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ON THE MAPPING H OF S. S. DRAGOMIR

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Abstract. In this paper, we establish some new results containing some inequalities for a convex and differentiable function f involving the mappings H and introduced by S. S. Dragomir in the papers [4], [5] and [6]. We give applications to some special means. This paper is a natural continuation to the paper [1].

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a convex function on the closed and finite interval $[a, b]$ of the real line. Then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known as Hadamard's inequalities. Many papers were recently devoted to their generalizations and their refinements, and now, there is a very rich literature on the subject. The reader can consult the list of papers in the references of this paper and their references. In his paper [5], S. S. Dragomir introduced two mappings associated to Hadamard's inequalities connected to convex functions. Precisely, if $f : [a, b] \to \mathbb{R}$ is given as above then one can define the two following mappings on $[0, 1]$ by setting:

$$\begin{aligned} F(t) &:= \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dx\,dy, \\ H(t) &:= \frac{1}{b-a}\int_a^b f\Big(tx+(1-t)\frac{a+b}{2}\Big)\,dx. \end{aligned}$$

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Concerning the mappings H, the following properties are known (see [4] and [5]):

1° H is convex and monotonous nondecreasing on $[0,1]$.

2° We have the bounds:

$$\inf_{t\in[0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right), \quad \sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

When f is differentiable, S.S. Dragomir established in [6] the following result which improves the right inequality of (1.1):

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be as above. Then one has the following inequalities:*

$$\begin{aligned}(1.2)\qquad 0 &\le \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{b-a}\int_a^b f\Big(tx+(1-t)\frac{a+b}{2}\Big)\,dx \\ &\le (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right],\end{aligned}$$

for all $t\in[0,1]$.

In this paper, we shall prove for a convex differentiable mapping f that its associated mappings H is differentiable and that its first derivative is bounded by the constant $(f(a)+f(b))/2-(b-a)^{-1}\int_a^b f(x)\,dx$. Therefore, H is Lipschitzian. Then we recapture the inequalities (1.2) as consequences of these properties of H. For more details, see Theorem 2.1 below. We prove also some new inequalities involving the mapping H (see Theorems 2.2, 2.3, 2.4 and Corollary 2.1 below). We point out that these results provide some refinements to the left and right inequalities of Hadamard inequalities (1.1). We end this paper by giving applications of these inequalities to some special means. This paper may be considered as a natural sequel to the papers [4], [5] [6] of S. S. Dragomir and to our paper [1] where some results concerning the mapping H have been established.

2. The Results

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be convex and differentiable. Then H is convex and differentiable, and for all $t\in(0,1)$, we have*

$$(2.1)\quad 0 = H'_+(0) \le H'(t) \le H'_-(1) = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In particular, H is Lipschitzian on $[0,1]$ and for all $t_1, t_2 \in [0,1]$ with $t_1 \le t_2$, we have

$$(2.2)\quad 0 \le H(t_2) - H(t_1) \le (t_2 - t_1)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right].$$

Proof. The convexity of H was proved in [5]. We remark that the assumptions made on f allow us to apply Lebesgue's theorem of differentiability of integrals depending on parameters to the mapping H. Therefore H is differentiable on $[0,1]$, and we have

$$H'(t) = \frac{1}{b-a}\int_a^b \Big(x - \frac{a+b}{2}\Big) f'\Big(tx + (1-t)\frac{a+b}{2}\Big)\,dx,$$

for all $t \in [0,1]$. Since f is convex then f' is monotonous nondecreasing on $[a,b]$, and we have the following observations:

i) If $a \le x \le (a+b)/2$, then we have

$$f'(x) \le f'\Big(tx + (1-t)\frac{a+b}{2}\Big) \le f'\Big(\frac{a+b}{2}\Big).$$

Therefore,

$$\Big(x-\frac{a+b}{2}\Big)f'\Big(\frac{a+b}{2}\Big) \le \Big(x-\frac{a+b}{2}\Big)f'\Big(tx+(1-t)\frac{a+b}{2}\Big) \le \Big(x-\frac{a+b}{2}\Big)f'(x).$$

ii) If $(a+b)/2 \le x \le b$, then we have

$$f'\Big(\frac{a+b}{2}\Big) \le f'\Big(tx + (1-t)\frac{a+b}{2}\Big) \le f'(x).$$

Therefore,

$$\Big(x-\frac{a+b}{2}\Big)f'\Big(\frac{a+b}{2}\Big) \le \Big(x-\frac{a+b}{2}\Big)f'\Big(tx+(1-t)\frac{a+b}{2}\Big) \le \Big(x-\frac{a+b}{2}\Big)f'(x).$$

From (i) and (ii) we deduce by integrating on $[0,1]$ that

$$\frac{f'(\frac{a+b}{2})}{b-a}\int_a^b \Big(x - \frac{a+b}{2}\Big)\,dx \le H'(t) \le \frac{1}{b-a}\int_a^b \Big(x - \frac{a+b}{2}\Big) f'(x)\,dx.$$

Because simple computations show that

$$\frac{1}{b-a}\int_a^b \Big(x - \frac{a+b}{2}\Big) f'(x)\,dx = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx,$$

and

$$\int_a^b \left(x - \frac{a+b}{2}\right) dx = 0$$

all the inequalities in (2.1) are proved.

The first inequality of (2.2) is clear. The proof of the second inequality is obvious by Lagrange's theorem. So the proof of Theorem 2.1 is complete. $\square$

Another result of this type is given by the following.

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be convex and differentiable. Then for all $t \in [0,1]$, we have the following inequalities:*

$$0 \le 2t(1-t)\left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)\right] \tag{2.3}$$

$$\le \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{t}{b-a}\int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)\,dx \tag{2.4}$$

$$-\frac{1-t}{b-a}\int_a^b f\Big((1-t)x + t\frac{a+b}{2}\Big)\,dx \tag{2.5}$$

$$\le 2t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right].$$

Proof. a) The inequality (2.3) is obvious. H being convex, for all $t \in [0,1]$ and $x \in [a,b]$, we have $H(t) \le tH(1) + (1-t)H(0)$, from which we obtain

$$H(t) - H(1) \ge (1-t)(H(1) - H(0)).$$

From this inequality and the following identity

$$H(1) - tH(t) - (1-t)H(1-t) = t(H(1) - H(t)) + (1-t)(H(1) - H(1-t)),$$

we obtain the inequality (2.4).

b) Since H is convex and nondecreasing, we may write

$$H(t^2 + (1-t)^2) \le tH(t) + (1-t)H(1-t) \le H(1),$$

for all $t \in [0,1]$. Then with the help of the inequality (1.2) of Theorem 1.1, we get

$$\begin{aligned} H(1) - tH(t) - (1-t)H(1-t) &\le H(1) - H(t^2 + (1-t)^2) \\ &\le 2t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]. \end{aligned}$$

Thus we have proved the inequality (2.5) and our theorem is completely proved. $\square$

Corollary 2.1. *Let $f : [a, b] \to \mathbb{R}$ be convex and differentiable, and let H be its associated mapping. Then for all $t \in [0, 1]$, we have the following inequalities:*

$$\begin{aligned} 0 \leq H(t) - H(0) &\leq t(H(t) - H(0)) \leq H(1) - H(1-t) \\ &\leq t\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right] \end{aligned}$$

In particular, for $t = 1/2$, we get

$$\begin{aligned} 0 &\leq \frac{2}{b-a}\int_{(3a+b)/4}^{(a+3b)/4} f(x)\,dx - f\Big(\frac{a+b}{2}\Big) \\ &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{2}{b-a}\int_{(3a+b)/4}^{(a+3b)/4} f(x)\,dx \\ &\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right] \end{aligned}$$

Another similar result is given by the following.

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be convex and differentiable. Then for all $t \in [0, 1]$, we have the following inequalities:*

$$\begin{aligned} 0 &\leq [t^2 + (1-t)^2]\left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{t}{b-a}\int_a^b f\Big((1-t)x + t\frac{a+b}{2}\Big)\,dx \\ &\qquad - \frac{(1-t)}{b-a}\int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)\,dx \\ &\leq \big(t^2 + (1-t)^2\big)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]. \end{aligned}$$

The proof of this theorem utilizes arguments which are similar to those used in the proof of Theorem 2.2. So we omit the details.

We end this section by giving the following result containing again some refinements both to the left and right inequalities of Hadamard's inequalities.

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be convex and differentiable. Then for all $t \in [0,1]$, we have the following inequalities:*

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f\Big(tx+(1-t)\frac{a+b}{2}\Big)\,dx - f\Big(\frac{a+b}{2}\Big) \\ &\le t\left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)\right], \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} 0 &\le \frac{t}{b-a}\int_a^b f(x) + (1-t)f\Big(\frac{a+b}{2}\Big) \\ &\qquad\qquad -\frac{1}{b-a}\int_a^b f\Big(tx+(1-t)\frac{a+b}{2}\Big)\,dx \\ &\le t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]. \end{aligned} \tag{2.7}$$

Proof. Since H is convex and nondecreasing on $[0,1$, we have

$$0 \le H(t) - H(0) \le t(H(1) - H(0)),$$

for all $t \in [0,1]$. Hence the inequalities (2.6) are proved. To prove (2.7), we use the properties of H and the inequality (1.2) of Theorem 1.1 which allow us to write the following inequalities:

$$\begin{aligned} 0 &\le tH(1) + (1-t)H(0) - H(t) \\ &= t(H(1) - H(t)) - (1-t)(H(t) - H(0)) \\ &\le t(H(1) - H(t)) \\ &\le t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]. \end{aligned}$$

Thus we are led to the desired result. This completes the proof. □

3. Applications to Some Special Means

We start by making some recalls on the means considered here.

1) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \quad a,b \geq 0;$$

2) The geometric mean:

$$G := G(a,b) := \sqrt{ab}, \quad a,b \geq 0;$$

3) The harmonic mean:

$$H = H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a,b \geq 0;$$

4) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a = b \\ \dfrac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b\ , a,b > 0; \end{cases}$$

5) The identric mean:

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a = b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)} & \text{if} \quad a \neq b\ , a,b > 0; \end{cases}$$

6) The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if} \quad a \neq b; \\ a & \text{if} \quad a = b \end{cases}$$

where, $p \in \mathbb{R} \setminus \{-1, 0\}$, $a, b > 0$.

The following inequalities involving these means are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

We recall also that the mean L_p is increasing in p with $L_0 = I$ and $L_{-1} = L$.

Now, we give some applications.

1° Consider the function $f : x \mapsto x^p$ with $p > 1$ on any subinterval $[a, b]$ of $[0, \infty[$, with $a < b$. Then by easy computations, for all $t \in [0, 1]$, we have

$$H(t) = L_p^p\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right).$$

In particular,

$$H(1) = L_p^p(a, b),\ H(0) = A^p(a, b) \text{ and } H\left(\frac{1}{2}\right) = L_p^p\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right).$$

By application of Theorem 2.1, for all $t_1, t_2 \in [0, 1]$ with $t_1 \le t_2$, we get the following inequalities:

$$\begin{aligned} 0 &\le L_p^p\left(\frac{a+b}{2} - t_2\frac{b-a}{2}, \frac{a+b}{2} + t_2\frac{b-a}{2}\right) \\ &\quad - L_p^p\left(\frac{a+b}{2} - t_1\frac{b-a}{2}, \frac{a+b}{2} + t_1\frac{b-a}{2}\right) \\ &\le (t_2 - t_1)\left[A(a^p, b^p) - L_p^p(a, b)\right]. \end{aligned}$$

By application of Corollary 2.1, we get the following inequalities:

$$\begin{aligned} 0 &\le L_p^p\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right) - A^p(a, b) \\ &\le t\left[L_p^p(a, b) - A^p(a, b)\right] \\ &\le L_p^p(a, b) - L_p^p\left(a + t\frac{b-a}{2}, b - t\frac{b-a}{2}\right) \\ &\le t\left[A(a^p, b^p) - L_p^p(a, b)\right], \end{aligned}$$

for all t in $[0, 1]$. In particular for $t = \frac{1}{2}$, we get

$$0 \le L_p^p\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) - A^p(a, b) \le \frac{1}{2}\left[L_p^p(a, b) - A^p(a, b)\right] \tag{3.1}$$

$$\le L_p^p(a, b) - L_p^p\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) \le \frac{1}{2}\left[A(a^p, b^p) - L_p^p(a, b)\right]. \tag{3.2}$$

As a consequence, from (3.1) and (3.2), we deduce the following inequality:

$$L_p^p(a, b) \le A(A^p(a, b), A(a^p, b^p)).$$

2° Consider the convex and differentiable function $f : x \mapsto 1/x$ on any subinterval $[a, b]$ of $(0, +\infty)$, with $a < b$. Then by easy computations, for all $t \in [0, 1]$, we have

$$H(t) = L^{-1}\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right).$$

In particular, we have

$$H(1) = L^{-1}(a, b),\ H(0) = A^{-1}(a, b) \ \text{ and } \ H\left(\frac{1}{2}\right) = L^{-1}\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right).$$

By application of Theorem 2.1, for all $t_1, t_2 \in [0, 1]$ with $t_1 \le t_2$, we get the following inequalities:

$$\begin{aligned} 0 \ &\le \ L^{-1}\left(\frac{a+b}{2} - t_2\frac{b-a}{2}, \frac{a+b}{2} + t_2\frac{b-a}{2}\right) \\ &\qquad - L^{-1}\left(\frac{a+b}{2} - t_1\frac{b-a}{2}, \frac{a+b}{2} + t_1\frac{b-a}{2}\right) \\ &\le \ (t_2 - t_1)\left[H^{-1}(a, b) - L^{-1}(a, b)\right]. \end{aligned}$$

By application of Corollary 2.1, we get the following inequalities:

$$\begin{aligned} 0 \ &\le \ L^{-1}\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right) - A^{-1}(a, b) \\ &\le \ t\left[L^{-1}(a, b) - A^{-1}(a, b)\right] \\ &\le \ L^{-1}(a, b) - L^{-1}\left(a + t\frac{b-a}{2}, b - t\frac{b-a}{2}\right) \\ &\le \ t\left[H^{-1}(a, b) - L^{-1}(a, b)\right], \end{aligned}$$

for all t in $[0, 1]$. In particular for $t = 1/2$, we get

$$\begin{aligned} 0 \ &\le \ L^{-1}\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) - A^{-1}(a, b) \le \frac{1}{2}\left[L^{-1}(a, b) - A^{-1}(a, b)\right] \\ &\le \ L^{-1}(a, b) - L^{-1}\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) \le \frac{1}{2}\left[H^{-1}(a, b) - L^{-1}(a, b)\right]. \end{aligned}$$

As a consequence of previous inequalities we deduce the following inequality:

$$H(H(a, b), A(a, b)) \le L(a, b).$$

3° Finally, let us consider the convex and differentiable function $f : x \mapsto -\ln(x)$ on any subinterval $[a, b]$ of the interval $(0, +\infty)$ with $a < b$. Then by easy computations, for all $t \in [0, 1]$, we get

$$H(t) = -\log I\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right).$$

In particular we have

$$H(1) = -\log I(a, b), \; H(0) = -\log A(a, b)$$

and

$$H\left(\frac{1}{2}\right) = -\log I\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right).$$

By application of Theorem 2.1, we get the following inequalities:

$$1 \leq \frac{I\left(\frac{a+b}{2} - t_1\frac{b-a}{2}, \frac{a+b}{2} + t_1\frac{b-a}{2}\right)}{I\left(\frac{a+b}{2} - t_2\frac{b-a}{2}, \frac{a+b}{2} + t_2\frac{b-a}{2}\right)} \leq \left[\frac{I(a,b)}{G(a,b)}\right]^{t_2-t_1}$$

By application of Corollary 2.1, we obtain the following inequalities:

$$\begin{aligned} 1 \quad &\leq \quad \frac{A(a,b)}{I\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right)} \leq \left[\frac{A(a,b)}{I(a,b)}\right]^t \\ &\leq \quad \frac{I\left(a + t\frac{b-a}{2}, b - t\frac{b-a}{2}\right)}{I(a,b)} \leq \left[\frac{I(a,b)}{G(a,b)}\right]^t \end{aligned}$$

for all t in $[0, 1]$. In particular, if we choose $t = 1/2$, then we get

$$I(a,b) \leq \frac{I(a,b)A^2(a,b)}{I^2\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right)} \leq A(a,b) \leq \frac{I^2\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right)}{I(a,b)} \leq \frac{I^2(a,b)}{G(a,b)}.$$

From previous inequalities we deduce the following inequality:

$$G(A(a,b), G(a,b)) \leq I(a,b).$$

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