \quad g(x,x) = \|x\|^2, \quad |g(x,y)| \le \|x\| \ \|y\|$$

and $g(x,y) = 0 \Leftrightarrow x \perp y \quad (x \perp y \Leftrightarrow \|x\| \le \|x + ty\|$ for all $t \in \mathbb{R})$.
(More at the functional g see in [3] and [4]).

If X is an inner product space (i.p. space) with i.p. $(\cdot,\cdot)$, we have

$$g(x,y) = (x,y) \quad (x, y \in X).$$

Received , 2004.
2000 *Mathematics Subject Classification.* Primary 46B20, 46C15, 51K05.

Lemma 1. *Let X be smooth and $x, y \in X$. Then the following assertions are valid:*

$$1^\circ \quad \left(\forall \alpha, \alpha' \in \mathbb{R}\setminus\{0\}\right)\ g\left(\alpha x + \beta y, \alpha' x + \beta' y\right) = 0 \Leftrightarrow \|\alpha x + \beta y\|^2 = \frac{\alpha'\beta - \alpha\beta'}{\alpha'} g\left(\alpha x + \beta y, y\right),$$

$$2^\circ \quad \left(\forall \beta, \beta' \in \mathbb{R}\setminus\{0\}\right)\ g\left(\alpha x + \beta y, \alpha' x + \beta' y\right) = 0 \Leftrightarrow \|\alpha x + \beta y\|^2 = \frac{\alpha\beta' - \beta\alpha'}{\beta'} g\left(\alpha x + \beta y, x\right).$$

Proof. 1° Using the properties of g we get equivalencies:

$$\begin{aligned}
& g\left(\alpha x + \beta y, \alpha' x + \beta' y\right) = 0 \\
\Leftrightarrow\quad & \frac{\alpha'}{\alpha} g\left(\alpha x + \beta y, \alpha x + \frac{\alpha\beta'}{\alpha'} y\right) = 0 \\
\Leftrightarrow\quad & g\left(\alpha x + \beta y, \alpha x + \beta y + \frac{\alpha\beta'}{\alpha'} y - \beta y\right) = 0 \\
\Leftrightarrow\quad & g\left(\alpha x + \beta y, \alpha x + \beta y\right) + g\left(\alpha x + \beta y, -\left(\beta - \frac{\alpha\beta'}{\alpha'}\right) y\right) = 0 \\
\Leftrightarrow\quad & \|\alpha x + \beta y\|^2 = \frac{\alpha'\beta - \alpha\beta'}{\alpha'} g\left(\alpha x + \beta y, y\right).
\end{aligned}$$

Similarly we get the statement 2°. □

Theorem 1. *Let X be smooth and uniformly convex, and let $x, y \in X\setminus\{0\}$ linearly independent. The vector ax is unique the best approximation of vector y with vectors from $[x]$, i.e. $P_{[x]}y = ax$, if and only if*

$$g(y - ax, x) = 0 \ \vee\ \|y - ax\|^2 = g(y - ax, y). \tag{3}$$

Proof. Let be $P_{[x]}y = ax$. Because X is uniformly convex, for fixed $x, y \in X\setminus\{0\}$, the real function $f(t) = \|y - tx\|$ is continuous on $\mathbb{R}$ and it achieves its unique minimum at $t = a$. Since X is smooth, using (2) we have

$$\begin{aligned}
f'(t) &= \lim_{t\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{t\to 0} \frac{\|y - (t+h)\| - \|y - tx\|}{h} \\
&= \lim_{t\to 0} \frac{\|y - tx - th\| - \|y - tx\|}{h} = -\frac{g(y - tx, x)}{\|y - tx\|}.
\end{aligned}$$

Since $\min f(t) = f(a)$ and f is differentiable, we obtain $f'(a) = 0$. So, $g(y - ax, x) = 0$.

The second equation we get immediately from 1° of Lemma 1.

Conversely, if there exists $a \in \mathbb{R}$ sash that $g(y-ax,x)=0$, it follows that $g(y-ax,\lambda x)=0$ for all $\lambda \in \mathbb{R}$. Then, using the properties of functional g we have:

$$g(y-ax, y-ax+\lambda x) = \|y-ax\|^2 \le \|y-ax\| \; \|y-ax+\lambda x\| .$$

So,

$$\|y-ax\| \le \|y-(a-\lambda)x\| ,$$

for all $\lambda \in \mathbb{R}$, i.e. $\min f(t) = \|y-ax\|$. $\square$

An importance of assertion of Theorem 1 is in therein why, in certain spaces, the equation $g(y-ax,x)=0$ is possibly resolve for a. For example, if X is an i.p. space we have

$$g(y-ax,x)=0 \Leftrightarrow (y-ax,x)=0 \Leftrightarrow (x,y)-a\,\|x\|^2=0 \Leftrightarrow a=\frac{(x,y)}{\|x\|^2},$$

i.e., $P_{[x]}y = \dfrac{(x,y)}{\|x\|^2}$.

One other example: The real Banach space $X=\ell^4$ is smooth and uniformly convex. (The space ℓ^4 is so called a quasi inner product space [3]). According to definition of the functional g we get

$$g(x,y) = \|x\|^{-2}\sum_k x_k^3 y_k$$

$(x=(x_1,x_2,...)\in \ell^4\backslash\{0\},\ y=(y_1,y_2,...)\in \ell^4\backslash\{0\})$.

Hence, in this case, we have

$$\begin{aligned} g(y-ax,x)=0 \quad &\Leftrightarrow \quad \|y-ax\|^{-2}\sum_k (y_k-ax_k)^3 x_k = 0 \\ &\Leftrightarrow \quad \sum_k y_k^3 x_k - 3a\sum_k y_k^2 x_k^2 + 3a^2\sum_k y_k x_k^3 - a^3\sum_k x_k^4 = 0. \end{aligned}$$

Since X is uniformly convex, this equation has uniquely solution for a. This solution is possible determined as solution an algebraic equation of the third degree.

Using the Theorem 1 it could be constitute the uniquely of solution certain equations. Par example: The real Banach space $L_p(X,S,\mu)$ $(1<p<$

∞) is uniformly smooth and uniformly convex. As thought in it $g(x,y) = \|x\|^{2-p}\int_X y\,|x|^{p-1}\operatorname{sgn} x\,d\mu$, ([2]), analogy as above, we conclude that the equation

$$\int_X |y-ax|^{p-1}\,x\operatorname{sgn}(y-ax)\,d\mu = 0 \quad (x,y\in L_p, x\neq y;\ a\in\mathbb{R})$$

has uniquely solution for a. This is not simply immediately demonstrate.

Immediate corollary of Theorem 1 is the following assertion.

Corollary 1. *Let X be smooth and uniformly convex, $x,y\in X\setminus\{0\}$ linearly independent, and $p=\{tx+(1-t)y \mid t\in\mathbb{R}\}$. Then, the vector $ax+(1-a)y$ is unique the best approximation of the vector z with the vectors of p if and only if*

$$g\,(z-y-a(x-y),x-y)=0. \tag{4}$$

Proof. $f(t)=\|z-(tx+(1-t)y\|=\|z-y-t(x-y)\|$. □

Let be $z=0$, $x,y\in S(X)$ and $a=1/2$. In this case, the equation (4) reduces on

$$g(x+y,x-y)=0. \tag{5}$$

Since in a smooth space we have $g(x,y)=0\Leftrightarrow x\perp y$, by the assertion 4.2 ([1]) we get

Corollary 2. *Let X is smooth and $x,y\in S(X)$ linearly independent. Then X is an i.p. space if and only if the equality* (5) *holds.*

Remark 1. Let $H:=\{t\in X \mid g(z-y-a(x-y),t)=0\}$. H is a hyperplane in X and $x-y\in H$. Then,

$$d(z,H)=\frac{g(z-y-a(x-y),z-y)}{\|z-y-a(x-y)\|},$$

i.e. $d(z,p)=d(z,H)=\|z-y-a(x-y)\|$. Thus $P_{[p]}z=P_Hz$.

What could say about a of the equation (3)?

Theorem 2. *Let X be smooth and $P_{[x]}y=ax$ ($x,y\in X\setminus\{0\}$ linearly independent). Then the following assertions hold.*

1° $\operatorname{sgn} a=\operatorname{sgn} g(y,x)$, 2° $\quad a \geq \dfrac{g(x,y) - \|x\| \, \|y\|}{\|x\|^2}$,

3° $\quad a = \dfrac{g(x,y)}{\|x\|^2}$ *if and only if X is an i.p. space,*

4° $\quad \left| a - \dfrac{g(x,y)}{\|x\|^2} \right| \leq \dfrac{\|y\|}{\|x\|}$.

Proof. 1° By (3) we have $a = 0 \Leftrightarrow g(y,x) = 0$. Let be $g(y,x) \neq 0$, then

$$g(y, y - ax) = \|y\|^2 - ag(y,x) \leq \|y\| \, \|y - ax\| \leq \|y\|^2 .$$

So, $ag(y,x) > 0$.

2° $g(x, y - ax) = g(x,y) - a\|x\|^2 \leq \|x\| \, \|y - ax\| \leq \|x\| \, \|y\| \Rightarrow a\|x\|^2 \geq g(x,y) - \|x\| \, \|y\|$.

3° If X is an i.p. space, we have seen that $a = \dfrac{g(x,y)}{\|x\|^2} = \dfrac{(x,y)}{\|x\|^2}$. Inversely, if

$$P_{[x]}y = a(x,y) = \frac{g(x,y)}{\|x\|^2}, \quad \text{for all} \quad x, y \in X \backslash \{0\},$$

we get

$$a(x, \alpha y) = \frac{g(x, \alpha y)}{\|x\|^2} = \alpha a(x,y) \qquad (\alpha \in \mathbb{R})$$

and

$$a(x, y_1 + y_2) = \frac{g(x, y_1 + y_2)}{\|x\|^2} = \frac{g(x, y_1)}{\|x\|^2} + \frac{g(x, y_2)}{\|x\|^2},$$

i.e. we have $P_{[x]}\alpha y = \alpha P_{[x]}y$ and

(6) $$P_{[x]}(y_1 + y_2) = P_{[x]}y_1 + P_{[x]}y_2 \qquad (x, y_1, y_2 \in X \backslash \{0\}).$$

By virtue of (6) and the assertion 13.2 in [1] we conclude that X is an i.p. space.

4° Let $H := \{z \in X \mid g(y - ax, z) = 0\}$. H is a hyperplane. Thus there exists $h \in H$ and $\lambda \in \mathbb{R}$ such that

$$y - \frac{g(x,y)}{\|x\|^2} = \lambda(y - ax) + h.$$

Therefore we have

$$g\left(y - ax, y - \frac{g(x,y)}{\|x\|^2}x\right) = \lambda \|y - ax\|^2,$$

i.e.

$$g(y-ax,y)-\frac{g(x,y)}{\|x\|^2}g(y-ax,x)=\lambda\|y-ax\|^2$$

or $\|y-ax\|^2=\lambda\|y-ax\|^2$. So, $\lambda=1$. Hence $h=\left(a-\frac{g(x,y)}{\|x\|^2}\right)x$ and

$$y-\frac{g(x,y)}{\|x\|^2}=y-ax+h. \tag{7}$$

Since $g\left(x,y-\frac{g(x,y)}{\|x\|^2}x\right)=0$, by (7), we get $g(h,y-ax)-\|h\|^2=0$. Thus

$$\left|a-\frac{g(x,y)}{\|x\|^2}\right|^2\|x\|^2\le\left|a-\frac{g(x,y)}{\|x\|^2}\right|\|x\|\,\|y-ax\|\le\left|a-\frac{g(x,y)}{\|x\|^2}\right|\|x\|\,\|y\|.$$

Hence 4° is valid. □

Corollary 3. *Let X be smooth and $z=y-\frac{g(x,y)}{\|x\|^2}x$, then $P_{[z]}y=1$.*

Proof. If $g(x,y)=0$, then $z=y$ and $P_{[y]}y=1$. Suppose $g(x,y)\neq 0$. Then we get $x=\frac{y-z}{g(x,y)}$. Since

$$g\left(x,y-\frac{g(x,y)}{\|x\|^2}\right)=0,$$

we obtain $g(y-z,z)=0$. So, by Theorem 1 we get $P_{[z]}y=1$. □

For some additional results we consider a linearly independent vectors $e_1,e_2,\dots,e_n\in S(X)$. Let $M:=\operatorname{span}\{e_1,e_2,\dots,e_n\}$ and $y\in X\backslash M$.

Theorem 3. *Let X be a smooth and uniformly convex Banach space. Then*

$$P_My=\sum_{k=1}^n a_ke_k$$

if and only if for all $i\in\{1,2,\dots,n\}$,

$$g\left(y-\sum_{k=1}^n a_ke_k,e_i\right)=0. \tag{8}$$

This system of equations has unique solution $a=(a_1,a_2,\dots,a_n)$.

Proof. Since X is smooth and uniformly convex Banach space, the real function

$$f(t_1, t_2, \ldots, t_n) = \left\| y - \sum_{k=1}^{n} t_k e_k \right\|$$

achieves its minimum at unique point $a = (a_1, a_2, \ldots, a_n)$ and there exists $\frac{\partial f}{\partial t_i}(a_1, a_2, \ldots, a_n)$ such that, for all $i \in \{1, 2, \ldots, n\}$, we have

$$\frac{\partial f}{\partial t_i}(a_1, a_2, ..., a_n) = 0.$$

The other side we have

$$\begin{aligned} \frac{\partial f}{\partial t_i}(t_1, t_2, \ldots, t_n) &= \lim_{h_i \to 0} \frac{\left\| y - \sum\limits_{k=1}^{n} t_k e_k - h_i e_i \right\| - \left\| y - \sum\limits_{k=1}^{n} t_k e_k \right\|}{h_i} \\ &= -\frac{g\left(y - \sum\limits_{k=1}^{n} t_k e_k, e_i \right)}{\left\| y - \sum\limits_{k=1}^{n} t_k e_k \right\|}. \end{aligned}$$

So,

$$g\left(y - \sum_{k=1}^{n} a_k e_k, e_i \right) = 0, \tag{9}$$

and, this system has the unique solution.

Conversely, let (9) holds. Then, for all $\lambda_i \in \mathbb{R}$, we have

$$g\left(y - \sum_{k=1}^{n} a_k e_k, \lambda_i e_i \right) = 0.$$

Using this, we obtain

$$g\left(y - \sum_{k=1}^{n} a_k e_k, \sum_{i=1}^{n} \lambda_i e_i \right) = 0 \qquad (\lambda_i \in \mathbb{R}).$$

Since $x_0 = \sum\limits_{k=1}^{n} a_k e_k \in M$ and $x = \sum\limits_{i=1}^{n} \lambda_i e_i \in M$, we conclude that, for all $x \in M$, we have $g(y - x_0, x) = 0$. Thus, it implies that $P_M y = x_0$. $\square$

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