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SOME IDENTITIES FOR THE RIEMANN ZETA-FUNCTION II

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Abstract. Several identities for the Riemann zeta-function $\zeta(s)$ are proved. For example, if $\varphi_1(x) := \{x\} = x - [x], \quad \varphi_n(x) := \int_0^{+\infty}\{u\}\varphi_{n-1}\left(\frac{x}{u}\right)\frac{\mathrm{d}u}{u}$ $(n \ge 2)$, then

$$\frac{\zeta^n(s)}{(-s)^n} = \int_0^{+\infty} \varphi_n(x)x^{-1-s}\,\mathrm{d}x \quad (s = \sigma + it,\ 0 < \sigma < 1)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|\zeta(\sigma+it)|^{2n}}{(\sigma^2+t^2)^n}\,\mathrm{d}t = \int_0^{+\infty}\varphi_n^2(x)x^{-1-2\sigma}\,\mathrm{d}x \qquad (0 < \sigma < 1).$$

Let as usual $\zeta(s) = \sum_{n=1}^{+\infty} n^{-s}$ $(\operatorname{Re} s > 1)$ denote the Riemann zeta-function. This note is the continuation of the author's work [6], where several identities involving $\zeta(s)$ were obtained. The basic idea is to use properties of the Mellin transform $(f : [0, +\infty) \to \mathbb{R}\ )$

$$F(s) = \mathcal{M}[f(x); s] := \int_0^{+\infty} f(x)x^{s-1}\,\mathrm{d}x \qquad (s = \sigma + it,\ \sigma > 0), \tag{1}$$

in particular the analogue of the Parseval formula for Mellin transforms, namely

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} |F(s)|^2\,\mathrm{d}s = \int_0^{+\infty} f^2(x)x^{2\sigma-1}\,\mathrm{d}x. \tag{2}$$

For the conditions under which (2) holds, see e.g., [5] and [11]. If $\{x\}$ denotes the fractional part of x ($\{x\} = x - [x]$, where $[x]$ is the greatest integer not

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exceeding x), we have the classical formula (see e.g., eq. (2.1.5) of E.C. Titchmarsh [12])

$$\frac{\zeta(s)}{s} = -\int_0^{+\infty} \{x\}x^{-1-s}\,\mathrm{d}x = -\int_0^{+\infty} \{1/x\}x^{s-1}\,\mathrm{d}x, \tag{3}$$

where $s = \sigma + it$, $0 < \sigma < 1$. A quick proof is as follows. We have

$$\begin{aligned}\zeta(s) &= \int_{1-0}^{+\infty} x^{-s}\,\mathrm{d}[x] = s\int_1^{+\infty} [x]x^{-s-1}\,\mathrm{d}x\\ &= s\int_1^{+\infty} ([x]-x)x^{-s-1}\,\mathrm{d}x + s\int_1^{+\infty} x^{-s}\,\mathrm{d}x\\ &= -s\int_1^{+\infty} \{x\}x^{-s-1}\,\mathrm{d}x + \frac{s}{s-1}.\end{aligned}$$

This holds initially for $\sigma > 1$, but since the last integral is absolutely convergent for $\sigma > 0$, it holds in this region as well by analytic continuation. Since

$$s\int_0^1 \{x\}x^{-s-1}\,\mathrm{d}x = s\int_0^1 x^{-s}\,\mathrm{d}x = \frac{s}{1-s} \qquad (0 < \sigma < 1),$$

we obtain (3) on combining the preceding two formulae. We note that (3) is a special case of the so-called Müntz's formula (with $f(x) = \chi_{[0,1]}(x)$, the characteristic function of the unit interval)

$$\zeta(s)F(s) = \int_0^{+\infty} Pf(x)\cdot x^{s-1}\,\mathrm{d}x, \tag{4}$$

where the Müntz operator P is the linear operator defined formally on functions $f : [0, +\infty) \to \mathbb{C}$ by

$$Pf(x) := \sum_{n=1}^{+\infty} f(nx) - \frac{1}{x}\int_0^{+\infty} f(t)\,\mathrm{d}t. \tag{5}$$

Besides the original proof of (4) by Müntz [8], proofs are given by E.C. Titchmarsh [12, Chapter 1, Section 2.11] and recently by L. Báez-Duarte [2]. The identity (4) is valid for $0 < \sigma < 1$ if $f'(x)$ is continuous, bounded in any finite interval and is $O(x^{-\beta})$ for $x \to \infty$ where $\beta > 1$ is a constant. The identity (3), which Báez-Duarte [2] calls the *proto-Müntz* identity, plays an important rôle in the approach to the Riemann Hypothesis (RH, that all complex zeros of $\zeta(s)$ have real parts equal to 1/2) via methods from functional analysis (see e.g., the works [1]–[4] and [9]).

Our first aim is to generalize (3). We introduce the convolution functions $\varphi_n(x)$ by

$$\varphi_1(x) := \{x\} = x - [x],\ \varphi_n(x) := \int_0^{+\infty} \{u\}\varphi_{n-1}\big(\frac{x}{u}\big)\,\frac{\mathrm{d}u}{u} \qquad (n \ge 2). \tag{6}$$

The asymptotic behaviour of the function $\varphi_n(x)$ is contained in

Theorem 1. *If $n \ge 2$ is a fixed integer, then*

$$\varphi_n(x) = \frac{x}{(n-1)!}\log^{n-1}(1/x) + O\Big(x\log^{n-2}(1/x)\Big) \qquad (0 < x < 1), \tag{7}$$

and

$$\varphi_n(x) = O(\log^{n-1}(x+1)) \qquad (x \ge 1). \tag{8}$$

Proof. Using the properties of $\{x\}$, namely $\{x\} = x$ for $0 < x < 1$ and $\{x\} \le x$, one easily verifies (7) and (8) when $n = 2$. To prove the general case we use induction, supposing that the theorem is true for some n. Then, when $0 < x < 1$,

$$\varphi_{n+1}(x) = \int_0^x + \int_x^1 + \int_1^{+\infty} = I_1 + I_2 + I_3,$$

say. We have, by change of variable,

$$I_1 = \int_0^x \{u\}\varphi_n\left(\frac{x}{u}\right)\frac{\mathrm{d}u}{u} = \int_0^x \varphi_n\left(\frac{x}{u}\right)\mathrm{d}u = x\int_1^{+\infty}\varphi_n(v)\,\frac{\mathrm{d}v}{v^2} = O(x).$$

By the induction hypothesis

$$\begin{aligned} I_2 &= \int_x^1 \{u\}\varphi_n\left(\frac{x}{u}\right)\frac{\mathrm{d}u}{u} = \int_x^1 \varphi_n\left(\frac{x}{u}\right)\mathrm{d}u \\ &= \int_x^1 \left\{\frac{x}{(n-1)!u}\log^{n-1}\left(\frac{u}{x}\right) + O\left(\frac{x}{u}\log^{n-2}\left(\frac{u}{x}\right)\right)\right\}\mathrm{d}u \\ &= \frac{x}{(n-1)!}\int_1^{1/x}\log^{n-1} y\frac{\mathrm{d}y}{y} + O\Big(x\log^{n-1}(1/x)\Big) \\ &= \frac{x}{n!}\log^n(1/x) + O(x\log^{n-1}(1/x)). \end{aligned}$$

Finally, since $\{x\} \le x$ and (8) holds,

$$I_3 = \int_1^{+\infty} \{u\}\varphi_n\left(\frac{x}{u}\right)\frac{\mathrm{d}u}{u} \ll x\int_1^{+\infty} \log^{n-1}\left(\frac{u}{x}\right)\frac{\mathrm{d}u}{u^2} \ll x\log^{n-1}\left(\frac{1}{x}\right).$$

The proof of (8) is on similar lines, when we write

$$\varphi_{n+1}(x) = \int_0^1 + \int_1^x + \int_x^{+\infty} = J_1 + J_2 + J_3 \qquad (x \ge 1),$$

say, so that there is no need to repeat the details. By more elaborate analysis (7) could be further sharpened. □

Theorem 2. *If $n \ge 1$ is a fixed integer, and $s = \sigma + it$, $0 < \sigma < 1$, then*

$$\frac{\zeta^n(s)}{(-s)^n} = \int_0^{+\infty} \varphi_n(x)x^{-1-s}\,\mathrm{d}x. \tag{9}$$

Clearly (9) reduces to (3) when $n = 1$. From Theorem 1 it transpires that the integral in (9) is absolutely convergent for $0 < \sigma < 1$. By using (2) (with $-s$ in place of s) we obtain the following

Corollary 1. *For $n \in \mathbb{N}$ we have*

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\zeta(\sigma+it)|^{2n}}{(\sigma^2+t^2)^n}\,\mathrm{d}t = \int_0^{+\infty} \varphi_n^2(x)x^{-1-2\sigma}\,\mathrm{d}x \qquad (0 < \sigma < 1). \tag{10}$$

Proof of Theorem 2. As already stated, (9) is true for $n = 1$. The general case is proved then by induction. Suppose that (9) is true for some n, and consider

$$\frac{\zeta^{n+1}(s)}{(-s)^{n+1}} = \int_0^{+\infty}\int_0^{+\infty} \{x\}\varphi_n(y)(xy)^{-1-s}\,\mathrm{d}x\ \mathrm{d}y \qquad (0 < \sigma < 1)$$

as a double integral. We make the change of variables $x = v, y = u/v$, noting that the absolute value of the Jacobian of the transformation is $1/v$. The above integral becomes then

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} \{v\}\varphi_n\left(\frac{u}{v}\right)u^{-1-s}v^{-1}\,\mathrm{d}u\,\mathrm{d}v &= \int_0^{+\infty}\left(\int_0^{+\infty}\{v\}\varphi_n\left(\frac{u}{v}\right)\frac{\mathrm{d}v}{v}\right)u^{-1-s}\,\mathrm{d}u\\ &= \int_0^{+\infty} \varphi_{n+1}(x)x^{-1-s}\,\mathrm{d}x,\end{aligned}$$

as asserted. The change of integration is valid by absolute convergence, which is guaranteed by Theorem 1. □

Remark 1. L. Báez-Duarte kindly pointed out to me that the above procedure leads in fact formally to a convolution theorem for Mellin transforms, namely (cf. (1))

$$\mathcal{M}\left[\int_0^{+\infty} f(u)g\left(\frac{x}{u}\right)\frac{\mathrm{d}u}{u}; s\right] = \mathcal{M}[f(x);s]\,\mathcal{M}[g(x);s] = F(s)G(s), \tag{11}$$

which is eq. (4.2.22) of I. Sneddon [10]. An alternative proof of (9) follows from the second formula in (3) and (11), but we need again a result like Theorem 1 to ensure the validity of the repeated use of (11). A similar approach via (modified) Mellin transforms and convolutions was carried out by the author in [7].

There is another possibility for the use of the identity (3). Namely, one can evaluate the Laplace transform of $\{x\}/x$ for real values of the variable. This is given by

Theorem 3. *If $M \ge 1$ is a fixed integer and γ denotes Euler's constant, then for $T \to +\infty$*

$$\begin{aligned}\int_0^{+\infty} \frac{\{x\}}{x}\mathrm{e}^{-x/T}\,\mathrm{d}x &= \tfrac12\log T - \tfrac12\gamma + \tfrac12\log(2\pi)\\ &\quad + \sum_{m=1}^{M}\frac{\zeta(1-2m)}{(2m-1)!(1-2m)}T^{1-2m} + O_M(T^{-1-2M}).\end{aligned}\tag{12}$$

Proof. We multiply (3) by $T^s\Gamma(s)$, where $\Gamma(s)$ is the gamma-function, integrate over s and use the well-known identity (e.g., see the Appendix of [5])

$$\mathrm{e}^{-z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}\Gamma(s)\,\mathrm{d}s \qquad (\operatorname{Re} z > 0,\ c > 0).$$

We obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(s)}{s}T^s\Gamma(s)\,\mathrm{d}s = -\int_0^{+\infty}\frac{\{x\}}{x}\,\mathrm{e}^{-x/T}\,\mathrm{d}x \qquad (0 < c < 1). \tag{13}$$

In the integral on the left-hand side of (13) we shift the line of integration to $\operatorname{Re} s = -N - 1/2$, $N = 2M + 1$ (i.e., taking $c = -N - 1/2$) and then apply the residue theorem. The gamma-function has simple poles at $s = -m$, $m = 0, 1, 2, \ldots$ with residues $(-1)^m/m!$. The zeta-function has simple (so-called "trivial zeros") at $s = -2m$, $m \in \mathbb{N}$, which cancel with the corresponding poles of $\Gamma(s)$. Thus there remains a pole of order two at $s = 0$, plus simple poles at $s = -1, -3, -5, \ldots$. The former produces the main term in (12), when we take into account that $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\frac{1}{2}\log(2\pi)$ (see [5, Chapter 1]) and $\Gamma'(1) = -\gamma$. The simple poles at $s = -1, -3, -5, \ldots$ produce the sum over m in (12), and the proof is complete. □

Remark 2. The method of proof clearly yields also, as $T \to +\infty$,

$$\int_0^{+\infty} \frac{\varphi_n(x)}{x} \mathrm{e}^{-x/T}\,\mathrm{d}x = P_n(\log T) + \sum_{m=1}^{M} c_{m,n} T^{1-2m} + O_M(T^{-1-2M}),$$

where $P_n(z)$ is a polynomial in z of degree n whose coefficients may be explicitly evaluated, and $c_{m,n}$ are suitable constants which also may be explicitly evaluated.

For our last result we turn to Müntz's identity (4)–(5) and choose $f(x) = \mathrm{e}^{-\pi x^2}$, which is a fast converging kernel function. Then

$$\begin{aligned} Pf(x) &= \sum_{n=1}^{+\infty} f(nx) - \frac{1}{x}\int_0^{+\infty} f(t)\,\mathrm{d}t = \sum_{n=1}^{+\infty} \mathrm{e}^{-\pi n^2 x^2} - \frac{1}{2x}, \\ F(s) &= \int_0^{+\infty} \mathrm{e}^{-\pi x^2} x^{s-1}\,\mathrm{d}x = \tfrac{1}{2}\pi^{-s/2}\Gamma(\tfrac{1}{2}s). \end{aligned}$$

From (2) and (4) it follows then that, for $0 < \sigma < 1$,

$$\begin{aligned} &\int_{-\infty}^{+\infty} |\zeta(\sigma + it)\Gamma(\tfrac{1}{2}\sigma + \tfrac{1}{2}it)|^2\,\mathrm{d}t \\ &\qquad = 8\pi^{1+\sigma}\int_0^{+\infty} \left(\sum_{n=1}^{+\infty} \mathrm{e}^{-\pi n^2 x^2} - \frac{1}{2x}\right)^2 x^{2\sigma-1}\,\mathrm{d}x. \end{aligned} \tag{14}$$

The series on the right-hand side of (14) is connected to Jacobi's theta function

$$\theta(z) := \sum_{n=1}^{+\infty} \mathrm{e}^{-\pi n^2 z} \qquad (\operatorname{Re} z > 0), \tag{15}$$

which satisfies the functional equation (proved easily by e.g., Poisson summation formula)

$$\theta(t) = \frac{1}{\sqrt{t}}\,\theta\left(\frac{1}{t}\right) \qquad (t > 0). \tag{16}$$

From (15)–(16) we infer that

$$\sum_{n=1}^{+\infty} \mathrm{e}^{-\pi n^2 x^2} = \frac{1}{2}\big(\theta(x^2) - 1\big) = \frac{1}{2x}\theta\left(\frac{1}{x^2}\right) - \frac{1}{2} \qquad (x > 0). \tag{17}$$

By using (17) it is seen that the right-hand side of (14) equals

(18)

$$8\pi^{1+\sigma}\int_0^{+\infty} \frac{1}{4}\left(\frac{1}{x}\theta\left(\frac{1}{x^2}\right) - 1 - \frac{1}{x}\right)^2 x^{2\sigma-1}\,\mathrm{d}x$$
$$= 2\pi^{1+\sigma}\int_0^{+\infty} (u\theta(u^2) - 1 - u)^2 u^{-1-2\sigma}\,\mathrm{d}u.$$

The (absolute) convergence of the last integral at infinity follows from

$$u\theta(u^2) - u = 2u\sum_{n=1}^{+\infty} \mathrm{e}^{-\pi n^2 u^2},$$

while the convergence at zero follows from

$$u\theta(u^2) = \theta\left(\frac{1}{u^2}\right) = 1 + O\left(\mathrm{e}^{-u^{-2}}\right) \qquad (u \to 0+).$$

Now we note that (14) remains unchanged when σ is replaced by $1-\sigma$, and then we use the functional equation (see e.g., [5, Chapter 1]) for $\zeta(s)$ in the form

$$\pi^{-s/2}\zeta(s)\Gamma(\tfrac{1}{2}s) = \pi^{-(1-s)/2}\zeta(1-s)\Gamma(\tfrac{1}{2}(1-s))$$

to transform the resulting left-hand side of (14). Then (14) and (18) yield the following

Theorem 4. *For $0 < \sigma < 1$ we have*

$$\int_{-\infty}^{+\infty} |\zeta(\sigma+it)\Gamma(\tfrac{1}{2}\sigma + \tfrac{1}{2}it)|^2\,\mathrm{d}t = 2\pi^{\sigma}\int_0^{+\infty} (u\theta(u^2) - 1 - u)^2 u^{2\sigma-3}\,\mathrm{d}u.$$

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