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ON HARDY'S INTEGRAL INEQUALITY*

James Adedayo Oguntuase and Emmanuel Oyeyemi Adeleke

Abstract. In this paper, transparent proofs of some Hardy-type integral inequalities were presented. These inequalities generalize some known results and simplify the proofs of some existing results.

1. Introduction

In 1920 G. H. Hardy obtained the following result:

Theorem 1.1. (Hardy [5]) *If* $p > 1, f(x) \geq 0,$ *and*

$$F(x) = \int_0^x f(t)dt,$$

then

$$\int_0^{+\infty} \left(\frac{F}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx, \tag{1.1}$$

unless $f \equiv 0$. *The constant* $\left(\frac{p}{p-1}\right)^p$ *is best possible.*

This appeared in the paper [5] which subsequently formed the basis of what is today known as Hardy's inequality. This inequality, which was discovered by Hardy [5] in the course of attempts to simplify the proofs then known of Hilbert's theorem:

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Theorem 1.2. (Hilbert) *If* $p > 1, p' = p/(p-1)$ *and*

$$\int_0^{+\infty} f^p(x)dx \le F, \quad \int_0^{+\infty} g^{p'}(y)dy \le G,$$

then

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y}\,dxdy < \frac{\pi}{\sin(\pi/p)} F^{1/p}G^{1/p'},$$

unless $f \equiv 0$ *or* $g \equiv 0$. *The constant* $\pi/\sin(\pi/p)$ *is the best possible.*

For further historical developments on Hardy's inequality, we refer interested reader to [7, Chapter IX] and the references cited therein.

In view of the usefulness of this inequality in analysis and its applications, it has received considerable attention in the past five decades and a number of papers have appeared which deals with its various generalizations and applications (see for instance [3, 8] and the references cited therein).

In particular, Hardy [6] in 1928 gave a generalized form of inequality (1.1) when he showed that for any $r \ne 1, p > 1$ and any integrable function $f(x) \ge 0$ on $(0, +\infty)$ for which

$$F(x) = \begin{cases} \displaystyle\int_0^x f(t)dt & \text{for } r > 1, \\ \displaystyle\int_x^{+\infty} f(t)dt & \text{for } r < 1, \end{cases}$$

then

$$\int_0^{+\infty} x^{-m}F^p(x)dx < \left(\frac{p}{|r-1|}\right)^p \int_0^{+\infty} x^{-m}\left[xf(x)\right]^p dx, \tag{1.2}$$

unless $f \equiv 0$, where the constant is also best possible.

The objective of this paper is to further obtain generalizations of the classical Hardy's integral inequality which will be useful in applications by using some elementary methods of analysis.

Throughout this paper, functions are assumed to be measurable, locally integrable and the left hand sides of the inequalities exist if the right hand sides exist.

2. Main results

The following theorems are the main results of the present paper.

Theorem 2.1. *If $p > 1$, $1/p + 1/q = 1$, $r > 1$ and $f(x) \geq 0$ so that*

$$1 + \frac{p}{r-1} \geq \frac{1}{\lambda} > 0$$

for some $\lambda > 0$. For any $a \in (0, +\infty)$, let

$$F(x) := \frac{1}{x} \int_a^x f(t)dt, \quad x \in (0, +\infty).$$

Then for $b \geq a$,

$$\int_a^b x^{-r} F^p(x)dx \leq \left(\frac{\lambda p}{r-1}\right)^p \int_a^b x^{-r} f^p(x)dx. \tag{2.1}$$

Proof. Integrating the left hand side of inequality (2.1) by parts, we have

$$\begin{aligned}
\int_a^b x^{-r} F^p(x)dx &= \frac{x^{-r+1}}{-r+1} F^p(x)\Big|_a^b \\
&\quad - \int_a^b \frac{x^{-r+1}}{-r+1} pF^{p-1}(x) \left[\frac{f(x)}{x} - \frac{1}{x^2}\int_a^x f(t)dt\right] dx \\
&= \frac{x^{-r+1}}{-r+1} F^p(x)\Big|_a^b - \int_a^b \frac{x^{-r+1}}{-r+1} p \frac{f(x)}{x} F^{p-1}(x)dx \\
&\quad + \int_a^b \frac{x^{-r+1}}{-r+1} pF^{p-1}(x) \frac{1}{x^2} \int_a^x f(t)dtdx \\
&= \frac{b^{-r+1}}{-r+1} F^p(b) - \int_a^b \frac{x^{-r}}{-r+1} pf(x)F^{p-1}(x)dx \\
&\quad + \int_a^b \frac{x^{-r}}{-r+1} pF^p(x)dx.
\end{aligned}$$

Using the fact that $r > 1, F(b) \geq 0$ and Holder's inequality, we have

$$\begin{aligned}
&\int_a^b x^{-r} F^p(x) \left[1 - \frac{1}{-r+1} p\right] dx \\
&\qquad = \frac{b^{-r+1}}{-r+1} F^p(b) + \int_a^b \frac{x^{-r}}{r-1} pf(x)F^{p-1}(x)dx \\
&\qquad \leq \int_a^b \frac{x^{-r}}{r-1} pf(x)F^{p-1}(x)dx \\
&\qquad = \frac{p}{r-1} \int_a^b \left[(x^r)^{1-p} x^{-r} f(x)\right] \left[(x^r)^{-(1-p)} F^{p-1}(x)\right] dx \\
&\qquad \leq \frac{p}{r-1} \left[\int_a^b x^{-r} f^p(x)dx\right]^{1/p} \left[\int_a^b x^{-r} F^p(x)dx\right]^{1/q}.
\end{aligned}$$

If we invoke the condition that

$$1+\frac{p}{r-1}\geq\frac{1}{\lambda}>0,$$

we easily obtain

$$\begin{aligned}\int_a^b x^{-r}F^p(x)\frac{1}{\lambda}dx &\leq \int_a^b x^{-r}F^p(x)\left[1+\frac{p}{r-1}\right]dx\\ &\leq \frac{p}{r-1}\left[\int_a^b x^{-r}f^p(x)dx\right]^{1/p}\left[\int_a^b x^{-r}F^p(x)dx\right]^{1/q}.\end{aligned}$$

Further simplification gives

$$\left[\int_a^b x^{-r}F^p(x)dx\right]^{1-1/q}\leq\frac{\lambda p}{r-1}\left[\int_a^b x^{-r}f^p(x)dx\right]^{1/p}.$$

From this, it follows that

$$\int_a^b x^{-r}F^p(x)dx\leq\left(\frac{\lambda p}{r-1}\right)^p\int_a^b x^{-r}f^p(x)dx.$$

This proves the theorem. □

Remark 2.1. Observe that the case $r=p$, inequality (2.1) reduces to

$$\int_a^b\left(\frac{F}{x}\right)^p dx\leq\left(\frac{\lambda p}{p-1}\right)^p\int_a^b\left(\frac{f}{x}\right)^p dx. \tag{2.2}$$

Theorem 2.2. *If $p>1$, $1/p+1/q=1$, $r>1$ and $f(x)\geq 0$ so that*

$$1+\frac{p}{r-1}\geq\frac{1}{\lambda}>0$$

almost everywhere for some $\lambda>0$. For any $a\in(0,+\infty)$, let

$$F(x):=\frac{1}{x}\int_{\frac{x}{2}}^{x}f(t)dt,\quad x\in(0,+\infty).$$

Then for $b\geq a$, we have

$$\int_a^b x^{-r}F^p(x)dx\leq\left(\frac{\lambda p}{r-1}\right)^p\int_a^b x^{-r}\left|f(x)-f\left(x/2\right)\right|^p(x)dx. \tag{2.3}$$

Proof. Integrating the left hand side of inequality (2.3) by parts gives,

$$\int_a^b x^{-r}F^p(x)dx = \frac{b^{-r+1}}{-r+1}F^p(b) + \int_a^b \frac{x^{-r}}{-r+1}pF^p(x)dx - \int_a^b \frac{x^{-r}}{-r+1}pF^{p-1}(x)f(x)dx + \int_a^b \frac{x^{-r}}{-r+1}pF^{p-1}(x)f\left(\frac{x}{2}\right)dx.$$

On further simplification and application of Holder's inequality gives

$$\begin{aligned}
&\int_a^b x^{-r}F^p(x)\left[1 - \frac{p}{-r+1}\right]dx \\
&\leq \int_a^b \frac{x^{-r}}{r-1}pF^{p-1}(x)\left|f(x) - f\left(\frac{x}{2}\right)\right|dx \\
&= \frac{p}{r-1}\int_a^b \left[(x^r)^{(p-1)/p}x^{-r}\left|f(x) - f\left(\frac{x}{2}\right)\right|\right]\left[(x^r)^{-(p-1)/p}F^{p/q}(x)\right]dx \\
&\leq \frac{p}{r-1}\left[\int_a^b x^{-r}\left|f(x) - f\left(\frac{x}{2}\right)\right|^p dx\right]^{1/p}\left[\int_a^b x^{-r}F^p(x)dx\right]^{1/q}.
\end{aligned}$$

Using the fact that

$$1 + \frac{p}{r-1} \geq \frac{1}{\lambda} > 0,$$

we have

$$\begin{aligned}
\int_a^b x^{-r}F^p(x)\frac{1}{\lambda}dx &\leq \int_a^b x^{-r}F^p(x)\left[1 + \frac{p}{r-1}\right]dx \\
&\leq \frac{p}{r-1}\left[\int_a^b x^{-r}\left|f(x) - f\left(\frac{x}{2}\right)\right|^p dx\right]^{1/p}\left[\int_a^b x^{-r}F^p(x)dx\right]^{1/q}.
\end{aligned}$$

From this, it follows that

$$\left[\int_a^b x^{-r}F^p(x)dx\right]^{1/p} \leq \frac{\lambda p}{r-1}\left[\int_a^b x^{-r}\left|f(x) - f\left(\frac{x}{2}\right)\right|^p dx\right]^{1/p}.$$

Hence

$$\int_a^b x^{-r}F^p(x)dx \leq \left(\frac{\lambda p}{r-1}\right)^p \int_a^b x^{-r}\left|f(x) - f\left(\frac{x}{2}\right)\right|^p dx,$$

and the proof is complete. □

Theorem 2.3. *If $p > 1$, $1/p + 1/q = 1$, $r > 1$ and $f(x) \geq 0$ so that*

$$1 + \frac{p}{r-1} - \frac{p}{r-1}\, x\, \frac{f'(t)}{f(t)} \geq \frac{1}{\lambda} > 0$$

almost everywhere for some $\lambda > 0$. For any $a \in (0, +\infty)$, let

$$F(x) := \frac{1}{xf(t)} \int_a^x f(t)g(t)dt, x \in (0, +\infty).$$

Then for $b \geq a$, we have

$$\int_a^b x^{-r} F^p(x)\, dx \leq \Big(\frac{\lambda p}{r-1}\Big)^p \int_a^b x^{-r} g(x)^p(x)\, dx. \tag{2.4}$$

Proof. Integration by parts yields,

$$\begin{aligned}
\int_a^b x^{-r} F^p(x) dx &= \frac{x^{-r+1}}{-r+1} F^p(x)\Big|_a^b - \int_a^b \frac{x^{-r+1}}{-r+1} p F^{p-1}(x) \\
&\quad \times \left[\frac{g(x)}{x} - \frac{1}{(xf(t))^2}\left\{f(t) + xf'(t)\right\} \int_a^x f(t)g(t)dt\right] dx \\
&= \frac{b^{-r+1}}{-r+1} F^p(b) - \int_a^b \frac{x^{-r}}{-r+1} p g(x) F^{p-1}(x) dx \\
&\quad + \int_a^b \frac{x^{-r}}{-r+1} p F^p(x) dx - \int_a^b \frac{x^{-r+1}}{-r+1}\, p\, \frac{f'(t)}{f(t)} F^p(x) dx.
\end{aligned}$$

Using the fact that $r > 1, F(b) \geq 0$ and Holder's integral inequality, we obtain

$$\begin{aligned}
&\int_a^b x^{-r} F^p(x) \left[1 - \frac{p}{-r+1} + \frac{p}{-r+1} x \frac{f'(t)}{f(t)}\right] dx \\
&\quad = \frac{b^{-r+1}}{-r+1} F^p(b) + \int_a^b \frac{x^{-r}}{r-1} p g(x) F^{p-1}(x) dx \\
&\quad \leq \int_a^b \frac{x^{-r}}{r-1} p g(x) F^{p-1}(x) dx \\
&\quad = \frac{p}{r-1} \int_a^b \left[(x^r)^{(p-1)/p} x^{-r} g(x)\right] \left[(x^r)^{-(p-1)/p} F^{p/q}(x)\right] dx \\
&\quad \leq \frac{p}{r-1} \left[\int_a^b x^{-r} g^p(x) dx\right]^{1/p} \left[\int_a^b x^{-r} F^p(x) dx\right]^{1/q}.
\end{aligned}$$

Thus, using the assumption that

$$1+\frac{p}{r-1}-\frac{p}{r-1}\,x\,\frac{f'(t)}{f(t)}\geq\frac{1}{\lambda}>0,$$

we have

$$\begin{aligned}\int_a^b x^{-r}F^p(x)\frac{1}{\lambda}dx &\leq \int_a^b x^{-r}F^p(x)\left[1-\frac{p}{-r+1}+\frac{p}{-r+1}\frac{f'(t)}{f(t)}\right]dx\\ &\leq \frac{p}{r-1}\left[\int_a^b x^{-r}g^p(x)dx\right]^{1/p}\left[\int_a^b x^{-r}F^p(x)dx\right]^{1/q}.\end{aligned}$$

On further simplification, we obtain

$$\left[\int_a^b x^{-r}F^p(x)dx\right]^{1-1/q}\leq\frac{\lambda p}{r-1}\left[\int_a^b x^{-r}g^p(x)dx\right]^{1/p}.$$

Hence

$$\left[\int_a^b x^{-r}F^p(x)dx\right]^{1/p}\leq\frac{\lambda p}{r-1}\left[\int_a^b x^{-r}g^p(x)dx\right]^{1/p}.$$

From this, it follows that

$$\int_a^b x^{-r}F^p(x)dx\leq\left(\frac{\lambda p}{r-1}\right)^p\int_a^b x^{-r}g^p(x)dx.$$

This completes the proof. □

Theorem 2.4. *Let $p>q>0, \alpha\geq 0$ and $r>1$ be real numbers. Let $f:(0,+\infty)\to(0,+\infty)$ be absolutely continuous and let $g:[0,+\infty)\to[0,+\infty)$ be integrable so that*

$$1+\frac{p}{q}\frac{1}{(r-1)}\,\alpha\,x\,\frac{f'(t)}{f(t)}\geq\frac{1}{\lambda}>0\quad a.e.$$

for some $\lambda>0$. For any $a\in(0,+\infty)$, let

$$F(x):=\frac{1}{f^{\alpha}(x)}\int_a^x\frac{f(t)g(t)}{t}\,dt,\quad x\in(0,+\infty).$$

Then for $b\geq a$, we have

$$\int_a^b x^{-r}F^{p/q}(x)\,dx\leq\left(\frac{p}{q}\frac{\lambda p}{(r-1)}\right)^{p/q}\int_a^b x^{-r}\frac{g^{p/q}(x)}{f^{(\alpha-1)p/q}(x)}\,dx.\tag{2.5}$$

Proof. Integration by parts yields,

$$
\begin{aligned}
\int_a^b x^{-r} F^{p/q}(x)dx &= \frac{x^{-r+1}}{-r+1} F^{p/q}(x)\Big|_a^b - \int_a^b \frac{x^{-r+1}}{-r+1} \frac{p}{q} F^{p/q-1}(x) \\
&\quad \times \left[\frac{f(x)g(x)}{x f^{\alpha}}(x) - \alpha \frac{f'(x)}{f^{(\alpha+1)}}(x) \int_a^x \frac{f(t)g(t)}{t} dt \right] dx \\
&= \frac{b^{-r+1}}{-r+1} F^{p/q}(b) - \int_a^b \frac{x^{-r}}{-r+1} \frac{p}{q} \frac{g(x)}{f^{(\alpha-1)}}(x) F^{p/q-1}(x)dx \\
&\quad + \int_a^b \frac{x^{-r}}{-r+1} \frac{p}{q}\, \alpha\, \frac{f'(x)}{f(x)} F^{p/q}(x)dx.
\end{aligned}
$$

Using the fact that $r > 1, F(b) \geq 0$ and the application of Holder's inequality with indices p/q and $p/(p-q)$ respectively, we have

$$
\begin{aligned}
&\int_a^b x^{-r} F^{p/q}(x) \left[1 - \frac{1}{-r+1}\, \alpha\, \frac{p}{q}\, x\, \frac{f'(t)}{f(t)} \right] dx \\
&= \frac{b^{-r+1}}{-r+1} F^{p/q}(b) + \int_a^b \frac{x^{-r}}{-r+1} \frac{p}{q} \frac{g(x)}{f^{(\alpha-1)}}(x) F^{p/q-1}(x)dx \\
&\leq \int_a^b \frac{x^{-r}}{-r+1} \frac{p}{q} \frac{g(x)}{f^{(\alpha-1)}}(x) F^{p/q-1}(x)dx \\
&= \frac{p}{q} \frac{1}{r-1} \int_a^b \left[(x^r)^{(p-q)/p} x^{-r} \frac{g(x)}{f^{(\alpha-1)}}(x) \right] \left[(x^r)^{-(p-q)/q} F^{(p-q)/q}(x) \right] dx \\
&\leq \frac{p}{q} \frac{1}{(r-1)} \left[\int_a^b x^{-r} \frac{g^{p/q}(x)}{f^{(\alpha-1)p/q}}(x)dx \right]^{q/p} \left[\int_a^b x^{-r} F^{p/q}(x)dx \right]^{(p-q)/p}.
\end{aligned}
$$

Using the condition that

$$
1 + \frac{p}{q} \frac{1}{(r-1)}\, \alpha\, x\, \frac{f'(t)}{f(t)} \geq \frac{1}{\lambda} > 0,
$$

we easily see that

$$
\begin{aligned}
&\int_a^b x^{-r} F^{p/q}(x) \frac{1}{\lambda} dx \leq \int_a^b x^{-r} F^{p/q}(x) \left[1 + \frac{1}{r-1}\, \alpha\, \frac{p}{q}\, x\, \frac{f'(t)}{f(t)} \right] dx \\
&\leq \frac{p}{q} \frac{1}{(r-1)} \left[\int_a^b x^{-r} \frac{g^{p/q}(x)}{f^{(\alpha-1)p/q}}(x)dx \right]^{q/p} \left[\int_a^b x^{-r} F^{p/q}(x)dx \right]^{(p-q)/p}.
\end{aligned}
$$

On simplification, we obtain

$$\begin{aligned}\left[\int_a^b x^{-r}F^{p/q}(x)dx\right]^{1-(p-q)/p} &\leq \frac{p}{q}\frac{\lambda}{(r-1)}\left[\int_a^b x^{-r}\frac{g^{p/q}(x)}{f^{(\alpha-1)p/q}}(x)dx\right]^{q/p}\\ &\quad\times\left[\int_a^b x^{-r}F^{p/q}(x)dx\right]^{(p-q)/p}.\end{aligned}$$

That is

$$\begin{aligned}\left[\int_a^b x^{-r}F^{p/q}(x)dx\right]^{q/p} &\leq \frac{p}{q}\frac{\lambda}{(r-1)}\left[\int_a^b x^{-r}\frac{g^{p/q}(x)}{f^{(\alpha-1)p/q}}(x)dx\right]^{q/p}\\ &\quad\times\left[\int_a^b x^{-r}F^{p/q}(x)dx\right]^{(p-q)/p}.\end{aligned}$$

From this, it follows that

$$\int_a^b x^{-r}F^{p/q}(x)dx \leq \Big(\frac{p}{q}\frac{\lambda}{(r-1)}\Big)^{p/q}\int_a^b x^{-r}\frac{g^{p/q}(x)}{f^{(\alpha-1)p/q}}(x)dx,$$

and the proof is complete. □

Remark 2.2. If we set $\alpha = 1$, then our result reduces to Lemma 1 in [2].

Theorem 2.5. *Let $p > q > 0, \alpha \geq 0$ and $r < 1$ be real numbers. Let $f : (0, +\infty) \to (0, +\infty)$ be absolutely continuous and let $g : [0, +\infty) \to [0, +\infty)$ be integrable so that*

$$1 - \frac{p}{q}\frac{1}{(r-1)}\alpha\, x\,\frac{f'(t)}{f(t)} \geq \frac{1}{\lambda} > 0 \quad a.e.$$

for some $\lambda > 0$. For any $b \in (0, +\infty)$, let

$$G(x) := \frac{1}{f^{\alpha}(x)}\int_x^b \frac{f(t)g(t)}{t}\,dt, \quad x \in (0, +\infty).$$

Then for $0 \leq a \leq b$, we have

$$\int_a^b x^{-r}F^{p/q}(x)dx \leq \left(\frac{p}{q}\frac{\lambda p}{(1-r)}\right)^{p/q}\int_a^b x^{-r}\frac{g^{p/q(x)}}{f^{(\alpha-1)p/q}(x)}dx. \tag{2.6}$$

Proof. This is similar to the proof of Theorem 2.4. □

Theorem 2.6. *Let $p > q > 0, \alpha \geq 0$ and $r > 1$ be real numbers. Let $f : (0, +\infty) \to (0, +\infty)$ be absolutely continuous and let $g : [0, +\infty) \to [0, +\infty)$ be integrable so that*

$$1 + \frac{p}{q} \frac{1}{(r-1)} \alpha \, x \frac{f'(t)}{f(t)} \geq \frac{1}{\lambda} > 0 \quad a.e.$$

for some $\lambda > 0$. For any $a \in (0, +\infty)$, let

$$F(x) := \frac{1}{x f^{\alpha}(x)} \int_a^x f(t) g(t) dt, \quad x \in (0, +\infty).$$

Then for $b \geq a$, we have

$$\int_a^b x^{-r} F^p(x) dx \leq \left(\frac{\lambda p}{r-1} \right)^p \int_a^b x^{-r} \frac{g(x)^p(x)}{f^{(\alpha-1)p}(x)} dx. \tag{2.7}$$

Proof. Using integration by parts, we obtain,

$$\begin{aligned}
\int_a^b x^{-r} F^p(x) dx = & \left. \frac{x^{-r+1}}{-r+1} F^p(x) \right|_a^b - \int_a^b \frac{x^{-r+1}}{-r+1} p F^{p-1}(x) \left[\frac{f(x)g(x)}{x f^{\alpha}}(x) \right. \\
& \left. - \frac{1}{(x f^{\alpha}(x))^2} \left\{ f\alpha(x) + \alpha x f^{\alpha-1}(x) f'(x) \right\} \int_a^x f(t) g(t) dt \right] dx \\
= & \frac{b^{-r+1}}{-r+1} F^p(b) - \int_a^b \frac{x^{-r}}{-r+1} p \frac{g(x)}{f^{(\alpha-1)}}(x) F^{p-1}(x) dx \\
& + \int_a^b \frac{x^{-r}}{-r+1} p F^p(x) dx + \int_a^b \frac{x^{-r+1}}{-r+1} p \frac{f'(x)}{f(x)} F^p(x) dx.
\end{aligned}$$

By Holder's inequality and the fact that $r > 1$ and $F(b) \geq 0$ we obtain

$$\begin{aligned}
& \int_a^b x^{-r} F^p(x) \left[1 - \frac{p}{-r+1} - \frac{p}{-r+1} x\alpha \frac{f'(x)}{f(x)} \right] dx \\
& = \frac{b^{-r+1}}{-r+1} F^p(b) + - \int_a^b \frac{x^{-r}}{-r+1} p \frac{g(x)}{f^{(\alpha-1)}}(x) F^{p-1}(x) dx \\
& \leq - \int_a^b \frac{x^{-r}}{-r+1} p \frac{g(x)}{f^{(\alpha-1)}}(x) F^{p-1}(x) dx \\
& = \frac{1}{r-1} p \int_a^b \left[(x^r)^{(p-1)/p} x^{-r} \frac{g(x)}{f^{(\alpha-1)}}(x) \right] \left[(x^r)^{-(p-1)/p} F^{p/q}(x) \right] dx \\
& \leq \frac{p}{r-1} \left[\int_a^b x^{-r} \frac{g^p(x)}{f^{(\alpha-1)p}}(x) dx \right]^{1/p} \left[\int_a^b x^{-r} F^p(x) dx \right]^{1/q}.
\end{aligned}$$

From the assumption that

$$1+\frac{p}{q}\frac{1}{(r-1)}\,\alpha\,x\,\frac{f'(t)}{f(t)}\geq\frac{1}{\lambda}>0,$$

it is clear that

$$\begin{aligned}\int_a^b x^{-r}F^p(x)\frac{1}{\lambda}\,dx &\leq \int_a^b x^{-r}F^p(x)\left[1-\frac{p}{-r+1}-\frac{p}{-r+1}\,x\,\alpha\,\frac{f'(x)}{f(x)}\right]dx\\ &\leq \frac{p}{r-1}\left[\int_a^b x^{-r}\frac{g^p(x)}{f^{(\alpha-1)p}}(x)dx\right]^{1/p}\left[\int_a^b x^{-r}F^p(x)dx\right]^{1/q}.\end{aligned}$$

On further manipulation, we obtain

$$\left[\int_a^b x^{-r}F^p(x)dx\right]^{1/p}\leq\frac{\lambda p}{r-1}\left[\int_a^b x^{-r}\frac{g^p(x)}{f^{(\alpha-1)p}(x)}\,dx\right]^{1/p}.$$

It follows from this that

$$\int_a^b x^{-r}F^p(x)dx\leq\left[\frac{\lambda p}{r-1}\right]^p\int_a^b x^{-r}\frac{g^p(x)}{f^{(\alpha-1)p}(x)}\,dx.$$

This completes the proof. □

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College of Natural Sciences
Department of Mathematical Sciences
University of Agriculture
P.M.B.2240
Abeokuta, Nigeria

e-mail: oguntuase@yahoo.com