Abstract. Several identities for the Riemann zeta-function $\zeta(s)$ are proved. For example, if $\varphi_1(x) := \{x\} = x - \lfloor x \rfloor$, $\varphi_n(x) := \int_0^\infty \{u\} \varphi_{n-1} \left( \frac{x}{u} \right) \frac{du}{u} \ (n \geq 2)$, then
$$
\frac{\zeta^n(s)}{(-s)^n} = \int_0^\infty \varphi_n(x)x^{-1-s} \, dx \quad (s = \sigma + it, \ 0 < \sigma < 1)
$$
and
$$
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{\zeta(\sigma + it)^2}{(\sigma^2 + t^2)^n} \right|^2 \, dt = \int_0^\infty \varphi_2^n(x)x^{-1-2\sigma} \, dx \quad (0 < \sigma < 1).
$$

Let as usual $\zeta(s) = \sum_{n=1}^{+\infty} n^{-s}$ (Re $s > 1$) denote the Riemann zeta-function. This note is the continuation of the author’s work [6], where several identities involving $\zeta(s)$ were obtained. The basic idea is to use properties of the Mellin transform ($f: [0, +\infty) \to \mathbb{R}$)

\begin{equation}
F(s) = \mathcal{M}[f(x); s] := \int_0^{+\infty} f(x)x^{s-1} \, dx \quad (s = \sigma + it, \ \sigma > 0),
\end{equation}

in particular the analogue of the Parseval formula for Mellin transforms, namely

\begin{equation}
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} |F(s)|^2 \, ds = \int_0^{+\infty} f^2(x)x^{2\sigma-1} \, dx.
\end{equation}

For the conditions under which (2) holds, see e.g., [5] and [11]. If $\{x\}$ denotes the fractional part of $x$ ($\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not
exceeding $x$), we have the classical formula (see e.g., eq. (2.1.5) of E.C. Titchmarsh [12])

\[
\zeta(s) = -\int_0^{+\infty} \{x\} x^{-1-s} \, dx = -\int_0^{+\infty} \{1/x\} x^{s-1} \, dx,
\]

where $s = \sigma + it$, $0 < \sigma < 1$. A quick proof is as follows. We have

\[
\zeta(s) = \int_{1-0}^{+\infty} x^{-s} d[x] = \int_{1}^{+\infty} [x] x^{-s-1} \, dx
\]

\[
= s \int_{1}^{+\infty} ([x] - x)x^{-s-1} \, dx + s \int_{1}^{+\infty} x^{-s} \, dx
\]

\[
= -s \int_{1}^{+\infty} \{x\} x^{-s-1} \, dx + \frac{s}{s-1}.
\]

This holds initially for $\sigma > 1$, but since the last integral is absolutely convergent for $\sigma > 0$, it holds in this region as well by analytic continuation. Since

\[
s \int_{0}^{1} \{x\} x^{-s-1} \, dx = s \int_{0}^{1} x^{-s} \, dx = \frac{s}{1-s} \quad (0 < \sigma < 1),
\]

we obtain (3) on combining the preceding two formulae. We note that (3) is a special case of the so-called Müntz’s formula (with $f(x) = \chi_{[0,1]}(x)$, the characteristic function of the unit interval)

\[
\zeta(s) F(s) = \int_{0}^{+\infty} Pf(x) \cdot x^{s-1} \, dx,
\]

where the Müntz operator $P$ is the linear operator defined formally on functions $f : [0, +\infty) \to \mathbb{C}$ by

\[
Pf(x) := \sum_{n=1}^{+\infty} f(nx) - \frac{1}{x} \int_{0}^{+\infty} f(t) \, dt.
\]

Besides the original proof of (4) by Müntz [8], proofs are given by E.C. Titchmarsh [12, Chapter 1, Section 2.11] and recently by L. Báez-Duarte [2]. The identity (4) is valid for $0 < \sigma < 1$ if $f'(x)$ is continuous, bounded in any finite interval and is $O(x^{-\beta})$ for $x \to \infty$ where $\beta > 1$ is a constant. The identity (3), which Báez-Duarte [2] calls the proto-Müntz identity, plays an important rôle in the approach to the Riemann Hypothesis (RH, that
all complex zeros of $\zeta(s)$ have real parts equal to $1/2$) via methods from functional analysis (see e.g., the works [1]–[4] and [9]).

Our first aim is to generalize (3). We introduce the convolution functions $\varphi_n(x)$ by

\[ \varphi_1(x) := \{x\} = x - \lfloor x \rfloor, \quad \varphi_n(x) := \begin{cases} \int_0^{+\infty} \{u\} \varphi_{n-1}(\frac{x}{u}) \frac{du}{u} & (n \geq 2), \end{cases} \]

The asymptotic behaviour of the function $\varphi_n(x)$ is contained in

**Theorem 1.** If $n \geq 2$ is a fixed integer, then

\[ \varphi_n(x) = \frac{x}{(n-1)!} \log^{n-1}(1/x) + O\left(x \log^{n-2}(1/x)\right) \quad (0 < x < 1), \]

and

\[ \varphi_n(x) = O(\log^{n-1}(x+1)) \quad (x \geq 1). \]

**Proof.** Using the properties of $\{x\}$, namely $\{x\} = x$ for $0 < x < 1$ and $\{x\} \leq x$, one easily verifies (7) and (8) when $n = 2$. To prove the general case we use induction, supposing that the theorem is true for some $n$. Then, when $0 < x < 1$, \[ \varphi_{n+1}(x) = \int_0^x + \int_x^1 + \int_1^{+\infty} = I_1 + I_2 + I_3, \]

say. We have, by change of variable,

\[ I_1 = \int_0^x \{u\} \varphi_n \left( \frac{x}{u} \right) \frac{du}{u} = \int_0^x \varphi_n \left( \frac{x}{u} \right) du = x \int_1^{+\infty} \varphi_n(v) \frac{dv}{v^2} = O(x). \]

By the induction hypothesis

\[ I_2 = \int_x^1 \{u\} \varphi_n \left( \frac{x}{u} \right) \frac{du}{u} = \int_x^1 \varphi_n \left( \frac{x}{u} \right) du \]

\[ = \int_x^1 \left\{ \frac{x}{u} - \frac{1}{(n-1)!} \log^{n-1} \left( \frac{x}{u} \right) + O\left( \frac{x}{u} \log^{n-2} \left( \frac{x}{u} \right) \right) \right\} \frac{du}{u} \]

\[ = \frac{x}{(n-1)!} \int_1^{1/x} \log^{n-1} \left( \frac{y}{x} \right) y \frac{dy}{y} + O\left( x \log^{n-1}(1/x) \right) \]

\[ = \frac{x}{n!} \log^n(1/x) + O(x \log^{n-1}(1/x)). \]
Finally, since \( \{ x \} \leq x \) and (8) holds,

\[
I_3 = \int_1^{+\infty} \{ u \} \varphi_n \left( \frac{x}{u} \right) \frac{du}{u} \ll x \int_1^{+\infty} \log^{n-1} \left( \frac{u}{x} \right) \frac{du}{u^2} \ll x \log^{n-1} \left( \frac{1}{x} \right).
\]

The proof of (8) is on similar lines, when we write

\[
\varphi_{n+1}(x) = \int_0^1 + \int_1^x + \int_x^{+\infty} = J_1 + J_2 + J_3 \quad (x \geq 1),
\]

say, so that there is no need to repeat the details. By more elaborate analysis (7) could be further sharpened. □

**Theorem 2.** If \( n \geq 1 \) is a fixed integer, and \( s = \sigma + it, \) \( 0 < \sigma < 1, \) then

\[
(9) \quad \frac{\zeta^n(s)}{(-s)^n} = \int_0^{+\infty} \varphi_n(x)x^{-1-s} \, dx.
\]

Clearly (9) reduces to (3) when \( n = 1. \) From Theorem 1 it transpires that the integral in (9) is absolutely convergent for \( 0 < \sigma < 1. \) By using (2) (with \(-s\) in place of \( s\)) we obtain the following

**Corollary 1.** For \( n \in \mathbb{N} \) we have

\[
(10) \quad \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\zeta((\sigma + it)^2)^2}{(\sigma^2 + t^2)^n} \, dt = \int_0^{+\infty} \varphi_n^2(x)x^{-1-2\sigma} \, dx \quad (0 < \sigma < 1).
\]

**Proof of Theorem 2.** As already stated, (9) is true for \( n = 1. \) The general case is proved then by induction. Suppose that (9) is true for some \( n, \) and consider

\[
\frac{\zeta^{n+1}(s)}{(-s)^{n+1}} = \int_0^{+\infty} \int_0^{+\infty} \{ x \} \varphi_n(y)(xy)^{-1-s} \, dy \, dx \quad (0 < \sigma < 1)
\]

as a double integral. We make the change of variables \( x = v, y = u/v, \) noting that the absolute value of the Jacobian of the transformation is \( 1/v. \) The above integral becomes then

\[
\int_0^{+\infty} \int_0^{+\infty} \{ v \} \varphi_n \left( \frac{u}{v} \right) u^{-1-s} v^{-1} \, du \, dv = \int_0^{+\infty} \left( \int_0^{+\infty} \{ v \} \varphi_n \left( \frac{u}{v} \right) \frac{dv}{v} \right) u^{-1-s} \, du
\]

\[
= \int_0^{+\infty} \varphi_{n+1}(x)x^{-1-s} \, dx,
\]
as asserted. The change of integration is valid by absolute convergence, which is guaranteed by Theorem 1. □

Remark 1. L. Báez-Duarte kindly pointed out to me that the above procedure leads in fact formally to a convolution theorem for Mellin transforms, namely (cf. (1))

\[
\mathcal{M} \left[ \int_0^\infty f(u)g\left(\frac{x}{u}\right) \frac{du}{u} ; s \right] = \mathcal{M}[f(x); s]\mathcal{M}[g(x); s] = F(s)G(s),
\]

which is eq. (4.2.22) of I. Sneddon [10]. An alternative proof of (9) follows from the second formula in (3) and (11), but we need again a result like Theorem 1 to ensure the validity of the repeated use of (11). A similar approach via (modified) Mellin transforms and convolutions was carried out by the author in [7].

There is another possibility for the use of the identity (3). Namely, one can evaluate the Laplace transform of \( \{x\}/x \) for real values of the variable.

Theorem 3. If \( M \geq 1 \) is a fixed integer and \( \gamma \) denotes Euler’s constant, then for \( T \to +\infty \)

\[
\int_0^\infty \frac{x^z}{x} e^{-x/T} \, dx = \frac{1}{2} \log T - \frac{1}{2} \gamma + \frac{1}{2} \log(2\pi)
\]

\[+ \sum_{m=1}^M \frac{\zeta(1-2m)}{(2m-1)!(1-2m)} T^{1-2m} + O(T^{-1-2M}).
\]

Proof. We multiply (3) by \( T^s \Gamma(s) \), where \( \Gamma(s) \) is the gamma-function, integrate over \( s \) and use the well-known identity (e.g., see the Appendix of [5])

\[e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} \Gamma(s) \, ds \quad (\text{Re } z > 0, \ c > 0).
\]

We obtain

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s)}{s} T^s \Gamma(s) \, ds = -\int_0^\infty \frac{x^z}{x} e^{-x/T} \, dx \quad (0 < c < 1).
\]

In the integral on the left-hand side of (13) we shift the line of integration to \( \text{Re } s = -N - 1/2, \ N = 2M + 1 \) (i.e., taking \( c = -N - 1/2 \)) and then apply the residue theorem. The gamma-function has simple poles at \( s = -m, \ m = 0, 1, 2, \ldots \) with residues \((-1)^m/m!\). The zeta-function has
simple (so-called “trivial zeros”) at \( s = -2m, m \in \mathbb{N} \), which cancel with the corresponding poles of \( \Gamma(s) \). Thus there remains a pole of order two at \( s = 0 \), plus simple poles at \( s = -1, -3, -5, \ldots \). The former produces the main term in (12), when we take into account that \( \zeta(0) = -\frac{1}{2}, \zeta'(0) = -\frac{1}{2} \log(2\pi) \) (see [5, Chapter 1]) and \( \Gamma'(1) = -\gamma \). The simple poles at \( s = -1, -3, -5, \ldots \) produce the sum over \( m \) in (12), and the proof is complete. □

**Remark 2.** The method of proof clearly yields also, as \( T \to +\infty \),

\[
\int_0^{+\infty} \frac{\varphi_n(x)}{x} e^{-x/T} \, dx = P_n(\log T) + \sum_{m=1}^{M} c_{m,n} T^{1-2m} + O_M(T^{-1-2M}),
\]

where \( P_n(z) \) is a polynomial in \( z \) of degree \( n \) whose coefficients may be explicitly evaluated, and \( c_{m,n} \) are suitable constants which also may be explicitly evaluated.

For our last result we turn to Müntz’s identity (4)–(5) and choose \( f(x) = e^{-\pi x^2} \), which is a fast converging kernel function. Then

\[
P f(x) = \sum_{n=1}^{+\infty} f(nx) - \frac{1}{x} \int_0^{+\infty} f(t) \, dt = \sum_{n=1}^{+\infty} e^{-\pi n^2 x^2} - \frac{1}{2x},
\]

\[
F(s) = \int_0^{+\infty} e^{-\pi x^2} x^{s-1} \, dx = \frac{1}{2} \pi^{-s/2} \Gamma(\frac{1}{2}s).
\]

From (2) and (4) it follows then that, for \( 0 < \sigma < 1 \),

\[
\int_{-\infty}^{+\infty} |\zeta(\sigma + it)\Gamma(\frac{1}{2}\sigma + \frac{1}{2}it)|^2 \, dt = 8\pi^{1+\sigma} \int_0^{+\infty} \left( \sum_{n=1}^{+\infty} e^{-\pi n^2 x^2} - \frac{1}{2x} \right)^2 x^{2\sigma-1} \, dx.
\]

The series on the right-hand side of (14) is connected to Jacobi’s theta function

\[
\theta(z) := \sum_{n=1}^{+\infty} e^{-\pi n^2 z} \quad (\text{Re } z > 0),
\]

which satisfies the functional equation (proved easily by e.g., Poisson summation formula)

\[
\theta(t) = \frac{1}{\sqrt{t}} \theta \left( \frac{1}{t} \right) \quad (t > 0).
\]
From (15)–(16) we infer that

\[(17) \sum_{n=1}^{\infty} e^{-\pi n^2 x^2} = \frac{1}{2} \left( \theta(x^2) - 1 \right) = \frac{1}{2x} \theta \left( \frac{1}{x^2} \right) - \frac{1}{2} \quad (x > 0).\]

By using (17) it is seen that the right-hand side of (14) equals

\[(18) 8\pi^{1+\sigma} \int_{0}^{+\infty} \frac{1}{4} \left( \frac{1}{x} \theta \left( \frac{1}{x^2} \right) - 1 - \frac{1}{x} \right)^2 x^{2\sigma - 1} \, dx = 2\pi^{1+\sigma} \int_{0}^{+\infty} (u\theta(u^2) - 1 - u)^2 u^{-1-2\sigma} \, du.\]

The (absolute) convergence of the last integral at infinity follows from

\[u\theta(u^2) - u = 2u \sum_{n=1}^{\infty} e^{-\pi n^2 u^2},\]

while the convergence at zero follows from

\[u\theta(u^2) = \theta \left( \frac{1}{u^2} \right) = 1 + O \left( e^{-u^{-2}} \right) \quad (u \to 0+).\]

Now we note that (14) remains unchanged when \(\sigma\) is replaced by \(1 - \sigma\), and then we use the functional equation (see e.g., [5, Chapter 1]) for \(\zeta(s)\) in the form

\[\pi^{-s/2} \zeta(s) \Gamma \left( \frac{1}{2} s \right) = \pi^{-(1-s)/2} \zeta(1-s) \Gamma \left( \frac{1}{2} (1-s) \right)\]

to transform the resulting left-hand side of (14). Then (14) and (18) yield the following

**Theorem 4.** For \(0 < \sigma < 1\) we have

\[
\int_{-\infty}^{+\infty} |\zeta(\sigma + it)\Gamma(\frac{1}{2} \sigma + \frac{1}{2} it)|^2 \, dt = 2\pi^{\sigma} \int_{0}^{+\infty} (u\theta(u^2) - 1 - u)^2 u^{2\sigma - 3} \, du.
\]
REFERENCES


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