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ASYMPTOTIC INVERTIBILITY OF CONTINUOUS FUNCTIONS OF ONE-SIDED INVERTIBLE OPERATORS

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Dedicated to my friend and colleague Giuseppe Mastroianni on the occasion of his 65th birthday

Abstract. Many concrete convolution operators such as for instance, Toeplitz operators, Wiener-Hopf integral operators, and Mellin integral operators can be treated as continuous functions of one-sided invertible operators. In this paper we establish that the problem of asymptotic invertibility of such operators is governed by at most 2 continuous functions of one-sided invertible elements in some Banach algebras. In particular the algebraization of the so-called Gohberg/Feldmann approach is presented.

1. Introduction

Asymptotic invertibility problems for convolution like operators have a long and interesting history. One of the corner-stones was built up by I. Gohberg and I. Feldmann in their by now classical work [4], where some axiomatic approach to Galerkin methods for convolution equations was proposed. This approach is now referred to as the Gohberg/Feldmann approach. Their method is heavily based on the fact that the convolution operators can be understood as continuous functions of one-sided invertible operators which fulfill some natural assumptions (see Section 2). Another key assumption was that the underlying one-sided invertible operators $V, V^{(-1)}$ ($V^{(-1)}V = I$, $VV^{(-1)} \neq I$) are connected with the defining sequence (P_n) of projections in a very simple way:

$$P_n V P_n = P_n V \, , \; P_n V^{(-1)} P_n = V^{(-1)} P_n \, .$$

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There are approximation sequences for convolution operators which do not fulfil such relations, but are close to them, for instance, those steaming from collocations, quadrature rules and so on. We propose a set of axioms which covers all these situations; this will basically be done in Section 4. Section 3 is devoted to an abstract version of the setting discussed in Section 4. The main tool we use is some theory concerning Banach algebras generated by one left-invertible element v and one of its left-inverses $v^{(-1)}$ (fulfilling some conditions). If v is a non-unitary isometry in a C^*-algebra and $v^{(-1)} = v^*$ then the C^*-algebra generated by v was already studied by Coburn [2]. The more general case we are concerned with in this paper is leant upon the paper [9] (see also [7]), where the case of operators is studied.

We do not consider concrete examples. All examples treated in [7], Chapter 4 and 5, are covered by the presented approach.

2. Algebras Generated by One-sided Invertible Elements

There are almost no papers devoted to Banach algebras generated by one-sided invertible elements. Only the paper [9] (see also [7]) dealt with such algebras in the special case where the elements are linear bounded operators. One more exception is the paper [2], where the case of C^*-algebras is considered. Since the proofs of the related corresponding in the general situation are similar to the operator case they are omitted (the interested reader is addressed to [7], Chapter 4, for getting insight).

Let $\mathcal{A}(v)$ be a unital Banach algebra and $v, v^{(-1)} \in \mathcal{A}$ elements such that

$$v^{(-1)}v = e\,,\; vv^{(-1)} \neq e\,, \tag{2.1}$$

where e is the unit element in $\mathcal{A}$ (the unit element in an unital algebra is always denoted by e). Thus v is invertible from the left. We have to impose a further condition in order to get a satisfactory theory, namely

$$\mathrm{sp}\; v\,, \mathrm{sp}\; v^{(-1)} \subset \{z \in \mathbb{C} : |z| \leq 1\}\,. \tag{2.2}$$

Let $\mathcal{A}$ stand for the smallest closed subalgebra of $\mathcal{A}$ containing v and $v^{(-1)}$. If $\mathcal{A}$ is a C^*-algebra and v is a nonunitary isometry, that is,

$$v^*v = e\,,\; vv^* \neq e\,,$$

then (2.1) and (2.2) are fulfilled. An important result of Coburn [2] says that all C^*-algebras generated by nonunitary isometries are $*$-isometrically isomorphic to each other. In particular, each algebra of that type is $*$-isometrically isomorphic to the C^*-subalgebra of $\mathcal{B}(l^2(\mathbb{Z}_+))$, generated by the forward shift $V : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$,

$$V(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots)\,,$$

and this algebra is nothing but the C^*-algebra $\operatorname{alg}\{T(a) : a \in C(\mathbb{T})\}$, where $\mathbb{T}$ is the complex unit circle and $T(a)$ denotes the familiar Toeplitz operator with continuous generating function.

Let us return to the general case. We shall use the notation $(n \in \mathbb{Z})$

$$v_n := \begin{cases} v^n\,, & \text{if} \quad n \geq 0, \\ (v^{(-1)})^{-n}, & \text{if} \quad n < 0\,. \end{cases}$$

For a given (trigonometric) polynomial $r(t) = \sum\limits_{j=-m}^{m} a_j t^j$ on the complex unit circle $\mathbb{T}$, we define an element $r(v) \in \mathcal{A}(v)$ by $\sum\limits_{j=-m}^{m} a_j v_j$ and call $r(v)$ a (trigonometric) polynomial of v. Let $L_0(v)$ stand for the set of all polynomials of v and let $L_0^+(v)$ and $L_0^-(v)$ stand for the set

$$\left\{ \sum_{j=0}^{m} a_j v_j : m \in \mathbb{Z}_+ \right\} \quad \text{and} \quad \left\{ \sum_{j=-m}^{0} a_i v_i : m \in \mathbb{Z}_+ \right\},$$

respectively.

It is worth mentioning that the mapping $r \mapsto r(v)$ is injective and that for $r_1(v) \in L_0^-(v)$, $r_2(v) \in L_0(v)$, and $r_3(v) \in L_0^+(v)$ the equality

$$(r_1\, r_2\, r_3)(v) = r_1(v) r_2(v) r_3(v)$$

holds.

Definition 2.1. The smallest closed two-sided ideal in $\mathcal{A}(v)$ containing all elements of the form

$$(r_1 r_2)(v) - r_1(v) r_2(v)$$

(with r_1, r_2 arbitrarily given polynomials) is denoted by $Q(v)$ and called the quasicommutator ideal of $\mathcal{A}(v)$.

Theorem 2.1. *The quotient algebra $\mathcal{A}(v)/Q(v)$ is commutative, the space of maximal ideals can be identified with $\mathbb{T}$, and the Gelfand transform can be chosen in such a way that $v + Q(v)$ is taken into the polynomial $p(t) = t$.*

Definition 2.2. For $a \in \mathcal{A}(v)$ the Gelfand transform of $a + Q(v)$ is called the *symbol* of a and is denoted by $\operatorname{smb} a$.

Thus, $\operatorname{smb} a$ is a continuous function on $\mathbb{T}$. It is easy to see that $\operatorname{smb} r(v) = r$ for any $r(v) \in L_0(v)$. It was already mentioned that the map $r(v) \mapsto \operatorname{smb} r(v)$ is injective for $r(v) \in L_0(v)$. However, this result is not true in general for elements $a \in L(v)$, $L(v)$ being the norm closure of $L_0(v)$ (an example can be found in [7], Section 4.3).

Definition 2.3. A commutative Banach algebra E is called v-dominating (more exactly $(v, v^{(-1)})$-dominating) if

(a) there is an invertible element $d \in E$ which generates with d^{-1} the algebra E,

(b) $\operatorname{sp} d, \operatorname{sp} d^{-1} \subset \{z \in \mathbb{C} : |z| \leq 1\}$,

(c) for any (trigonometric) polynomial p,

$$\|p(v)\|_{\mathcal{A}} \leq M\|p(d)\|, \tag{2.3}$$

where M is some positive constant not depending on p.

It is not hard to see that the space of maximal ideals of E can be identified with $\mathbb{T}$, and that the Gelfand transform can be chosen in such a way that d is taken into the polynomial t.

Since $p(d)$ is uniquely defined by its Gelfand transform (even in the case where E has nontrivial radical) the mapping $p(d) \mapsto p(v)$ is well-defined, linear, and by (2.3), bounded. Hence, this mapping can be extended continuously to the whole algebra E, and the image, denoted by $L_E(v)$, is contained in $L(v)$. For an element $a \in E$ let $a(v)$ be the image under this mapping. Note that

$$\|a(v)\|_{\mathcal{A}} \leq M\|a\|_E$$

for all $a \in E$ and that the symbol of $a(v)$ coincides with the Gelfand transform of a. If the radical of E is trivial then an element of $L_E(v)$ is uniquely determined by its symbol (even in the case where this is unknown for elements from $L(v)$). The importance of $L_E(v)$ is given by the next theorem which basically goes back to I. Gohberg (see [3] or [4]).

Theorem 2.2. *Let E be v-dominating.*

(i) *An element $a \in L_E(v)$ is at least one-sided invertible if $(\operatorname{smb} a)(t) \neq 0$ for all $t \in \mathbb{T}$. If the symbol of a does not vanish on $\mathbb{T}$ then the invertibility of a corresponds with the winding number $\kappa = \operatorname{wind} \operatorname{smb} a$, i.e., a is invertible, invertible only from the left or only from the right if κ is zero, positive or negative, respectively.*

(ii) *If* $a \in L(v)$ *and* $(\operatorname{smb} a)(t_0) = 0$ *for some* $t_0 \in \mathbb{T}$ *then* a *is not one-sided invertible.*

Now, we give some examples of v-dominating algebras:

1° We call a sequence of positive numbers $(u_n)_{n\in\mathbb{Z}}$ a weight if

$$\lim_{n\to+\infty} (u_n)^{1/n} = \lim_{n\to+\infty} (u_{-n})^{1/n} = 1 \quad \text{and} \quad u^* := \sup_{k,n\in\mathbb{Z}} \frac{u_{n+k}}{u_n u_k} < +\infty.$$

Denote by $W(u)$ the collection of all functions a on $\mathbb{T}$ the Fourier coefficients of which satisfy

$$\sum_{k=-\infty}^{+\infty} |a_k| u_k < +\infty$$

and put $\|a\|_{W(u)} := u^* \sum\limits_{k\in\mathbb{Z}} |a_k| u_k$.

The set $W(u)$ actually forms a commutative Banach algebra whose maximal ideal space is $\mathbb{T}$. If v and $v^{(-1)}$ fulfill (2.1) and (2.2) then

$$W(u)\,,\ u = (u_n)_{n\in\mathbb{Z}} \quad \text{with} \quad u_n = \|v_n\|$$

is v-dominating.

2° If $\sup\limits_{n\in\mathbb{Z}} \|v_n\| < +\infty$ then $\mathcal{A}(v)/Q(v)$ is v-dominating. Indeed the mentioned condition ensures the decomposition

$$\mathcal{A}(v) = L(v) + Q(v)\,,$$

(see [7], Chapter 4) and Theorem 2.1 gives the claim.

3° If $\mathcal{A}$ is unital C^*-algebra and v a nonunitary isometry (i.e., $v^*v = e\,,\ vv^* \neq e$) then

$$\mathcal{A}(v)/Q(v) \cong C(\mathbb{T})$$

and $C(\mathbb{T})$ is v-dominating (use Example 2, Theorem 2.1 and the description of commutative unital C^*-algebras); moreover, $\|a(v)\|_{\mathcal{A}} = \|a\|_{C(\mathbb{T})}$.

Now we take up the problem of inverse closedness. Recall that a subalgebra B of a unital $\mathcal{A}$ Banach algebra with $e \in B$ is called inverse closed in A if the spectrum of every $a \in B$ with respect to B coincides with the spectrum of a with respect to A.

Given a Banach algebra A let $A^{(r)}$ stand for the collection of all $r \times r$-matrices with entries from A. The set $A^{(r)}$ actually forms a Banach algebra under the norm

$$\|(a_{ij})\| = r \max_{1\le i,j\le r} \|a_{ij}\|\,.$$

Theorem 2.3. *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{A}(v)$ be as above. Then $\mathcal{A}^{(r)}(v)$ is inverse closed in $\mathcal{A}^{(r)}$ for every $r \in \mathbb{N}$.*

Sketch of proof. Let

$$Z^{(r)}(v) := \left\{ \sum_{l=1}^{m} \prod_{j=1}^{k} r_{lj}(v) : r_{lj}(v) \text{ are } r \times r\text{-matrices with entries from } L_0(v) \right\}.$$

Obviously, $Z^r(v)$ is dense subalgebra of $\mathcal{A}^{(r)}(v)$ for each $r \in \mathbb{N}$. It is easy to see that is remains to show the assertion for $a \in Z^{(r)}(v)$. So let $a \in Z^{(r)}(v)$ be invertible in $\mathcal{A}^{(r)}$. Then form a linear extension $\tilde{a}$ of a exactly as it is done in [4], Chapter VIII, § 10. The properties of $\tilde{a}$ are:

1. $\tilde{a}$ belongs to $\mathcal{A}^{(s)}(v) \subset \mathcal{A}^{(s)}$ for some (sufficiently large) s.
2. $\tilde{a}$ is invertible in $\mathcal{A}^{(s)}(\mathcal{A}^{(s)}(v))$ if and only if a is invertible in $\mathcal{A}^{(r)}(\mathcal{A}^{(r)}(v))$.
3. The entries of $\tilde{a}$ are $r \times r$-matrices the entries of which belong to $L_0(v)$.

To $\tilde{a} = (a_{ij})_{i,j=1}^{s} \in \mathcal{A}^{(s)}(v)$ assign $\operatorname{smb} \tilde{a} := \operatorname{smb} a_{ij}$); note that this map is one-to-one by construction (for $s = 1$ it was already mentioned). Moreover, $\operatorname{smb} \tilde{a}$ belongs to $W^{(s)}(u)$ (see Example 1°). Now one has to use Wiener-Hopf factorization (see [4], Chapter VIII) in order to represent $\tilde{a}$ as $\tilde{a} = \tilde{a}_{-}\tilde{a}_{+}$, where the factors $\tilde{a}_{-}, \tilde{a}_{+}$ are of a special form, what allows to conclude that $\tilde{a}$ is invertible in $\mathcal{A}^{(s)}(v)$ if and only if $\tilde{a}$ is invertible in $\mathcal{A}^{(s)}$, and we are done (one has of course to check that the invertibility of a implies that $\det \operatorname{smb} \tilde{a}$ does not vanish; but this can be proved analogously to Theorem 4.98 in [7]).

Remark 2.1. The proof of Theorem 2.3 shows that the following conclusion is valid: Let $\mathcal{A}_1, \mathcal{A}_2$ be unital Banach algebras and $w, w^{(-1)} \in \mathcal{A}_1$, $v, v^{(-1)} \in \mathcal{A}_2$ elements which fulfill (2.1) and (2.2). If p_{ij}, $i = 1, \ldots, l, j = 1, \ldots, m$ are arbitrarily given polynomials on $\mathbb{T}$ then $\sum\limits_{i=1}^{l} \prod\limits_{j_1}^{m} p_{ij}(w) \in \mathcal{A}_1(w)$ is invertible in $\mathcal{A}_1(w)$ if and only if $\sum\limits_{i=1}^{l} \prod\limits_{j=1}^{m} p_{ij}(v) \in \mathcal{A}_2(v)$ is invertible in $\mathcal{A}_2(v)$. This fact yields a proof of Coburns theorem cited above.

The theory developed can be used to study various classes of convolution operators (see [4], [7]).

3. An Invertibility Problem

The problem of asymptotic invertibility consists (roughly spoken) in the following: approximate an invertible operator by matrices in the strong operator topology (or in weaker topologies) so that these matrices are invertible with the exception of a finite number and that the inverses of the given matrices converge (in the given topology) to the inverse of the operator. The approximations are usually obtained by some rules, and only the invertibility of the operator at hand is in general not sufficient to ensure its asymptotic invertibility.

For operators from $L(V)$ (V is now an only left-invertible operator in a Banach space with left-inverse fulfilling (2.1) and (2.2)) the following set of axioms is a very abstract form of the requirements we shall use later on.

Let $v, v^{(-1)}$ be as above (that is let (2.1) and (2.2) be fulfilled), and let $\mathcal{F}_0$ be some unital Banach algebra. Suppose that there is a linear bounded operator $D : L(v) \to \mathcal{F}_0$ such that: (i) $D(e) = e$; (ii)

$$D(v_k) = (D(v_{\operatorname{sgn} k}))^{|k|}; \tag{3.1}$$

(iii) $J_1 \cap J_2 = \{0\}$, where $J_1 := \operatorname{id}(D(e) - D(v)D(v^{(-1)}))$, $J_2 := \operatorname{id}(D(e) - D(v^{(-1)})D(v))$ ($\operatorname{id} b$ means the smallest closed two-sided ideal of a Banach algebra containing b).

Let $\mathcal{A}_D(v) \subset \mathcal{F}_0$ stand for the smallest closed subalgebra of $\mathcal{F}_0$ containing all elements $D(a), a \in L(v)$. We shall consider only the case $J_1 \neq \{0\}$ since in applications J_1 is always non-trivial, but J_2 can be trivial.

The case $J_2 = \{0\}$ is simple because then $D(v)$ is invertible only from the left and the assumptions (3.1) imply that (2.1) and (2.2) is fulfilled. Therefore, $\mathcal{A}_D(v)$ is again an algebra of the type $\mathcal{A}(v)$. Notice that if E is v-dominating then E is also $D(v)$-dominating.

Proposition 3.1. *Suppose $J_2 \neq \{0\}$. An element $a \in \mathcal{A}_D(v)$ is invertible in $\mathcal{A}_D(v)$ if and only if the cosets $a + J_i$, $i = 1, 2$, are invertible in $\mathcal{A}_D(v)/J_i$, respectively.*

Sketch of the proof: If $a + J_i$ are invertible ($i = 1, 2$) then there are elements $b_i \in \mathcal{A}_D(v)$, $j_i \in J_i$ such that $ab_i = e + j_i$. Now, we consider $(ab_1 - e)(ab_2 - e) = j_1 j_2 = 0$ what gives $a(b_1 + b_2 - b_1 a b_2) = e$. The left-invertibility can be shown in an analogous way. □

Proposition 3.1 indicates that one has to handle the algebras $\mathcal{A}_D(v)/J_i$.

Proposition 3.2. *Let $J_2 \neq \{0\}$. Then $\mathcal{A}_D(v)/J_i$ is an algebra of the type $\mathcal{A}(w_i)$, $i = 1, 2$, with*

$$\begin{aligned} w_1 &= D(v^{(-1)}) + J_1\,, & w_1^{(-1)} &= D(v) + J_1\,, \\ w_2 &= D(v) + J_2\,, & w_2^{(-1)} &= D(v^{(-1)}) + J_2\,. \end{aligned}$$

Proof. By our assumptions $w_i^{(-1)} w_i = e$, $w_i w^{(-1)} \neq e$, $i = 1, 2$, and for instance

$$\|w_1^k\| = \|D(v^{(-1)})^k + J_1\| = \|D(v^{(-1)})^k + J_1\| \leq M\|v^{(-1)^k}\|;$$

hence the spectral radius of w_1 is less or equal to 1. That $\mathcal{A}_D(v)/J_i$ is generated by w_i and $w_i^{(-1)}$, respectively, is a consequence of (3.1). □

Remark 3.1. Suppose that the conditions of Proposition 3.2 are fulfilled and that E is v-dominating. Then E is w_2-dominating and $\tilde{E}$ is w_1-dominating. The Banach algebra $\tilde{E}$ is defined as follows: Let p be any (trigonometric) polynomial, $p(t) = \sum_{j=-k}^{k} a_j t^j$, $|t| = 1$. Put

$$\tilde{p}(t) := \sum_{j=-k}^{k} a_{-j} t^j,$$

that is

$$\tilde{p}(t) = p(t^{-1}), \quad \tilde{p}(d) := \sum_{j=-k}^{k} a_{-j} d^j$$

and define

$$\|\tilde{p}(d)\| := \|p(d)\|.$$

The collection $\tilde{E}_0$ of all elements $\tilde{p}(d)$ with the above defined norm forms an algebra which is non–closed. Take its completion $\tilde{E}$. Clearly, $\tilde{E}$ is a Banach algebra which is w_1-dominating. Moreover, E and $\tilde{E}$ are isometrically isomorphic, the isometry given by $d \mapsto d^{(-1)}$.

Let W_i, $i = 1, 2$, denote the canonical homomorphism from $\mathcal{A}_D(v)$ into $\mathcal{A}_D(v)/J_i$ ($J_2 \neq \{0\}$).

Proposition 3.1 and 3.2 can be summarized as follows.

Theorem 3.1. *Suppose conditions* (3.1) *are fulfilled and* $J_2 \neq \{0\}$. *Then* $a \in \mathcal{A}_D(v)$ *is invertible if and only if* $W_i(a)$ *is invertible in* $\mathcal{A}_D(v)/J_i$, $i = 1, 2$.

Corollary 3.1. *Suppose that conditions* (3.1) *are fulfilled, that* E *is* v-*dominating and* $J_2 \neq \{0\}$. *If* $a \in L_E(v)$, *then* $D(a)$ ($\in \mathcal{A}_D(v)$) *is invertible in* $\mathcal{A}_D(v)$ *if and only if*

$$(\operatorname{smb} a)(t) \neq 0 \quad \textit{for all} \;\; t \in \mathbb{T}$$

and

$$\operatorname{wind} \operatorname{smb} a = 0.$$

Remark 3.2. It is by no means clear whether the elements $W_i(a)$ occurring in Theorem 3.1 are invertible or not. There are no practicable invertibility criterions in general and this is a serious obstacle, which causes difficulties in practice.

4. Asymptotic Invertibility of Continuous Functions of One-sided Invertible Operators

Let X be a Banach-space and let $\mathcal{B}(X)$ denote the Banach algebra of all bounded and linear operators acting in X, and let $\mathcal{K}(X) \subset \mathcal{B}(X)$ be the ideal of all compact operators. Suppose we are given a sequence of projections $(P_n) \subset \mathcal{B}(X)$ such that the strong limits

$$s - \lim P_n\,,\ s - \lim P_n^*$$

exist and equal the identity operators in X and X^*, respectively. Introduce the collection $\mathcal{F}$ of all bounded sequences (A_n), $A_n \in \mathcal{B}(\text{im } P_n)$ and define the norm by

$$\|(A_n)\| := \sup_n \|A_n P_n\|\,.$$

Under componentwise operations, $\mathcal{F}$ actually forms a unital Banach algebra with unit $e := (P_n)$.

A discretization of $A \in \mathcal{B}(X)$ is by definition a sequence $(A_n) \subset \mathcal{F}$ which converges strongly to A:

$$A = s - \lim A_n P_n\,.$$

Especially important are discretizations by matrices. In this case norm convergence does not coincide with strong convergence if X is infinite dimensional.

It is also possible to replace strong convergence by weaker convergence (see [7] or [8]). We will not deal with such situations for simplicity.

The problem of asymptotic invertibility of a given invertible operator A is to find a discretization $(A_n) \in \mathcal{F}$ of A such that the operators $A_n \in \mathcal{B}(\text{im } P_n)$ are invertible for n large enough and $\sup\limits_{n \geq n_0} \|A_n^{-1} P_n\| < +\infty$ (in this case (A_n) is called stable). Then the stability (as it is well-known) ensures $s - \lim A_n^{-1} P_n = A^{-1}$. Hence, the invertible operator A is asymptotically invertible by the sequence (A_n), and the problem is indeed the stability problem of (A_n).

Let $\mathcal{N}$ stand for the ideal of all sequences $(A_n) \in \mathcal{F}$ with $\|A_n P_n\| \to 0$ as $n \to +\infty$. The following proposition is well-known.

Proposition 4.1. *$(A_n) \in \mathcal{F}$ is stable if and only if the coset $(A_n) + \mathcal{N}$ is invertible in $\mathcal{F}/\mathcal{N}$.*

Denote by $\mathcal{F}^c$ the closed subalgebra of $\mathcal{F}$ consisting of all sequences such that $s - \lim A_n P_n$, $s - \lim A_n^* P_n^*$ exist. The following proposition is also well-known (see [5], page 9).

Proposition 4.2. *$\mathcal{N} \subset \mathcal{F}^c$ and $\mathcal{F}_0 := \mathcal{F}^c/\mathcal{N}$ is inverse closed in $\mathcal{F}/\mathcal{N}$.*

We are now going to study the stability problem for elements from $L(V)$, where $V, V^{(-1)} \in \mathcal{B}(X)$ are operators fulfilling (2.1) and (2.2).

Suppose we are given a bounded and linear map $D : L(V) \to \mathcal{F}^c$ such that

(i) $D(I) - (P_n) \in \mathcal{N}$ and $s - \lim D(a) = a$ for any $a \in L(V)$.

(ii) $D(V_k) - D(V_{\mathrm{sgn}k})^{|k|} \in \mathcal{N}$ for any $k \in \mathbb{Z}$.

Denote by $\mathcal{A}_D(V)$ the smallest closed subalgebra of $\mathcal{F}^c$ containing all elements $D(a)$, $a \in L(V)$. Suppose further

(iii) $J_1 \cap J_2 \subset \mathcal{N}(J_1, J_2, \subset \mathcal{A}_D(v))$ where

$$J_1 := \mathrm{id}\,(D(I) - D(V)D(V^{(-1)}), \quad J_2 := \mathrm{id}\,(D(I) - D(V^{(-1)})D(V)).$$

Notice that in [6] there was presented a related set of axioms in the C^*-algebra setting.

The following result, based on Section 3, is almost obvious (use $\mathcal{F}_0 := \mathcal{F}^c/\mathcal{N}$ and Corollary 3.1).

Theorem 4.1. *Let E be V-dominating.*

1.) *If $J_2 \subset \mathcal{N}$ then $D(a)$, $a \in L_E(V)$, is stable if and if only* $(\mathrm{smb}\, a)(t) \neq 0$ *for all $t \in \mathbb{T}$ and* $\mathrm{wind}\,\mathrm{smb}\, a = 0$.

2.) *If $D(I) - D(V^{(-1)})D(V) \notin \mathcal{N}$, then $D(a)$, $a \in L_E(V)$, is stable if* $(\mathrm{smb}\, a)(t) \neq 0$ *for all $t \in \mathbb{T}$ and* $\mathrm{wind}\,\mathrm{smb}\, a = 0$.

That the conditions of Theorem 4.1, 2.) are also necessary is not clear at all. But under the assumption (iii)′ this is the case (see Theorem 4.2 below):

(iii)′ $D(I) - D(V)D(V^{(-1)}) = (P_n(I - VV^{(-1)}P_n)) + (C_n)$, with $(C_n) \in \mathcal{N}$ and dim ker $V^{(-1)} < +\infty$.

The very last condition ensures that $Q(V) \subset \mathcal{K}(X)$.

Proposition 4.1. *If* (iii)$'$ *is fulfilled then*

$$J_1 \cap J_2 \subset \mathcal{N} \quad \textit{and} \quad \hat{J}_1 \cap \hat{J}_2 \subset \mathcal{N}\,,$$

where

$$\hat{J}_1 = \mathrm{id}\,(D(I) - D(V)D(V^{(-1)})), \quad \hat{J}_2 = \mathrm{id}\,(D(I) - D(V^{(-1)})D(V))$$

with respect to $\mathcal{F}^c$.

Proof. It needs only to prove the second claim. Because of our assumption we have $s - \lim D(I) - D(V^{(-1)})D(V) = I - V^{(-1)}V = 0$. This implies in fact that every sequence $(A_n) \in \hat{J}_2$ tends strongly to zero. On the other hand, J_1 consists of sequences (A_n) such that $A_n = P_nKP_n + C_n$ with $K \in \mathcal{K}(X)$, $(C_n) \in \mathcal{N}$ (see [5] for instance). Notice that then the sequence (A_n) tends in the norm to K. If $(A_n) \in \hat{J}_1 \cap \hat{J}_2$ then $s - \lim A_n = 0$. On the other hand, this sequence tends in the norm to the zero operator. Thus, $(A_n) \in \mathcal{N}$. $\square$

Let $M \subset \mathcal{F}^c$ some set. We denote by $M^{\mathcal{N}}$ the image of M under the canonical homomorphism $a \mapsto a + \mathcal{N}$.

Theorem 4.2. *If* (i), (ii), *and* (iii)$'$ *are fulfilled, then $\mathcal{A}_D^{\mathcal{N}}(v)$ is inverse closed in $\mathcal{F}_0$.*

Proof. If $J_2 \subset \mathcal{N}$ then the result follows from Theorem 2.3 (under the conditions (i) – (iii) only). Now let $D(I) - D(V^{(-1)})D(V) \notin \mathcal{N}$. Then

$$\hat{J}_1^{\mathcal{N}} \cap \hat{J}^{\mathcal{N}} = \{0\}$$

by Proposition 4.1. As in the proof of Proposition 3.1 one proves with help of Proposition 4.1 that $(A_n)^{\mathcal{N}}$, $(A_n) \in \mathcal{F}_0$, is invertible in $\mathcal{F}_0$ if and only if the cosets $(A_n)^{\mathcal{N}} + \hat{J}_1^{\mathcal{N}}$, $(A_n)^{\mathcal{N}} + \hat{J}_2^{\mathcal{N}}$ are invertible in $\mathcal{B}_i := \mathcal{F}_0/\hat{J}_i^{\mathcal{N}}$, respectively.

Take now $(A_n)^{\mathcal{N}} \in \mathcal{A}_D^{\mathcal{N}}(v)$. Then the cosets $(A_n)^{\mathcal{N}} + \hat{J}_i^{\mathcal{N}}$, $i = 1, 2$, belong to algebras $\mathcal{D}_i$ of the type studied in Section 2, which are inverse closed in $\mathcal{B}_i$, respectively. It is now easy to show (by help of Remark 3.2) that $\mathcal{A}_D^{\mathcal{N}}(v)$ is inverse closed in $\mathcal{F}_0$. $\square$

Remark 4.1. Theorem 4.2 is also valid in the block case, that is for $r \geq 1$.

We shall denote by W_1, W_2 also the homomorphisms $\mathcal{A}_D^{(r)}(v) \ni (A_n) \mapsto (A_n)^{\mathcal{N}} + (J_i^{(r)})^{\mathcal{N}}$, $i = 1, 2$. The following theorem is now a consequence of what was said above.

Theorem 4.3. *Suppose If* (i), (ii), (iii)$'$ *are fulfilled. Then* $(A_n) \in \mathcal{A}_D^{(r)}(V)$ *is stable if and only if*

$$W_1(A_n)\,, \quad W_2(A_n)$$

are invertible in $(\mathcal{A}_D^{(r)}(V))^{\mathcal{N}}/(J_i^{(r)})^{\mathcal{N}}\,,\ i = 1, 2.$

Remark 4.2. If $D(V)$, is a sequence the members of which are Fredholm operators of index zero for any $a \in L(V)$ then $D(I) - D(V^{(-1)})D(V) \notin \mathcal{N}$.

Indeed suppose $D(I) - D(V^{(-1)})D(V) \in \mathcal{N}$. Then $\mathcal{A}_D^{\mathcal{N}}(V)$ is an algebra of the type studied in Section 2 and $D(V^{(-1)})D(V) - (P_n) \in \mathcal{N}$. That is the sequence $D(V)$ consists of operators which are invertible from the left for n sufficiently large, say for $n \geq n_0$. But then the members A_n of $D(V)$ are invertible for $n \geq n_0$ because of the index condition and the inverses are given by the members B_n of $D(V^{(-1)}) + (C_n)$, where $(C_n) \in \mathcal{N}$ is constructed in an obvious way $(n \geq n_0)$. Hence, $D(V)D(V^{(-1)}) - (P_n) \in \mathcal{N}$. But this would imply $VV^{(-1)} - I = 0$ (take strong limits) and we arrive at a contradiction.

Notice that for all practicable approximation methods for convolution equations for which $\mathcal{F}^c$ is the appropriate algebra, the set of axioms (i) – (iii) is fulfilled. See for instance [4] and [7], Chapter 4 and 5. More precisely, in [4] there are considered projection methods for continuous functions of one-sided invertible operators $V, V^{(-1)}$ under the condition dim ker $V^{(-1)} < +\infty$ and

$$P_n V P_n = P_n V\,, \quad P_n V^{(-1)} P_n = V^{(-1)} P_n\,.$$

If $(P_n), (P_n^*)$ tend to identity operators, respectively, then (i) – (iii) (even (i), (ii), (iii)$'$) are fulfilled and we get by the theory developed here the wanted picture both in the scalar and in the system case.

Now we are going to explain the example announced in Remark 3.2.

Example. Consider the matrix function

$$p(t) = \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix}\,,\ t \in \mathbb{T}\,,$$

and $p(V) \in L^{(2)}(V)$, where $V : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is the forward shift and $V^{(-1)} := V^*$. It is directly computed in [1], Chapter 6, that (in our language) $p(V)$ is invertible but $\tilde{p}(V)$ not. Using Remark 2.1 it is now easy to see that $D(p(V))$ is not stable in the general setting (under the condition $D(I) - D(V^{(-1)})D(V) \notin \mathcal{N}$). Hence, the invertibility of $W_1(A_n)$ does not imply the invertibility of $W_2(A_n)$ and vice versa.

Final remark. If (P_n) converges to identity in weaker topologies then the main ideas of this paper are also applicable. However additional work has to be done. Maybe, this will be the subject of a forthcoming paper.

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