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PROJECTION METHODS FOR CAUCHY SINGULAR INTEGRAL EQUATIONS ON THE BOUNDED INTERVALS *

A. S. Cvetković and M. C. De Bonis

Dedicated to Prof. G. Mastroianni for his 65th birthday

Abstract. The authors propose a numerical method to approximate the solutions of particular Cauchy singular integral equations (CSIE). It is based on interpolation processes and it is stable and convergent. Error estimates and numerical tests are shown.

1. Introduction

We consider Cauchy singular integral equations (CSIE) of the following type

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}dx+\lambda\int_{-1}^{1}f(x)k(x,y)\varphi(x)dx=g(y), \tag{1.1}$$

where the first integral is understood in the principal value sense, $y\in[-1,1]$, $\lambda\in\mathbb{R}$, $\varphi(x)=\sqrt{1-x^2}$, k and g are known functions and f is the unknown solution.

Letting $F(x):=f(x)\varphi(x)$,

$$(\tilde{K}F)(y):=\lambda\int_{-1}^{1}k(x,y)F(x)dx \quad \text{and} \quad H(f,y):=\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}dx,$$

(1.1) can be rewritten as follows

$$H(f,y)+(\tilde{K}F)(y)=g(y).$$

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Since for all $f \in L^2$, with the property

$$\int_{-1}^{1} f(x)dx = 0, \tag{1.2}$$

the left inverse operator of $H(f, y)$ exists and has the following form

$$\bar{H}(f, y) := -\frac{1}{\pi\varphi(y)} \int_{-1}^{1} \frac{f(x)\varphi(x)}{x - y} dx, \tag{1.3}$$

(1.1) can be expressed as an equivalent integral equation. In fact, applying (1.3) to the both sides of (1.1) and multiplying by φ, (1.1) becomes

$$F(y) - \lambda \int_{-1}^{1} \Gamma(x, y)F(x)dx = G(y), \tag{1.4}$$

with

$$G(y) := -H(g\varphi, y)$$

and

$$\Gamma(x, y) := \frac{1}{\pi} \int_{-1}^{1} \frac{k(x, t)\varphi(t)}{t - y} dt. \tag{1.5}$$

Note that (1.4) is equivalent to (1.1), therefore if we assume that, for every choice of g, (1.1) has a unique solution $f^* \in L^2$ satisfying (1.2), then $F^* = f^*\varphi$ is the unique solution of (1.4).

In this paper, under suitable assumptions on the kernel k and the known function g, we introduce a numerical method to approximate the solution of (1.4). It is based on interpolation processes related to Legendre zeros and it is stable and convergent. The approximate solution is represented by means of polynomials whose coefficients are computed solving a well-conditioned linear system. Error estimates and numerical tests are shown.

2. Notations and Preliminary Results

In the following $\mathcal{C}$ denotes a positive constant which may have different values in different formulas and, if $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$, *if and only if* there exists a positive constant $\mathcal{C}$, independent of the parameters of A and B, such that

$$\frac{1}{\mathcal{C}} \le \frac{A}{B} \le \mathcal{C}.$$

We define the Hölder-Zygmund-type subspaces of L^2 as follows

$$Z_s^2 = \Big\{ f \in L^2 : \|f\|_{Z_s^2} < +\infty \Big\},$$

where

$$\|f\|_{Z_s^2} := \|f\|_2 + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_2}{t^s}, \quad r > s > \frac{1}{2},$$

and

$$\Omega_\varphi^r(f,t)_2 := \sup_{0<h\le t} \left\| \Delta_{h\varphi}^r f \right\|_{L^2(I_{rh})}$$

is the main part of the φ-modulus of smoothness on $[-1,1]$, with $I_{rh} = \big[-1+4r^2h^2, 1-4r^2h^2\big]$, $0<t<1$, and

$$\Delta_{h\varphi}^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\Big(x + \frac{h\varphi(x)}{2}(r-2k)\Big).$$

Note that if s is an integer, we have

$$\Omega_\varphi^r(f,t)_2 \le \sup_{0<h\le t} h^r \|f^{(r)}\varphi^r\|_{L^2(I_{rh})} \le \mathcal{C} t^s \|f^{(r)}\varphi^r\|_2.$$

We define the error of best approximation of a function $f \in L^2$ as

$$E_m(f)_2 := \inf_{P \in \mathcal{P}_m} \|f-P\|_2 = \|f - S_m(f)\|_2,$$

$S_m(f)$ being the $m-$th Fourier sum of f related to the system of the Legendre polynomials $\{P_m\}_m$. In [4], we find the following inequalities

$$\Omega_\varphi^r(f,t)_2 \le \mathcal{C} t^r \sum_{k=0}^{[1/t]} (1+k)^{r-1} E_k(f)_2 \tag{2.1}$$

and

$$E_m(f)_2 \le \mathcal{C} \int_0^{1/m} \frac{\Omega_\varphi^r(f,t)_2}{t} dt. \tag{2.2}$$

Applying (2.1) and (2.2), we deduce the following equivalence

$$\Big(\forall f \in Z_s^2\Big) \qquad \sup_m \Big[(m+1)^s E_m(f)_2\Big] \sim \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_2}{t^s}. \tag{2.3}$$

We want to search for the solution of the integral equation (1.1) in the following set

$$Z^2_{s,0} := \left\{ f \in Z^2_s \ : \ \int_{-1}^{1} f(x)dx = 0 \right\}, \quad s > \frac{1}{2}.$$

We assume the following properties of the kernel $k(x,y)$ and the function g:

$$\sup_{|y|\le 1} \sup_{t>0} \frac{\Omega^r_\varphi(k_y,t)_2}{t^s} < +\infty, \quad \sup_{|y|\le 1} \sup_{t>0} \frac{\Omega^r_\varphi\left(\frac{\partial}{\partial y}k_y,t\right)_2}{t^s} < +\infty, \tag{2.4}$$

$$\sup_{|x|\le 1} \frac{\Omega^r_\varphi(k_x,t)_2}{t^s} < +\infty, \tag{2.5}$$

$$\sup_{|x|\le 1} \frac{\Omega^r_\varphi(g,t)_2}{t^s} < +\infty, \tag{2.6}$$

with $r > s > 1/2$ and $k_x(y) = k_y(x) = k(x,y)$. Under the above assumptions the integral equation (1.1) can be regularized as shown in Introduction (applying (1.3) to its both sides). The resulting integral equation (1.4) is equivalent to (1.1) and, if $f^* \in Z^2_{s,0}$ is the unique solution of (1.1), then $F^* = f^*\varphi$ is the unique solution of (1.4). Therefore, we consider the integral equation (1.4) instead of (1.1).

Letting

$$(KF)(y) = \lambda \int_{-1}^{1} \Gamma(x,y)F(x)dx, \tag{2.7}$$

(1.1) can be rewritten as follows

$$(I-K)F = G.$$

The following lemma deals with the smoothness of the new kernel $\Gamma(x,y)$ and the new right-hand side G.

Lemma 2.1. *If $k(x,y)$ satisfies (2.4), then we have*

$$\begin{aligned} \sup_{|y|\le 1} \sup_{t>0} \frac{\Omega^r_\varphi(\Gamma_y,t)_2}{t^s} & \\ \le \mathcal{C} \sup_{|y|\le 1} & \left[\sup_{t>0} \frac{\Omega^r_\varphi(k_y,t)_2}{t^s} + \sup_{t>0} \frac{\Omega^r_\varphi\left(\frac{\partial}{\partial y}k_y,t\right)_2}{t^s} \right] < +\infty; \end{aligned} \tag{2.8}$$

If $k(x, y)$ satisfies (2.5), then we get

$$\sup_{|x|\le 1}\sup_{t>0}\frac{\Omega_\varphi^r(\Gamma_x, t)_2}{t^s}\le \mathcal{C}\sup_{|x|\le 1}\sup_{t>0}\frac{\Omega_\varphi^r(k_x, t)_2}{t^s}<+\infty. \tag{2.9}$$

Moreover, if g satisfies (2.6), then we have

$$\sup_{t>0}\frac{\Omega_\varphi^r(G, t)_2}{t^s}\le \mathcal{C}\sup_{t>0}\frac{\Omega_\varphi^r(g, t)_2}{t^s}<+\infty. \tag{2.10}$$

Here $\mathcal{C}$ is a positive constant independent of k, x, y, g and t and $r > s > 1/2$.

For the operator K the following result holds.

Theorem 2.1. *If $k(x, y)$ satisfies (2.5) then*

$$\Omega_\varphi^r(KF, t)_2\le \mathcal{C}t^s\|F\|_2\sup_{|x|\le 1}\|k_x\|_{Z_s^2},\quad r>s>\frac{1}{2}, \tag{2.11}$$

where $\mathcal{C}$ is a positive constant independent of F and t. Consequently, the operator $K : L^2 \to L^2$ is compact.

For a continuous function f on $(-1, 1)$, let

$$L_m(f; x)=\sum_{k=1}^{m}\ell_k(x)f(x_k)$$

be the Lagrange polynomial interpolating the function f on the zeros $-1 < x_1 < x_2 < \cdots < x_m < 1$ of the m-th orthonormal Legendre polynomial P_m, where

$$\ell_k(x):=\ell_{m,k}(x)=\frac{P_m(x)}{P_m'(x_k)(x-x_k)}$$

is the k-th fundamental Lagrange polynomial.

Using the above defined Lagrange polynomial we introduce the following operator

$$(K^*F):=(K_m^*F)(y)=\lambda\int_{-1}^{1}L_m(\Gamma_y; x)F(x)dx. \tag{2.12}$$

The next theorem shows that K and K^* have the same behaviour.

Theorem 2.2. *If $k(x,y)$ satisfies* (2.5)*, then*

$$\Omega_\varphi^r(K^*F,t)_2 \le \mathcal{C}t^s\|F\|_2 \sup_{|x|\le 1}\|k_x\|_{Z_s^2}, \quad r>s>\frac{1}{2}, \tag{2.13}$$

where $\mathcal{C}$ is a positive constant independent of F and t.

Moreover, we define the following sequence of operators

$$(K_mF)(y) = L_m(K^*F;y) = \sum_{k=1}^{m} \ell_k(y)(K^*F)(x_k), \quad m=1,2,\dots\ . \tag{2.14}$$

Analogously, we define the polynomial sequence

$$G_m(y) = L_m(G;y), \quad m=1,2,\dots\ .$$

The next theorem is crucial for an introduction of the numerical method.

Theorem 2.3. *If $k(x,y)$ satisfies* (2.5) *and* (2.4) *and g satisfies* (2.6)*, then*

$$\|K-K_m\|_{L^2} \le \frac{\mathcal{C}}{m^s} \tag{2.15}$$

and

$$\|G-G_m\|_2 \le \frac{\mathcal{C}}{m^s}\|g\|_{Z_s^2}, \tag{2.16}$$

where $\mathcal{C}$ is a positive constant independent of m and $\|D\|_{L^2} := \|D\|_{L^2\to L^2}$ denotes the norm of the operator D as map from L^2 into itself.

3. Numerical Method

Using the previous definitions, we consider the following sequence of equations

$$(I-K_m)F_m = G_m, \quad m=1,2,\dots, \tag{3.1}$$

where $\{F_m\}_m$ is an unknown sequence of polynomials of degree at most m. Note that both $(I-K_m)F_m$ and G_m are polynomials of the same degree m. The following theorem holds:

Theorem 3.1. *If $k(x,y)$ satisfies* (2.5) *and* (2.4) *and g satisfies* (2.6) *then* (3.1) *has a unique solution F_m^* for any m sufficiently large* (*say $m > m_0$*) *and, denoting by F^* the unique solution of* (1.4), *we have*

$$(3.2)\ \|F^* - F_m^*\|_2 \le \mathcal{C}\Big[\|G - G_m\|_2 + \|G\|_2\|K - K_m\|_{L^2\to L^2}\Big] \le \frac{\mathcal{C}}{m^s}\|g\|_{Z_s^2}.$$

Consequently, $F^* \in Z_s^2$. Moreover,

$$(3.3) \qquad |\mathrm{cond}(I - K) - \mathrm{cond}(I - K_m)| = \mathcal{O}(m^{-s}),$$

where $\mathrm{cond}(A) = \|A\|_{L^2}\|A^{-1}\|_{L^2}$ *is the condition number of an invertible operator $A : L^2 \to L^2$.*

Remark 3.1. By (3.2) we deduce that the smoothness of the solution F^* depends on the smoothness of g and k. In particular, if $g \in Z_{s_1}^2$ and k, with respect to x and y, belongs to $Z_{s_2}^2$ then $F^* \in Z_s^2$, where $s = \min\{s_1, s_2\}$.

In order to construct the coefficients of F_m^*, we consider the polynomial equality

$$(3.4) \qquad (I - K_m)F_m^* = G_m,$$

and expand both K_m and G_m using the same basis.

We denote by λ_i and x_i, $i = 1, \ldots, m$, nodes and weights in the Gauss-Legendre quadrature rule, respectively. Since we are working in L^2, the most natural basis is the orthonormal basis

$$\varphi_i(x) = \frac{\ell_i(x)}{\sqrt{\lambda_i}}, \quad i = 1, \ldots, m, \quad \ell_i(x) = \frac{P_m(x)}{P_m'(x_i)(x - x_i)}.$$

Thus, we write

$$(3.5) \qquad G_m(x) = \sum_{i=1}^m \varphi_i(x) b_i, \quad b_i = G(x_i)\sqrt{\lambda_i},$$

$$(3.6) \qquad F_m^*(x) = \sum_{i=1}^m \varphi_i(x) a_i$$

and

$$(3.7) \qquad (K_m F_m^*)(y) = \lambda \sum_{i=1}^m \varphi_i(y)\sqrt{\lambda_i} \sum_{k=1}^m a_k \Gamma(x_k, x_i)\sqrt{\lambda_k}.$$

Substituting (3.5), (3.6) and (3.7) into (3.4) and comparing the coefficients, the equality (3.4) is equivalent to the following system of linear equations

$$\sum_{k=1}^{m}\Big[\delta_{ki}-\lambda\sqrt{\lambda_k}\sqrt{\lambda_i}\Gamma(x_k,x_i)\Big]a_k \;=\; b_i, \quad i=1,\dots,m, \tag{3.8}$$

where $a_k, k=1,\dots,m$, are the unknown coefficients of F_m^*. Denoting by

$$M_m=\Big[\delta_{ki}-\lambda\sqrt{\lambda_k}\sqrt{\lambda_i}\Gamma(x_k,x_i)\Big]_{k,i=1,\dots,m}:=[C_{k,i}]_{k,i=1,\dots,m},$$

the matrix of the system (3.8), its condition number is given by

$$\mathrm{cond}(M_m)=\|M_m\|_2\|M_m^{-1}\|_2,$$

where

$$\|B\|_2=\max_{x\neq 0}\frac{\|Bx\|_{l^2}}{\|x\|_{l^2}}=\left[\sum_{i=1}^{m}\sum_{j=1}^{m}|b_{i,j}|^2\right]^{1/2}$$

and $\|\mathbf{a}\|_{l^2}$ denotes the l^2-norm of the array $\mathbf{a}$.

Now, we formulate the following result:

Theorem 3.2. *The matrix M_m of the linear system* (3.8) *satisfies*

$$\mathrm{cond}(M_m)=\mathrm{cond}(I-K)+\mathcal{O}(m^{-s}).$$

Therefore, the system (3.8) is well-conditioned.

3.1. Computational aspects. The resolution of the linear system (3.8) requires the computation of the quantities

$$\Gamma(x_k,x_i)=\frac{1}{\pi}\int_{-1}^{1}\frac{k(x_k,t)\varphi(t)}{t-x_i}dt, \quad k,i=1,\dots,m, \tag{3.9}$$

and

$$G(x_i)=\frac{1}{\pi}\int_{-1}^{1}\frac{G(t)\varphi(t)}{t-x_i}dt, \quad i=1,\dots,m. \tag{3.10}$$

Unfortunately, their closed analytical expressions are often not known and thus it is necessary to approximate them using a suitable quadrature rule. Following an idea from [1], we compute them using the following Gaussian-type quadrature rules based on m points

$$\Gamma_m(x_k,x_i)=\frac{1}{\pi}\sum_{j=1}^{m}\frac{k(x_k,t_j)-k(x_k,x_i)}{t_j-x_i}\lambda_{m,j}(\varphi)-x_i\; k(x_k,x_i) \tag{3.11}$$

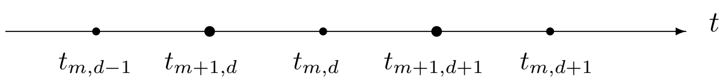
and

$$(3.12) \qquad G_m(x_i) = \frac{1}{\pi} \sum_{j=1}^{m} \frac{G(t_j) - G(x_i)}{t_j - x_i} \lambda_{m,j}(\varphi) - x_i \, G(x_i),$$

$t_j := t_{m,j} = \cos \frac{j\pi}{m+1}$, $j = 1, \ldots, m$, are the zeros of the m-th orthonormal Chebyshev polynomial of second kind U_m and $\lambda_{m,j}(\varphi) = \frac{\pi}{m+1} \sin^2 \frac{j\pi}{m+1}$, $j = 1, \ldots, m$, are the corresponding Christoffel numbers. Since for any fixed x_i, $i = 1, \ldots, m$, there could be a t_j, $j = 1, \ldots, m$, too close to x_i, the above quadrature rules might not be stable. In order to control the distances $t_j - x_i$, $i = 1, \ldots, m$, $j = 1, \ldots, m$, and, therefore, to avoid the numerical cancellation, we apply the quadrature rules (3.11) and (3.12) together with the following algorithm.

For every fixed $x_i, i = 1, \ldots, m$, choose $m_0 = m_0(t) \in \mathbb{N}$ such that, for $m \geq m_0$, we have $t_{m,d} \leq x_i \leq t_{m,d+1}$ for some $d \in \{1, \ldots, m\}$.

Moreover, because of the interlacing properties of the zeros $t_{m+1,j}$, $j = 1, \ldots, m+1$, of the $(m+1)$-orthonormal Chebyshev polynomial of second kind U_{m+1}, we have



Thus, two cases are possible:

$$\text{(a) } t_{m+1,d+1} \leq x_i \leq t_{m,d+1} \quad \text{or} \quad \text{(b) } t_{m,d} \leq x_i \leq t_{m+1,d+1}.$$

In case (a),

if $x_i < \frac{1}{2}(t_{m+1,d+1} + t_{m,d+1})$, then we use the quadrature rule $\Gamma_m(x_k, x_i)$;

if $x_i \geq \frac{1}{2}(t_{m+1,d+1} + t_{m,d+1})$, then we use the quadrature rule $\Gamma_{m+1}(x_k, x_i)$.

Similarly in the case (b).

Thus, for every fixed $x_i, i = 1, \ldots, m$, we have defined the numerical sequences $\{\Gamma_{m^*}(x_k, x_i)\}$ and $\{G_{m^*}(x_i)\}$, $m^* \in \{m, m+1\}$. The following theorem (see [2]) shows that such numerical sequences are stable and convergent. Till now we have assumed that $k_x, g \in Z_s^2$. For the sake of simplicity of notation and proof we slightly strengthen this assumptions, more precisely we assume that k_x, $g \in Z_s^\infty$, $s > 1/2$.

Theorem 3.3. *If*

$$\sup_{|x|\le 1}\sup_{t>0}\frac{\Omega_\varphi^r(k_x,t)_\infty}{t^s}<+\infty \quad \textit{and} \quad \sup_{|x|\le 1}\sup_{t>0}\frac{\Omega_\varphi^r(g,t)_\infty}{t^s}<+\infty, \qquad r>s>0,$$

then we have

$$\sup_{1\le i,k\le m}\sqrt{1-x_i^2}|\Gamma(x_k,x_i)-\Gamma_{m^*}(x_k,x_i)|\le\frac{\mathcal{C}}{m^s}\log m\sup_{1\le k\le m}\|k_{x_k}\|_{Z_s^\infty}, \tag{3.13}$$

$$\sup_{1\le i,k\le m}\sqrt{1-x_i^2}|G(x_i)-G_{m^*}(x_i)|\le\frac{\mathcal{C}}{m^s}\log m\|g\|_{Z_s^\infty}, \tag{3.14}$$

and, consequently,

$$\sup_{1\le i,k\le m}\sqrt{1-x_i^2}|\Gamma_{m^*}(x_k,x_i)|\le\mathcal{C}\sup_{1\le k\le m}\|k_{x_k}\|_{Z_s^\infty}\log m, \tag{3.15}$$

$$\sup_{1\le i,k\le m}\sqrt{1-x_i^2}|G_{m^*}(x_i)|\le\mathcal{C}\|g\|_{Z_s^\infty}\log m, \tag{3.16}$$

where $\mathcal{C}$ is a positive constant independent of m and k.

Consequently the quadrature rules (3.11) and (3.12), together with the above described algorithm, are efficient tools for the computation of the quantities (3.9) and (3.10).

Approximating Γ with Γ_{m^*}, we solve the following linear system

$$\sum_{k=1}^{m}[\delta_{ki}-\lambda\sqrt{\lambda_k}\sqrt{\lambda_i}\Gamma_{m^*}(x_k,x_i)]\bar{a}_k \;=\; \bar{b}_i, \quad i=1,\dots,m, \tag{3.17}$$

where $\bar{b}_i=G_{m^*}(x_i)\sqrt{\lambda_i}$. Its matrix is given by

$$\begin{aligned} M_m^* \;&=\; \Big[\delta_{ki}-\lambda\sqrt{\lambda_k}\sqrt{\lambda_i}\Gamma_{m^*}(x_k,x_i)\Big]_{k,i=1,\dots,m} \\ &=\; \Big[\delta_{ki}-\lambda\sqrt{\lambda_k}\sqrt{\lambda_i}\Gamma(x_k,x_i)\Big]_{k,i=1,\dots,m} \\ &\qquad\qquad +\Big[\lambda\sqrt{\lambda_k}\sqrt{\lambda_i}\Big[\Gamma(x_k,x_i)-\Gamma_{m^*}(x_k,x_i)\Big]\Big]_{k,i=1,\dots,m} \\ &=\; [C_{k,i}]_{k,i=1,\dots,m}+[\varepsilon_{k,i}]_{k,i=1,\dots,m}=:M_m+\mathcal{E}_m, \end{aligned} \tag{3.18}$$

where M_m is the matrix of the linear system (3.8) and $\mathcal{E}_m$ is the matrix of the perturbations induced on M_m. Moreover, the right-hand side of (3.17) can be written as follows

$$
\begin{aligned}
(3.19)\quad \bar{\mathbf{b}} &= \Big[G_{m^*}(x_i)\sqrt{\lambda_i}\Big]_{i=1,\dots,m} \\
&= \Big[G(x_i)\sqrt{\lambda_i}\Big]_{i=1,\dots,m} + \Big[\sqrt{\lambda_i}\Big[G_{m^*}(x_i) - G(x_i)\Big]\Big]_{i=1,\dots,m} \\
&= [b_i]_{i=1,\dots,m} + [\theta_i]_{i=1,\dots,m} =: \mathbf{b} + \boldsymbol{\theta},
\end{aligned}
$$

where $\mathbf{b}$ is the right-hand side of the linear system (3.8) and $\boldsymbol{\theta}$ is the array of the perturbations induced on $\mathbf{b}$.

In virtue of Theorem 3.3, the perturbations $\varepsilon_{k,i}$ and $\theta_i, k, i = 1, \dots, m$, are of the order of $m^{-s}\log m$, i.e., essentially of the same order as the global error of the method. Moreover, the following theorem gives an estimate of the condition number of the matrix M_m^*.

Theorem 3.4. *If*

$$\sup_{|x|\le 1}\sup_{t>0}\frac{\Omega_\varphi^r(k_x,t)_\infty}{t^s} < +\infty, \quad r > s > \frac{1}{2},$$

then, for m sufficiently large (say $m > m_0$), the matrix M_m^* of the linear system (3.17) *satisfies*

$$\mathrm{cond}(M_m^*) \le 4\ \mathrm{cond}(M_m),$$

where M_m is the matrix of the linear system (3.8).

Therefore, since by Theorem 3.2, the matrix M_m is well-conditioned, the matrix M_m^* is well-conditioned too.

The solution of the linear system (3.17) can be written as follows

$$\bar{\mathbf{a}}^T := (\bar{a}_1, \dots, \bar{a}_m)^T = (a_1, \dots, a_m)^T + (\eta_1, \dots, \eta_m)^T =: \mathbf{a}^T + \boldsymbol{\eta}^T,$$

where $\mathbf{a}$ is the array of the solutions of the linear system (3.8) and $\boldsymbol{\eta}$ is the array of the perturbations induced on $\bar{\mathbf{a}}$ due to the perturbations induced in the matrix M_m and the array $\mathbf{b}$. The following proposition gives an estimate of the relative error induced on the solution $\mathbf{a}$.

Proposition 3.1. *If*

$$\sup_{|x|\le 1}\sup_{t>0}\frac{\Omega_\varphi^r(k_x,t)_\infty}{t^s} < +\infty \quad \textit{and} \quad \sup_{|x|\le 1}\sup_{t>0}\frac{\Omega_\varphi^r(g,t)_\infty}{t^s} < +\infty, \quad r > s > \frac{1}{2},$$

then

$$\frac{\|\bar{\mathbf{a}}\|_{l^2}}{\|\mathbf{a}\|_{l^2}} \le 1 + 2\ \mathrm{cond}(M_m)\left(\frac{\|\mathcal{E}_m\|_2}{\|M_m\|_2} + \frac{\|\boldsymbol{\theta}\|_{l^2}}{\|\mathbf{b}\|_{l^2}}\right) \le 1 + \mathcal{O}\left(\frac{\log m}{m^{s-1/2}}\right).$$

Therefore, the relative error induced in the solution of the linear system (3.8), due to the approximations of Γ with Γ_{m^*} and of G with G_{m^*} is, essentially, of the same order as the global error of the method.

4. Proofs

Proof of Lemma 2.1. We first prove (2.8). If $y = \pm 1$ or $y = 0$ the proof easily follows. Then we assume $-1 < y < 0$, and the case $0 < y < 1$ is similar. By definition (1.5) of Γ, we have

$$\Delta^r_{h\varphi(x)}\Gamma(x,z) = \frac{1}{\pi}\int_{-1}^{1} \Delta^r_{h\varphi(x)} k(x,y)\frac{\varphi(y)}{y-z}dy.$$

We consider the following decomposition

$$\begin{aligned} \Delta^r_{h\varphi(x)}\Gamma(x,z) &= \frac{1}{\pi}\varphi(z)\int_{-1}^{2z+1} \Delta^r_{h\varphi(x)} k(x,y)\frac{dy}{y-z} \\ &+ \frac{1}{\pi}\int_{-1}^{2z+1} \Delta^r_{h\varphi(x)} k(x,y)\frac{\varphi(y)-\varphi(z)}{y-z}dy \\ &+ \frac{1}{\pi}\int_{2z+1}^{1} \Delta^r_{h\varphi(x)} k(x,y)\frac{\varphi(y)}{y-z}dy \\ &:= A_1(x,z) + A_2(x,z) + A_3(x,z). \end{aligned} \tag{4.1}$$

Applying Buniakowski inequality we have

$$\begin{aligned} \|A_2(\cdot,z)\|_{L^2(I_{rh})} &\le \frac{1}{\pi}\int_{-1}^{2z+1} \left|\frac{\varphi(y)-\varphi(z)}{y-z}\right| \left(\int_{I_{rh}} [\Delta^r_{h\varphi(x)} k(x,y)]^2 dx\right)^{1/2} dy \\ &\le \mathcal{C}\int_{-1}^{2z+1} \frac{1}{\sqrt{|z-y|}} \|\Delta^r_{h\varphi} k_y\|_{L^2(I_{rh})} dy. \end{aligned}$$

Taking the supremum for $h \le t$, we get

$$\|A_2(\cdot,z)\|_{L^2(I_{rh})} \le \mathcal{C} \sup_{|z|\le 1} \Omega^r_\varphi(k_z,t)_2. \tag{4.2}$$

Analogously, we obtain

$$\begin{aligned} \|A_3(\cdot,z)\|_{L^2(I_{rh})} &\le \frac{1}{\pi}\int_{2z+1}^{1} \frac{\varphi(y)}{y-z}\left(\int_{I_{rh}} [\Delta^r_{h\varphi(x)} k(x,y)]^2 dx\right)^{1/2} dy \\ &\le \mathcal{C} \sup_{|z|<1} \Omega^r_\varphi(k_z,t)_2. \end{aligned} \tag{4.3}$$

It remains to evaluate $A_1(x,z)$. We have

$$\begin{aligned}
\|A_1(\cdot,z)\|_{L^2(I_{rh})} &= \frac{1}{\pi}\left(\int_{I_{rh}}\left[\varphi(z)\Delta^r_{h\varphi(x)}\left(\int_{-1}^{2z+1}\frac{k(x,y)}{y-z}dy\right)\right]^2 dx\right)^{1/2} \\
&= \frac{1}{\pi}\left(\int_{I_{rh}}\left[\varphi(z)\Delta^r_{h\varphi(x)}\left(\int_{-1}^{2z+1}\frac{k(x,y)-k(x,z)}{y-z}dy\right)\right]^2 dx\right)^{1/2} \\
&\leq \frac{4}{\pi}\left(\int_{I_{rh}}\left[\sup_{|y|\leq 1}\left|\Delta^r_{h\varphi(x)}\frac{\partial}{\partial y}k(x,y)\right|\right]^2 dx\right)^{1/2}.
\end{aligned}$$

Therefore, taking the supremum for $h \leq t$,

$$\|A_1(\cdot,z)\|_{L^2(I_{rh})} \leq \mathcal{C}\sup_{|z|\leq 1}\Omega^r_\varphi\left(\frac{\partial}{\partial z}k_z,t\right)_2 \tag{4.4}$$

holds. Then, combining (4.4), (4.2) and (4.3) with (4.1), (2.8) follows.

Now we prove (2.9). We note that for all $Q_m \in \mathcal{P}_m$ (the set of all polynomials of degree at most m) it results

$$\begin{aligned}
\frac{1}{\pi}\int_{-1}^{1}Q_m(x)\frac{\varphi(x)}{x-t}\,dx &= -tQ_m(t)+\frac{1}{\pi}\int_{-1}^{1}\frac{Q_m(x)-Q_m(t)}{x-t}\varphi(x)\,dt \\
&= T_{m+1}\in\mathcal{P}_{m+1}.
\end{aligned} \tag{4.5}$$

Then, by definition (1.5) of Γ,

$$\begin{aligned}
|\Gamma(x,y)-H(Q_m\varphi,y)| &= \frac{1}{\pi}\left|\int_{-1}^{1}k(x,t)\frac{\varphi(t)}{t-y}dt-\int_{-1}^{1}Q_m(t)\frac{\varphi(t)}{t-y}dt\right| \\
&= |H((k_x-Q_m)\varphi,y)|
\end{aligned} \tag{4.6}$$

holds and consequently, since $\|Hf\|_2 = \|f\|_2$, we have

$$\|\Gamma_x - T_{m+1}\|_2 \leq \|H((k_x-Q_m)\varphi)\|_2 = \|\varphi(k_x-Q_m)\|_2.$$

Taking the infimum on $Q_m \in \mathcal{P}_m$ we get $E_{m+1}(\Gamma_x)_2 \leq E_m(k_x)_2$. Finally, by (2.3) it results

$$\frac{\Omega^r_\varphi(\Gamma_x,t)_2}{t^s} \leq \mathcal{C}\frac{\Omega^r_\varphi(k_x,t)_2}{t^s}, \quad r > s > \frac{1}{2},$$

and (2.9) follows. Analogously we can prove (2.10).

Proof of Theorem 2.1. Since

$$\|\Delta_{h\varphi}^r KF\|_{L^2(I_{rh})} = \lambda \left(\int_{I_{rh}} \left[\int_{-1}^{1} \Delta_{h\varphi(y)}^r \Gamma(x,y) F(x) dx \right]^2 dy \right)^{1/2},$$

applying Buniakowski inequality we get

$$\|\Delta_{h\varphi}^r KF\|_{L^2(I_{rh})} \le \mathcal{C} \int_{-1}^{1} |F(x)| \left(\int_{I_{rh}} \left[\Delta_{h\varphi(y)}^r \Gamma(x,y) \right]^2 dy \right)^{1/2} dx.$$

Taking the supremum on h to both sides and applying Cauchy inequality, we get

$$\Omega_\varphi^r(KF,t)_2 \le \mathcal{C} \int_{-1}^{1} |F(x)| \Omega_\varphi^r(\Gamma_x,t)_2 dx \le \mathcal{C} \|F\|_2 \sup_{|x|\le 1} \Omega_\varphi^r(\Gamma_x,t)_2.$$

Then, applying (2.9), (2.11) follows. It remains to prove the compactness of the linear operator $K : L^2 \to L^2$, i.e.,

$$\lim_{m\to+\infty} \sup_{\|F\|_2=1} E_m(KF)_2 = 0.$$

But, substituting (2.11) into (2.2) we get $E_m(KF)_2 \le \mathcal{C} m^{-s} \|F\|_2$, therefore

$$\lim_{m\to+\infty} \sup_{\|F\|_2=1} E_m(KF)_2 \le \mathcal{C} \lim_{m\to+\infty} \frac{1}{m^s} = 0$$

and the theorem follows.

Proof of Theorem 2.2. By (2.12), we deduce

$$\Delta_{h\varphi(y)}^r (K^*F)(y) = \lambda \int_{-1}^{1} L_m(\Delta_{h\varphi}^r \Gamma_y; x) F(x) dx.$$

Using Cauchy inequality and the Gaussian quadrature rule, we get

$$\begin{aligned} \Delta_{h\varphi(y)}^r (K^*F)(y) &\le \mathcal{C} \|F\|_2 \left(\int_{-1}^{1} \left[L_m(\Delta_{h\varphi}^r \Gamma_y; x) \right]^2 dx \right)^{1/2} \\ &= \mathcal{C} \|F\|_2 \left(\sum_{i=1}^{m} [\Delta_{h\varphi(y)}^r \Gamma(x_i,y)]^2 \lambda_{m,i} \right)^{1/2}. \end{aligned}$$

Thus,

$$\begin{aligned}\|\Delta^r_{h\varphi}(K^*F)\|_{L^2(I_{rh})} &\leq \mathcal{C}\|F\|_2\left(\int_{I_{rh}}\sum_{i=1}^{m}[\Delta^r_{h\varphi(y)}\Gamma(x_i,y)]^2\lambda_{m,i}dy\right)^{1/2}\\ &= \mathcal{C}\|F\|_2\left(\sum_{i=1}^{m}\lambda_{m,i}\|\Delta^r_{h\varphi}\Gamma_{x_i}\|^2_{L^2(I_{rh})}\right)^{1/2}.\end{aligned}$$

Taking the supremum on h of the both sides, we obtain

$$\Omega^r_\varphi(K^*F,t)_2 \leq \mathcal{C}\|F\|_2\left(\sum_{i=1}^{m}\lambda_{m,i}[\Omega^r_\varphi(\Gamma_{x_i},t)_2]^2\right)^{1/2} \leq \mathcal{C}\sqrt{2}\|F\|_2\sup_{|x|\leq 1}\Omega^r_\varphi(\Gamma_x,t)_2.$$

Finally, using (2.9), (2.13) follows.

Proof of Theorem 2.3. We first prove (2.15). We can write

$$\|(K-K_m)F\|_2 \leq \|(K-K^*)F\|_2 + \|(K-K_m)F\|_2 := A+B. \tag{4.7}$$

Applying Cauchy inequality, for all $y \in [-1,1]$, we have

$$(K-K^*)F(y) = \lambda\int_{-1}^{1}[\Gamma(x,y)-L_m(\Gamma_y;x)]F(x)dx \leq \mathcal{C}\|F\|_2\|\Gamma_y - L_m(\Gamma_y)\|_2.$$

Recalling that (see [5])

$$\|f-L_m(f)\|_2 \leq \frac{\mathcal{C}}{\sqrt{m}}\int_0^{1/m}\frac{\Omega^r_\varphi(f,t)_2}{t^{\frac{3}{2}}}dt \tag{4.8}$$

and using (2.8), we get

$$A \leq \frac{\mathcal{C}}{\sqrt{m}}\|F\|_2\int_0^{1/m}\frac{\Omega^r_\varphi(\Gamma_y,t)_2}{t^{3/2}}dt \leq \frac{\mathcal{C}}{m^s}\|F\|_2\sup_{|y|\leq 1}\left(\|k_y\|_{Z^2_s} + \left\|\frac{\partial}{\partial y}k_y\right\|_{Z^2_s}\right). \tag{4.9}$$

Analogously, since $B = \|K^*F - L_m(K^*F)\|_2$, using (4.8) and (2.13), we obtain

$$B \leq \frac{\mathcal{C}}{\sqrt{m}}\int_0^{1/m}\frac{\Omega^r_\varphi(K^*F,t)_2}{t^{3/2}}dt \leq \frac{\mathcal{C}}{m^s}\|F\|_2\sup_{|x|\leq 1}\|k_x\|_{Z^2_s}. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.7), (2.15) follows.

Concerning (2.16), by (4.8) we get

$$\|G - G_m\|_2 = \|G - L_m(G)\|_2 \leq \frac{\mathcal{C}}{\sqrt{m}} \int_0^{1/m} \frac{\Omega_\varphi^r(G,t)_2}{t^{3/2}}\, dt.$$

Moreover, using the assumption (2.10) on g, we deduce

$$\|G - G_m\|_2 \leq \frac{\mathcal{C}}{m^s} \sup_{t>0} \frac{\Omega_\varphi^r(g,t)_2}{t^s},$$

i.e., (2.16).

Proof of Theorem 3.1. Since we assume that $(I-K)^{-1}$ exists and it is bounded and, by Theorem 2.3, K_m converges to K, applying Von Neumann Theorem it is easy to prove that $(I-K_m)^{-1}$ exists and it is bounded, too. Then (3.1) has a unique solution. Moreover, we have

$$\begin{aligned}(I-K_m)(F^* - F_m^*) &= (I-K_m)F^* - (I-K_m)F_m^* \\ &= (I-K)F^* - (K_m - K)F^* - (I-K_m)F_m^* \\ &= G - G_m - (K_m - K)F^*,\end{aligned}$$

from which we deduce

$$\begin{aligned}F^* - F_m^* &= (I-K_m)^{-1}[(G-G_m) - (K_m-K)F^*] \\ &= (I-K_m)^{-1}[(G-G_m) - (K_m-K)(I-K)^{-1}G]\end{aligned}$$

and

$$\begin{aligned}\|F^* - F_m^*\|_2 &\leq \|(I-K_m)^{-1}\|_{L^2}\|(G-G_m) + (K_m-K)(I-K)^{-1}G\|_2 \\ &\leq \|(I-K_m)^{-1}\|_{L^2}\left[\|G-G_m\|_2 + \|(I-K)^{-1}\|_{L^2}\|K_m-K\|_{L^2}\|G\|_2\right] \\ &\leq \mathcal{C}\left[\|G-G_m\|_2 + \|(I-K)^{-1}\|_{L^2}\|K_m-K\|_{L^2}\|G\|_2\right].\end{aligned}$$

Applying Theorem 2.3 and taking into account that $\|G\|_2 = \|H(g\varphi)\|_2 = \|g\varphi\|_2 \leq \|g\|_2$, we get

$$E_m(F^*)_2 \leq \|F^* - F_m^*\|_2 \leq \frac{\mathcal{C}}{m^s}\|g\|_{Z_s^2},$$

i.e., (3.2). Moreover, by (2.3), we deduce that F^* belongs to Z_s^2.

In order to prove (3.3), note that for $C_m := |\mathrm{cond}(I-K) - \mathrm{cond}(I-K_m)|$ we have

$$\begin{aligned}C_m &= \left|\|I-K_m\|_{L^2}\|(I-K_m)^{-1}\|_{L^2} - \|I-K\|_{L^2}\|(I-K)^{-1}\|_{L^2}\right| \\ &\leq \left|\|I-K_m\|_{L^2}\|(I-K_m)^{-1}\|_{L^2} - \|I-K\|_{L^2}\|(I-K_m)^{-1}\|_{L^2}\right| \\ &\quad + \left|\|I-K\|_{L^2}\|(I-K_m)^{-1}\|_{L^2} - \|I-K\|_{L^2}\|(I-K)^{-1}\|_{L^2}\right|,\end{aligned}$$

i.e.,

$$C_m = \|(I-K_m)^{-1}\|_{L^2}\,\big|\|I-K_m\|_{L^2}-\|I-K\|_{L^2}\big| \\ +\|I-K\|_{L^2}\left|\|(I-K_m)^{-1}\|_{L^2}-\|(I-K)^{-1}\|_{L^2}\right|. \tag{4.11}$$

Using (2.15), it is easy to prove that

$$\Big|\,\|I-K_m\|_{L^2}-\|I-K\|_{L^2}\Big| \le \|K-K_m\|_{L^2} \le \frac{\mathcal{C}}{m^s}. \tag{4.12}$$

Moreover, it results

$$\begin{aligned}\Big|\,\|(I-K_m)^{-1}\|_{L^2} \;&-\; \|(I-K)^{-1}\|_{L^2}\Big| \\ &= \left\|(I-K)^{-1}[(I-K)(I-K_m)^{-1}-I]\right\|_{L^2} \\ &= \left\|(I-K)^{-1}[(I-K)-(I-K_m)](I-K_m)^{-1}\right\|_{L^2} \\ &\le \|(I-K)^{-1}\|_{L^2}\|K_m-K\|_{L^2}\|(I-K_m)^{-1}\|_{L^2}.\end{aligned}$$

Thus, using (2.15) and taking into account that both $\|(I-K)^{-1}\|_{L^2}$ and $\|(I-K_m)^{-1}\|_{L^2}$ are bounded, we get

$$\left|\|(I-K_m)^{-1}\|_{L^2}-\|(I-K)^{-1}\|_{L^2}\right| \le \frac{\mathcal{C}}{m^s}. \tag{4.13}$$

Finally, substituting (4.12) and (4.13) into (4.11), (3.3) follows.

Proof of Theorem 3.2. Recalling the definition (3.6) of F_m^* and the definition (3.5) of G_m, we set $\mathbf{a} := (a_1,\dots,a_m)$, with $a_j = F_m^*(x_j)\sqrt{\lambda_j}$ and $\mathbf{b} := (b_1,\dots,b_m)$, with $b_j = G_m(x_j)\sqrt{\lambda_j}$. Thus, system (3.8) can be written as $M_m\mathbf{a} = \mathbf{b}$. Assuming that (1.4) has a unique solution for every G, by Theorem 3.1, (3.1) has a unique solution too and then $M_m\mathbf{a} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$. Therefore, for all $\theta = (\theta_1,\dots,\theta_m)$ there exists $\eta = (\eta_1,\dots,\eta_m)$ such that $M_m\theta = \eta$ if and only if $(I-K)\tilde\theta(y) = \tilde\eta(y)$, where

$$\tilde\theta(y) = \sum_{i=1}^m \varphi_i(y)\theta_i, \quad \theta_i = \tilde\theta(x_i)\sqrt{\lambda_i}$$

and

$$\tilde\eta(y) = \sum_{i=1}^m \varphi_i(y)\eta_i, \quad \eta_i = \tilde\eta(x_i)\sqrt{\lambda_i}.$$

For all $\tilde\theta$ it results

$$\|\tilde\theta\|_2^2 = \int_{-1}^1 \tilde\theta^2(y)dy = \sum_{k=1}^m \lambda_k\tilde\theta^2(x_k) = \sum_{k=1}^m \lambda_k\left(\sum_{i=1}^m \varphi_i(x_k)\theta_i\right)^2$$

and, recalling that $\varphi_i(x_k) = \dfrac{\ell_{m,i}(x_k)}{\sqrt{\lambda_i}} = \dfrac{\delta_{i,k}}{\sqrt{\lambda_i}}$, $i, k = 1, \ldots, m$, we obtain

$$\|\tilde{\theta}\|_2^2 = \sum_{i=1}^{m} \lambda_i \frac{\theta_i^2}{\lambda_i} = \sum_{i=1}^{m} \theta_i^2 = \|\theta\|_{l^2}^2.$$

Analogously, for all $\tilde{\eta}$, we obtain $\|\tilde{\eta}\|_2 = \|\eta\|_{l^2}$. Thus, we get

$$\|M_m\theta\|_{l^2} = \|\eta\|_{l^2} = \|\tilde{\eta}\|_2 = \|(I - K_m)\tilde{\theta}\|_2 \le \|(I - K_m)|_{\mathcal{P}_{m-1}}\|\|\tilde{\theta}\|_2,$$

i.e., $\|M_m\theta\|_{l^2} \le \|I - K_m\|_{L^2 \to L^2}\|\theta\|_{l^2}$. Similarly, we have

$$\begin{aligned} \|M_m^{-1}\eta\|_{l^2} = \|\theta\|_{l^2} &= \|\tilde{\theta}\|_2 = \|(I - K_m)^{-1}\tilde{\eta}\|_2 \\ &\le \|(I - K_m)^{-1}\|_{L^2 \to L^2}\|\eta\|_{l^2}. \end{aligned} \tag{4.14}$$

Consequently, $\mathrm{cond}(M_m) \le \mathrm{cond}(I - K_m)$. By (3.3), the theorem follows.

Proof of Theorem 3.4. Recalling (3.18), we can write

$$M_m^* = M_m + \mathcal{E}_m = M_m(I_m + M_m^{-1}\mathcal{E}_m),$$

where I_m is the identity matrix of order m, and then

$$\mathrm{cond}(M_m^*) \le \mathrm{cond}(M_m)\mathrm{cond}(I_m + M_m^{-1}\mathcal{E}_m).$$

Therefore, it is sufficient to prove that for m sufficiently large we have

$$\mathrm{cond}(I_m + M_m^{-1}\mathcal{E}_m) = \|I_m + M_m^{-1}\mathcal{E}_m\|_2\|(I_m + M_m^{-1}\mathcal{E}_m)^{-1}\|_2 \le 4. \tag{4.15}$$

If we proved that, for m sufficiently large,

$$\|M_m^{-1}\mathcal{E}_m\|_2 \le \|M_m^{-1}\|_2\|\mathcal{E}_m\|_2 < \frac{1}{2}, \tag{4.16}$$

the proof of (4.15) is rather easy, in fact

$$\|I_m + M_m^{-1}\mathcal{E}_m\|_2 \le \|I_m\|_2 + \|M_m^{-1}\|_2\|\mathcal{E}_m\|_2 \le 1 + \frac{1}{2} < 2$$

and

$$\|(I_m + M_m^{-1}\mathcal{E}_m)^{-1}\|_2 \le \frac{1}{1 - \|M_m^{-1}\mathcal{E}_m\|_2} \le 2.$$

Therefore, it remains to prove (4.16). By (4.14) we have

$$\|M_m^{-1}\|_2 \le \|(I - K_m)^{-1}\|_{L^2} \le \mathcal{C}. \tag{4.17}$$

Moreover, since

$$\mathcal{E}_m = \left[\lambda\sqrt{\lambda_k}\sqrt{\lambda_i}\Big[\Gamma_{m^*}(x_k,x_i) - \Gamma(x_k,x_i)\Big]\right]_{k,i=1,\dots,m},$$

we have

$$\|\mathcal{E}_m\|_2 = |\lambda| \left\{ \sum_{k=1}^m \sum_{i=1}^m \lambda_k \frac{\lambda_i}{1-x_i^2} \left[\sqrt{1-x_i^2}\Big(\Gamma_{m^*}(x_k,x_i) - \Gamma(x_k,x_i)\Big)\right]^2 \right\}^{1/2}$$

and, using (3.13), we obtain

$$(4.18)\quad \|\mathcal{E}_m\|_2 \le |\lambda| \frac{\mathcal{C}}{m^s} \log m \sup_{1\le k\le m} \|k_{x_k}\|_{Z_s^\infty} \left\{ \sum_{k=1}^m \sum_{i=1}^m \frac{\lambda_i}{1-x_i^2}\lambda_k \right\}^{1/2}.$$

Since $|x_i| < 1 - \frac{\mathcal{C}}{m^2}$ it results $\frac{\mathcal{C}}{m} < \sqrt{1-|x_i|} \sim \sqrt{1-x_i^2}$ and then we deduce

$$(4.19)\quad \left\{ \sum_{i=1}^m \frac{\lambda_i}{1-x_i^2} \right\}^{1/2} \le \sqrt{m} \left\{ \sum_{i=1}^m \frac{\lambda_i}{\sqrt{1-x_i^2}} \right\}^{1/2} \le \mathcal{C}\sqrt{m} \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}.$$

Therefore,

$$\begin{aligned}(4.20)\quad \|\mathcal{E}_m\|_2 &\le \frac{\mathcal{C}}{m^{s-1/2}} \log m \sup_{1\le k\le m} \|k_{x_k}\|_{Z_s^\infty} \left\{ \int_{-1}^1 \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} dy \right\}^{1/2} \\ &\le \frac{\mathcal{C}}{m^{s-1/2}} \log m \sup_{|x|\le 1} \|k_x\|_{Z_s^\infty}\end{aligned}$$

holds, where $s > 1/2$. Then, combining (4.17) and (4.20), for m sufficiently large, (4.16) follows.

Proof of Proposition 3.1. The first inequality follows by a well-known theorem of numerical linear algebra (see for example [3, p. 249]). For the second one, by (3.14) and (4.19) we get

$$\begin{aligned}\|\boldsymbol{\theta}\|_{l^2} &= \left\{ \sum_{i=1}^m \frac{\lambda_i}{1-x_i^2} \left[\sqrt{1-x_i^2}\Big(G_{m^*}(x_i) - G(x_i)\Big)\right]^2 \right\}^{1/2} \\ &\le \frac{\mathcal{C}}{m^{s-1/2}} \log m \|g\|_{Z_s^\infty} \left\{ \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} \right\}^{1/2} \\ &\le \frac{\mathcal{C}}{m^{s-1/2}} \log m \|g\|_{Z_s^\infty},\end{aligned}$$

where $s > 1/2$. Moreover, taking into account (4.20) the proposition easily follows.

5. Numerical Examples

Example 5.1. We consider

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}dx+\int_{-1}^{1}f(x)\frac{(x^2+y^2)\sqrt{1-x^2}}{(1+\cos^2 x+e^x)}dx=y^2\sin y, \tag{5.1}$$

and we solve

$$F(y)-\int_{-1}^{1}\Gamma(x,y)F(x)dx=G(y), \tag{5.2}$$

where

$$\Gamma(x,y)=\frac{1}{\pi}\int_{-1}^{1}\frac{(x^2+t^2)\sqrt{1-t^2}}{(1+\cos^2 x+e^x)}\frac{dt}{(t-y)}$$

and

$$G(y)=-\frac{1}{\pi}\int_{-1}^{1}\frac{t^2\sin t}{t-y}\sqrt{1-t^2}dt$$

According to the fact that functions k and g are analytic the convergence is rather fast. Using only 16 points, in (3.8), we are able to achieve relative error of order 10^{-15}. In Table 5.1 we present obtained values for the function F^*, at the point $y=0.1$, also column cond holds condition number of the matrix os the linear system (3.8). As it can be seen condition number of the matrix is quite small. Figure 5.1 represents solution F_{16}^*.

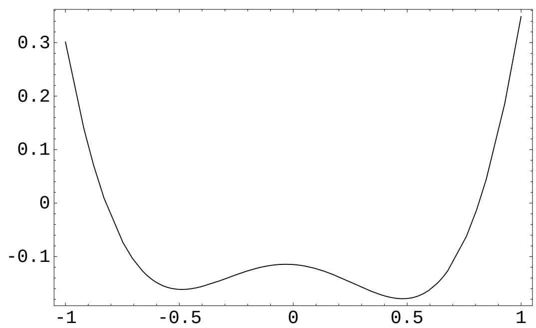


FIG. 5.1: Solution $F_{16}^*(x)$ of the Cauchy singular integral equation (5.1)

Example 5.2. Let

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}dx+\int_{-1}^{1}f(x)\sin(x+y)\sqrt{1-x^2}dx=\left|y-\frac{1}{2}\right|^{7/2}. \tag{5.3}$$

Table 5.1: Numerical results for F_m^* in Example 5.1

m	$y = 0.1$	cond
8	-0.1226	2.0
16	-0.122605465878427	2.6

We solve again (5.2), with

$$\Gamma(x,y) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sin(x+t)\sqrt{1-t^2}}{t-y}dt, \; G(y) = -\frac{1}{\pi}\int_{-1}^{1} \frac{\left|t-\frac{1}{2}\right|^{7/2}}{t-y}\sqrt{1-t^2}dt.$$

In this case, the function g is not analytic; it belongs to the class Z_s^2, with $s < 7/2$. Applying the estimate from Theorem 3.1 it can be easily seen that for $m = 256$ we have an error of order $m^{-s} \sim 10^{-8}$. This is in a good correspondence with experimental results. Table 5.2 holds results, at the point $y = 0.5$ and the condition number of the matrix of the linear system of equations (3.8).

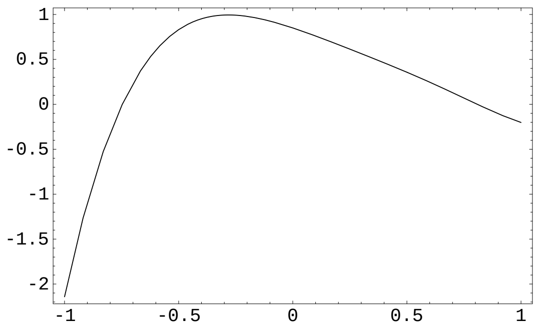


FIG. 5.2: Solution $F_{256}^*(x)$ of the Cauchy singular integral equation (5.3)

Table 5.2: Numerical results for F_m^* in Example 5.2

m	$y = 0.5$	cond
16	0.357	4.2
32	0.35725	4.2
64	0.357251	4.28
256	0.3572518	4.285

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Faculty of Electronic Engineering
Department of Mathematics
P.O. Box 73
18000 Niš, Serbia
e-mail: aca@elfak.ni.ac.yu

Dipartimento di Matematica
Università della Basilicata
Campus Macchia Romana
Potenza 85100, Italy
e-mail: mdebonis@unibas.it