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APPROXIMATION OF BIVARIATE FUNCTIONS BY OPERATORS OF STANCU–HURWITZ TYPE

Ioana Taşcu

Abstract. The aim of this paper is to introduce and study a linear positive approximation operator of Stancu-Hurwitz type [1], depending on several nonnegative parameters, useful in the approximation of functions of two variables.

The corresponding approximation formula (3.1) has the degree of exactness $(1,1)$. For the remainder of this formula we give several representations, by starting from a method of T. Popoviciu [7] for representation of the remainder term in linear approximation formulas, by using the divided differences.

1. Introduction

In a paper published in 2002 by D. D. Stancu [11] there has been constructed an approximation linear positive operator, denoted by $S_m^{(\beta_1,\dots,\beta_m)}$, which was defined, for any function $f \in C[0,1]$, by the following formula

$$\left(S_m^{(\beta_1,\dots,\beta_m)} f\right)(x) = \sum_{k=0}^{m} w_{m,k}^{(\beta_1,\dots,\beta_m)}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where

$$\begin{aligned} &(1+\beta_1+\cdots+\beta_m)^{m-1} w_{m,k}^{(\beta_1,\dots,\beta_m)}(x) \\ &\quad = \sum x(x+\beta_1+\cdots+\beta_{i_k})^{k-1}(1-x)(1-x+\beta_{j_1}+\cdots+\beta_{j_{m-k}})^{m-k-1}. \end{aligned} \tag{1.2}$$

On the other hand, $\beta_1,\dots,\beta_m$ are m nonnegative parameters.

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These basis polynomials were constructed in [11] by starting from an identity of Hurwitz [3], generalizing the classical identity of Abel-Jensen, namely

$$(u+v)(u+v+\beta_1+\cdots+\beta_m)^{m-1} \\ = \sum u(u+\beta_{i_1}+\cdots+\beta_{i_k})^{k-1} v(v+\beta_{j_1}+\cdots+\beta_{j_{m-k}})^{m-k-1},$$

which in the special case $\beta_1=\beta_2=\cdots=\beta_m=\beta$ reduces to the identity of Abel-Jensen [4]:

$$(u+v)(u+v+m\beta)^{m-1}=\sum_{k=0}^{m}\binom{m}{k}u(u+k\beta)^{k-1}v(v+(m-k)\beta)^{m-k-1}.$$

2. The Bivariate Polynomial Operator of Stancu-Hurwitz Type

In this paper we consider the space of real-valued bivariate functions $C(D)$, continuous on the unit square $D=[0,1]\times[0,1]$, and we associate the Stancu-Hurwitz type bivariate polynomials

$$\left(S_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(x,y) \\ = \sum_{k=0}^{m}\sum_{\nu=0}^{n} w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)v_{n,\nu}^{(\gamma_1,\ldots,\gamma_n)}(y)f\left(\frac{k}{m},\frac{\nu}{n}\right), \tag{2.1}$$

where $w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)$ is defined by formula (1.2) and $v_{n,\nu}^{(\gamma_1,\ldots,\gamma_n)}(y)$ is given by a similar formula

$$(1+\gamma_1+\cdots+\gamma_n)v_{n,\nu}^{(\gamma_1,\ldots,\gamma_n)}(y) \\ = \sum y\left(y+\gamma_1+\cdots+\gamma_{s_\nu}\right)^{\nu-1}(1-y)(1-y+\gamma_{t_1}+\cdots+\gamma_{t_{n-\nu}}^{n-\gamma-1}),$$

$\gamma_1,\gamma_2,\ldots,\gamma_n$ being nonnegative parameters.

In the special cases $\beta_1=\cdots=\beta_m=\beta$ and $\gamma_1=\cdots=\gamma_n=\gamma$, we obtain the Cheney-Sharma-Stancu type bivariate linear positive operator defined by the following formula

$$\left(S_{m,n}^{(\beta;\gamma)}f\right)(x,y)=\sum_{k=0}^{m}\sum_{\nu=0}^{n} w_{m,k}^{(\beta)}(x)v_{n,\nu}^{(\gamma)}(y)f\left(\frac{k}{m},\frac{\nu}{n}\right),$$

where

$$(1+m\beta)^{m-1}w_{m,k}^{(\beta)}(x)=\binom{m}{k}x(x+k\beta)^{k-1}(1-x)(1-x+(m-k)\beta)^{m-k-1}$$

and

$$(1+n\gamma)^{n-1}v_{n,\nu}^{(\gamma)}(y)=\binom{n}{\gamma}y(y+\nu\gamma)^{\nu-1}(1-y+(n-\nu)\gamma)^{n-\nu-1}.$$

The operator $S_{m,n}^{(\beta;\gamma)}$ represents an extension to two variables of the second operator of Cheney-Sharma [1].

3. Approximation of Bivariate Functions by Means of a Polynomial Operator of Stancu-Hurwitz Type

It is easy to see that the polynomial defined at (2.1) is interpolatory in the corners of the square D, that is it reproduces de values of the function $f \in C(D)$ in the four points: $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Consequently, the approximation formula

$$(3.1)\quad f(x,y)=\left(S_{m,n}^{(\beta_1,\dots,\beta_m;\gamma_1,\dots,\gamma_n)}f\right)(x,y)+\left(R_{m,n}^{(\beta_1,\dots,\beta_m;\gamma_1,\dots,\gamma_n)}f\right)(x,y)$$

has the degree of exactness $(1,1)$.

Now if we use a theorem of Peano-Milne-Stancu type, given in D. D. Stancu [9], we can give an integral representation for the remainder term of the approximation formula (3.1).

Theorem 3.1. *If the function f has continuous second-order partial derivatives on the square D, then the remainder of the approximation formula* (3.1) *can be represented under the following integral form*

$$(3.2)\quad \left(R_{m,n}^{(\beta_1,\dots,\beta_m;\gamma_1,\dots,\gamma_n)}f\right)(x,y)$$
$$=\int_0^1 G_m^{(\beta_1,\dots,\beta_m)}(t,x)f^{(2,0)}(t,y)dt+\int_0^1 H_n^{(\gamma_1,\dots,\gamma_n)}(z,y)f^{(0,2)}(x,z)dz$$
$$-\int_0^1\int_0^1 G_m^{(\beta_1,\dots,\beta_m)}(t,x)H_n^{(\gamma_1,\dots,\gamma_n)}(z,y)f^{(2,2)}(t,z)dtdz$$

where the Peano kernels are

$$G_m^{(\beta_1,\dots,\beta_m)}(t,x)=\left(R_m^{(\beta_1,\dots,\beta_m)}\varphi_x\right)(t),$$

with

$$\varphi_x(t)=\frac{x-t+|x-t|}{2}=(x-t)_+,$$

and

$$H_n^{(\gamma_1,\dots,\gamma_n)}(z,y) = \left(R_n^{(\gamma_1,\dots,\gamma_n)}\psi_y\right)(z),$$

with

$$\psi_y(z) = \frac{y-z+|y-z|}{2} = (y-z)_+\,.$$

We have used above the notation

$$f^{(r,s)}(u,v) = \frac{\partial^{r+s} f(u,v)}{\partial u^r \partial v^s}\,.$$

It follows that we can write explicitly

$$G_m^{(\beta_1,\dots,\beta_m)}(t,x) = (x-t)_+ - \sum_{k=0}^{m} w_{m,k}^{(\beta_1,\dots,\beta_m)}(x)\left(\frac{k}{m}-t\right)_+,$$

$$H_n^{(\gamma_1,\dots,\gamma_n)}(z,y) = (y-z)_+ - \sum_{\nu=0}^{n} v_{n,\nu}^{(\gamma_1,\dots,\gamma_n)}(y)\left(\frac{\nu}{n}-z\right)_+.$$

Using these explicit expressions for the partial Peano kernels, we can see that they represent polygonal lines situated beneath the t-axis, respectively the z-axis, which joins the points $(0,0)$ and $(0,1)$, respectively the points $(0,0)$ and $(1,0)$.

If we assume that $x \in \left[\frac{r-1}{m}, \frac{r}{m}\right]$, we can give for the first Peano kernel the following expression:

$$\begin{aligned} G_m^{(\beta_1,\dots,\beta_m)}(t,x) = \\ & -\sum_{k=0}^{i-1} w_{m,k}^{(\beta_1,\dots,\beta_m)}(x)\left(t-\frac{k}{m}\right) \quad \text{if } t \in \left[\frac{i-1}{m}, \frac{i}{m}\right] \quad (1 \le i \le r-1), \\ & -\sum_{k=0}^{r-1} w_{m,k}^{(\beta_1,\dots,\beta_m)}(x)\left(t-\frac{k}{m}\right) \quad \text{if } t \in \left[\frac{r-1}{m}, x\right], \\ & -\sum_{k=r}^{m} w_{m,k}^{(\beta_1,\dots,\beta_m)}(x)\left(\frac{k}{m}-t\right) \quad \text{if } t \in \left[x, \frac{r}{m}\right], \\ & -\sum_{k=i}^{m} w_{m,k}^{(\beta_1,\dots,\beta_m)}(x)\left(\frac{k}{m}-t\right) \quad \text{if } t \in \left[\frac{i-1}{m}, \frac{i}{m}\right] \quad (r \le i \le m). \end{aligned}$$

The dual Peano kernel $H_n^{(\gamma_1,\dots,\gamma_n)}(z,y)$ has a similar expression.

Now if we take into account that on the square D, we have $G_m^{(\beta_1,\dots,\beta_m)}(t,x) \le 0$ and $H_n^{(\gamma_1,\dots,\gamma_n)}(z,y) \le 0$, then we can apply the first law of the mean to the integrals and we can find that

$$\begin{aligned}\left(R_{m,n}^{(\beta_1,\dots,\beta_m;\gamma_1,\dots,\gamma_n)}f\right)(x,y) &= f^{(2,0)}(\xi,y)\int_0^1 G_n^{(\beta_1,\dots,\beta_m)}(t,x)dt\\ &+f^{(0,2)}(x,\eta)\int_0^1 H_n^{(\gamma_1,\dots,\gamma_n)}(z,y)dz\\ &-f^{(2,2)}(\xi,\eta)\left[\int_0^1 G_m^{(\beta_1,\dots,\beta_m)}(t,x)dt\right]\left[\int_0^1 H_n^{(\gamma_1,\dots,\gamma_n)}(z,y)dz\right],\end{aligned}$$

where ξ and η are certain points from the interval $(0,1)$.

It is easy to see that we have

$$\int_0^1 G_m^{(\beta_1,\dots,\beta_m)}(t,x)dt = \frac{1}{2}\left(R_m^{(\beta_1,\dots,\beta_m)}e_{2,0}\right)(x)$$

and

$$\int_0^1 H_n^{(\gamma_1,\dots,\gamma_n)}(z,y)dz = \frac{1}{2}\left(R_n^{(\gamma_1,\dots,\gamma_n)}e_{0,2}\right)(y),$$

where we have considered the univariate remainders

$$R_m^{(\beta_1,\dots,\beta_m)} = I - S_m^{(\beta_1,\dots,\beta_m)}, \qquad R_n^{(\gamma_1,\dots,\gamma_n)} = I - S_n^{(\gamma_1,\dots,\gamma_n)}.$$

Now we can state the following result:

Theorem 3.2. *If $f \in C^{2,2}(D)$, then the remainder of the approximation formula* (3.1) *can be represented under the following Cauchy form:*

$$\begin{aligned}\left(R_{m,n}^{(\beta_1,\dots,\beta_m;\gamma_1,\dots,\gamma_n)}f\right)(x,y) &= \frac{1}{2}\left(R_m^{(\beta_1,\dots,\beta_m)}e_{2,0}\right)(x)f^{(2,0)}(\xi,y)\\ &+\frac{1}{2}\left(R_n^{(\gamma_1,\dots,\gamma_n)}e_{0,2}\right)(y)f^{(0,2)}(x,\eta)\\ &-\frac{1}{4}\left(R_m^{(\beta_1,\dots,\beta_m)}e_{2,0}\right)(x)\left(R_n^{(\gamma_1,\dots,\gamma_n)}e_{0,2}\right)(y)f^{(2,2)}(\xi,\eta).\end{aligned} \tag{3.3}$$

Because $\left(S_m^{(\beta_1,\ldots,\beta_m)}f\right)(x)$ and $\left(S_n^{(\gamma_1,\ldots,\gamma_n)}f\right)(y)$ are interpolatory at both sides of the interval $[0,1]$, we can conclude that $\left(R_m^{(\beta_1,\ldots,\beta_m)}e_{2,0}\right)(x)$ contains the factor $x(x-1)$, while $\left(R_n^{(\gamma_1,\ldots,\gamma_n)}e_{0,2}\right)(y)$ has the factor $y(y-1)$.

Since $\left(R_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}e_{0,0}\right)(x,y) = 0$ and the remainder is different from zero for any convex function f of the first order, we can apply a criterion of T. Popoviciu [7] and we find that the remainder of the approximation formula (3.1) is of simple form.

Consequently, we can state:

Theorem 3.3. *If the second-order divided differences of the function $f \in C(D)$ are bounded on the square D, then we can give an expression of the remainder of the formula* (3.1) *in terms of divided differences under the following form*

$$\begin{aligned}(3.4)\quad &(R_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f)(x,y)\\ &= (R_m^{(\beta_1,\ldots,\beta_m)}e_{2,0})(x)\,[x_{m,1},x_{m,2},x_{m,3};f(t,y)]\\ &\quad +(R_n^{(\gamma_1,\ldots,\gamma_n)}e_{0,2})(y)\,[y_{n,1},y_{n,2},y_{n,3};f(x,z)]\\ &\quad -(R_m^{(\beta_1,\ldots,\beta_m)}e_{2,0})(x)(R_n^{(\gamma_1,\ldots,\gamma_n)}e_{0,2})(y)\left[\begin{matrix}x_{m,1},x_{m,2},x_{m,3}\\ y_{n,1},y_{n,2},y_{n,3}\end{matrix}\;;f(t,z)\right],\end{aligned}$$

where $x_{m,1},x_{m,2},x_{m,3}$, respectively $y_{n,1},y_{n,2},y_{n,3}$ are certain points in the interval $[0,1]$.

Now if we consider that $f \in C^{2,2}(D)$, then we can apply the mean value theorems to the divided differences and we arrive at the expression (3.3) for the remainder of approximation formula (3.1).

Finally, we mention that formulas (3.2), (3.3) and (3.4) can be extended to functions more than two variables without any difficulty.

REFERENCES

1. E.W. Cheney and A. Sharma: *On a generalization of Bernstein polynomials.* Rev. Mat. Univ. Parma. **5** (1964), 77–84.
2. I. Horova and M. Budikova: *A note on D. D. Stancu operators.* Ricerche di Matem. **44** (1995), 397–407.

3. A. HURWITZ: *Über Abel's Vereingemeinerung der Binomischen Formel.* Acta Math. **26** (1902), 199–203.
4. L.W. JENSEN: *Sur une identité d'Abel et sur d'autres formules analogues.* Acta Math. **26** (1902), 307–318.
5. W.E. MILNE: *The remainder in linear methods of approximation.* J. Res. Mat. Bur. Standards **43** (1949), 501–511.
6. G. PEANO: *Resto nelle formule di quadratura expresso con un integrale definito.* Atti Acad. Naz. Lincei Rend. **22** (1913), 562–569.
7. T. POPOVICIU: *Sur le reste dans certaines formules linéaires d'approximation de l'analyse.* Mathematica, Cluj **1**(**24**) (1959), 95–142.
8. D.D. STANCU: *Evaluation of the remainder term in approximation formulas by Bernstein polynomials.* Math. Comp. **17** (1963), 270–278.
9. D.D. STANCU: *The remainder of certain linear approximation formulas in two variables.* J. SIAM Numer. Anal. B, **1** (1964), 137–163.
10. D.D. STANCU: *Methods for construction of linear positive operators of approximation.* In: Numerical Analysis and Approximation Theory, Proc. of the International Symposium (R. T. Trîmbiţaş, ed.), Cluj University Press, 2002, pp. 23–45.
11. D.D. STANCU: *Use of an identity of A. Hurwitz for construction of a linear positive operator of approximation.* Rev. Anal. Numér. Théor. Approx. **31** (2002), 115–118.
12. D.D. STANCU and C. CIŞMAŞIU: *On an approximating linear positive operator of Cheney-Sharma.* Rev. Anal. Numér. Théor. Approx. **26** (1997), 221–227.

North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 43012 Baia Mare, Romania

itascu@yahoo.com