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KRYLOV SUBSPACE METHODS FOR CAUCHY SINGULAR INTEGRAL EQUATIONS

P. Junghanns and K. Rost

Dedicated to our friend Giuseppe Mastroianni on the occasion of his 65th birthday

Abstract. We discuss the applicability of Krylov subspace methods to the solution of linear systems of equations occurring in the application of collocation methods to the numerical solution of Cauchy singular integral equations on an interval. The aim is to obtain fast algorithms for the approximate solution of such equations with $O(n \log n)$ computational complexity.

1. Introduction

During the last 15 years, the problem of the construction of fast solvers for linear systems of algebraic equations occurring in the numerical solution of singular integral and related equations have been played an important role. Since, usually, the matrices of such systems are not sparse one has to try to employ some structural and/or smoothing properties of the operators (or matrices) involved in the original equation. The classes of singular integral equations, for which there exist essential contributions in this direction can be divided into two cases, the periodic case and a case, which we call quasi-periodic.

As an example for the periodic case, let us consider a Cauchy singular integral equation on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ of the form

$$(Au)(t) := a(t)u(t) + \frac{b(t)}{\pi \mathrm{i}} \int_{\mathbb{T}} \frac{u(s)\, ds}{s-t} = f(t)\,, \quad t \in \mathbb{T}\,, \tag{1.1}$$

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where $a, b, f : \mathbb{T} \longrightarrow \mathbb{C}$ are given, say, continuous functions satisfying $[a(t)]^2 - [b(t)]^2 \neq 0$, $t \in \mathbb{T}$. A collocation method for equation (1.1) can be described in the following way. Let $t_{nj} = e^{2\pi \mathrm{i} j/(2n+1)}$, $j = -n, \ldots, n$, be the collocation points, $e_j(t) = t^j$, and

$$P_n^{\mathbb{T}} u = \sum_{j=-n}^{n} \langle u, e_j \rangle_{\mathbb{T}} \, e_j \,,$$

the respective orthoprojection from $\boldsymbol{L}^2(\mathbb{T})$ onto span $\{e_{-n}, \ldots, e_n\}$, where

$$\langle u, v \rangle_{\mathbb{T}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{\mathrm{i}s}) \overline{v(e^{\mathrm{i}s})} \, ds \,.$$

Look for an approximate solution $u_n \in \operatorname{im} P_n^{\mathbb{T}}$, such that

$$(Au_n)(t_{nj}) = f(t_{nj}) \,, \quad j = -n, \ldots, n \,. \tag{1.2}$$

If we define the interpolation operator $L_n^{\mathbb{T}}$ by

$$L_n^{\mathbb{T}} f \in \operatorname{im} P_n^{\mathbb{T}} \,, \quad (L_n^{\mathbb{T}} f)(t_{nj}) = f(t_{nj}) \,, \ j = -n, \ldots, n \,,$$

the collocation conditions (1.2) can be written equivalently as the operator equation

$$L_n^{\mathbb{T}} A u_n = L_n^{\mathbb{T}} f \,, \quad u_n \in \operatorname{im} P_n^{\mathbb{T}} \,.$$

Write the operator A of (1.1) in the form $A = aI + bS_{\mathbb{T}}$. It is well known that $S_{\mathbb{T}} : \boldsymbol{L}^2(\mathbb{T}) \longrightarrow \boldsymbol{L}^2(\mathbb{T})$ is continuous with $S_{\mathbb{T}}^2 = I$, such that

$$P_{\mathbb{T}} = \frac{1}{2}(I + S_{\mathbb{T}}) \quad \text{and} \quad Q_{\mathbb{T}} = \frac{1}{2}(I - S_{\mathbb{T}}) = I - P_{\mathbb{T}}$$

are projections and $A = cP_{\mathbb{T}} + dQ_{\mathbb{T}}$, where $c = a + b$, $d = a - b$. Hence the collocation method (1.2) is equivalent to

$$L_n^{\mathbb{T}} (cP_{\mathbb{T}} + dQ_{\mathbb{T}}) u_n = L_n^{\mathbb{T}} f \,, \quad u_n \in \operatorname{im} P_n^{\mathbb{T}} \,. \tag{1.3}$$

The application of Amosov's idea (see [1, 5]) to this situation leads to the following method.

Choose a positive integer m such that $m < n$ and $(2n+1)/(2m+1)$ is an integer, and look for an approximate solution v_n as $v_n = P_m^{\mathbb{T}} w_m + Q_m^{\mathbb{T}} z_n$, where $Q_m^{\mathbb{T}} = I - P_m^{\mathbb{T}}$, $w_m \in \operatorname{im} P_m^{\mathbb{T}}$, $z_n \in \operatorname{im} P_n^{\mathbb{T}}$, and

$$z_n = L_n^{\mathbb{T}} B L_n^{\mathbb{T}} f \,, \quad B = c^{-1} P_{\mathbb{T}} + d^{-1} Q_{\mathbb{T}} \,,$$

$$L_m^{\mathbb{T}}(cP_{\mathbb{T}} + dQ_{\mathbb{T}})w_m = L_m^{\mathbb{T}} g\,, \quad g = f - (cP_{\mathbb{T}} + dQ_{\mathbb{T}})Q_m^{\mathbb{T}} z_n\,.$$

Using the Fourier matrix $\boldsymbol{F}_n = \left[\, t_{nj}^k \,\right]_{j,k=-n}^n$ of order $2n+1$ together with its inverse $\boldsymbol{F}_n^{-1} = \dfrac{1}{2n+1}\left[\, t_{nj}^{-k} \,\right]_{j,k=-n}^n$ the method can be realized with $O(n \log n)$ complexity if m is chosen in such a way that $m \sim n^{1/3}$ (see the following algorithm). Let $\widetilde{f}_n = \left[\, f(t_{nj}) \,\right]_{j=-n}^n$ and $\widehat{f}_n = \left[\, \langle f, e_j \rangle_{\mathbb{T}} \,\right]_{j=-n}^n$.

ALGORITHM:

1. Compute $\widehat{z}_n = \boldsymbol{F}_n^{-1}\Big(\boldsymbol{C}_n^{-1}\boldsymbol{F}_n\boldsymbol{I}_n^+ + \boldsymbol{D}_n^{-1}\boldsymbol{F}_n\boldsymbol{I}_n^-\Big)\boldsymbol{F}_n^{-1}\widetilde{f}_n$, where $\boldsymbol{C}_n = \operatorname{diag}\widetilde{c}_n$, $\boldsymbol{D}_n = \operatorname{diag}\widetilde{d}_n$,
$$\boldsymbol{I}_n^+ = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{n+1} \end{bmatrix}, \quad \boldsymbol{I}_n^- = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

2. Compute $\widetilde{g}_m = \widetilde{f}_m - (\boldsymbol{C}_m\boldsymbol{I}_{m,n}\boldsymbol{F}_n\boldsymbol{I}_n^+ + \boldsymbol{D}_m\boldsymbol{I}_{m,n}\boldsymbol{F}_n\boldsymbol{I}_n^-)\boldsymbol{I}_n^m\widehat{z}_n$, where
$$\boldsymbol{I}_n^m = \operatorname{diag}[\underbrace{1,\dots,1}_{n-m},0,\dots,0,\underbrace{1,\dots,1}_{n-m}], \quad [\boldsymbol{I}_{m,n}]_{jk} = \begin{cases} 1 & : \; t_{mj} = t_{nk}, \\ 0 & : \; \text{otherwise.} \end{cases}$$

3. Solve $L_m^{\mathbb{T}}\Big(cP_{\mathbb{T}} + dQ_{\mathbb{T}}\Big)w_m = L_m^{\mathbb{T}} g$.

In case of stability of the collocation method (1.3) and of appropriate smoothness properties of a, b, f the solution v_n of the above fast algorithm satisfies the same convergence rate as the solution u_n of the usual collocation method (1.3). A consequence of this is that this fast algorithm is restricted to the case of smooth coefficients a, b. For more details and further fast algorithms we refer the reader to [5] and two the overview paper [14] as well as to [4, 18].

The Cauchy singular integral operator $S_{\mathbb{T}}$ on the unit circle satisfies the nice relations

$$(S_{\mathbb{T}}e_j)(t) = \frac{1}{\pi \mathrm{i}}\int_{\mathbb{T}} \frac{e_j(s)\,ds}{s-t} = \begin{cases} e_j(t) & : \; j = 0,1,2,\dots, \\ -e_j(t) & : \; j = -1,-2,\dots \end{cases}$$

Cases, in which analogous (trigonometric) relations are available, we call quasi-periodic. One of such relations, for example, is the relation between the normed Chebyshev polynomials $W_n(x)$ and $V_n(x)$ of degree n with respect to the weights $(1-x)^{-1/2}(1+x)^{1/2}$ and $(1-x)^{1/2}(1+x)^{-1/2}$, respectively, namely

$$\frac{1}{\pi}\int_{-1}^{1}\frac{W_n(y)}{y-x}\sqrt{\frac{1+y}{1-y}}\,dy = V_n(x)\,, \quad -1<x<1\,,\ n=0,1,2,\ldots\,.$$

Based on this relation on can design an analogous fast algorithm as in the periodic case to solve the equation

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+h(x,y)\right]u(y)\sqrt{\frac{1+y}{1-y}}\,dy = f(x)\,, \quad -1<x<1\,,$$

if the kernel $h(x,y)$ and the right hand side $f(x)$ are sufficiently smooth. In such an algorithm there are used fast trigonometric transforms instead of the discrete Fourier transform. For the interested reader we refer to [2, 9, 14]. Analogous ideas are applicable, for example, to singular integro-differential equations of Prandtl's type (see [6]).

In the present paper we discuss the applicability of Krylov subspace methods to solve systems of linear algebraic equations generated by collocation methods applied to equations of the form (2.1). Here only the structure of the matrix is used and no smoothness properties on the data are necessary. Algorithms with $O(n^2)$ complexity were already used (see, for example, [3, 8, 10]). In this paper our aim is to design algorithms of $O(n\log n)$ complexity.

The paper is organized as follows. In the next section we discuss the stability of the collocation methods under consideration together with the condition of the respective sequences of systems of linear equations. In Section 3 the structure of the system matrices is described and representations of these matrices are presented, which involve only fast trigonometric transforms and diagonal matrices. In Section 4 we shortly describe the Krylov subspace methods, for which we present numerical examples in Section 5.

2. The Collocation Method and Its Stability

For given piecewise continuous functions $a,b:[-1,1]\longrightarrow\mathbb{C}$, let us consider the Cauchy singular integral equation

$$a(x)u(x)+\frac{b(x)}{\pi\mathrm{i}}\int_{-1}^{1}\frac{u(y)\,dy}{y-x} = f(x)\,, \quad -1<x<1\,, \tag{2.1}$$

where also $f:[-1,1]\longrightarrow\mathbb{C}$ is a given function belonging to the Hilbert space $\boldsymbol{L}^2_\sigma$ of square integrable with respect to the Chebyshev weight $\sigma(x)=$

$1/\sqrt{1-x^2}$ functions on $[-1,1]$. The inner product in $\boldsymbol{L}^2_\sigma$ is defined by

$$\langle u, v\rangle_\sigma = \int_{-1}^{1} u(x)\overline{v(x)}\sigma(x)\,dx\,.$$

Using the operators aI and bI of multiplication by $a(x)$ and $b(x)$, respectively, as well as the Cauchy singular integral operator S on the interval $[-1,1]$ equation (2.1) can be written as operator equation

$$au + bSu = f\,.$$

The operators of multiplication by a piecewise continuous function as well as the singular integral operator S are linear bounded operators in $\boldsymbol{L}^2_\sigma$.

Let us solve equation (2.1) numerically. Firstly we look for an approximate solution $u_n(x)$ as weighted polynomial

$$u_n(x) = \sum_{k=1}^{n} \xi_{nk}\widetilde{\ell}^{\omega}_{nk}(x) = \sum_{j=0}^{n-1} a_{nj}\widetilde{u}_j(x)\,, \tag{2.2}$$

where

$$\widetilde{\ell}^{\omega}_{nk}(x) = \frac{\varphi(x)\ell^{\omega}_{nk}(x)}{\varphi(x^{\omega}_{nk})}\,, \quad \widetilde{u}_j(x) = \varphi(x)U_j(x)\,, \quad \varphi(x) = \sqrt{1-x^2}\,,$$

and $\ell^{\omega}_{nk}(x)$, $k = 1,\ldots,n$, denote the usual fundamental Lagrange interpolation polynomials with respect to the nodes x^{ω}_{nk} (the zeros of the nth orthogonal polynomial with respect to the weight $\omega(x)$). By $U_j(x)$ we denote the normalized Chebyshev polynomial of second kind and degree j, i.e.,

$$U_j(\cos s) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(j+1)s}{\sin s}\,.$$

For ω we choose $\omega = \sigma$ or $\omega = \varphi$. Remark that

$$x^{\sigma}_{nk} = \cos\frac{2k-1}{2n}\pi \quad \text{and} \quad x^{\varphi}_{nk} = \cos\frac{k\pi}{n+1}\,.$$

Secondly we apply a collocation method

$$a(x^{\omega}_{nj})u_n(x^{\omega}_{nj}) + \frac{b(x^{\omega}_{nj})}{\pi \mathrm{i}}\int_{-1}^{1}\frac{u_n(y)\,dy}{y - x^{\omega}_{nj}} = f(x^{\omega}_{nj})\,, \quad j = 1,\ldots,n\,, \tag{2.3}$$

to determine the unknown coefficients ξ_{nk}, $k = 1,\ldots,n$, or a_{nj}, $j = 0,1,\ldots$, $n-1$, in (2.2). If we define the weighted interpolation operator $M^{\omega}_n =$

$\varphi L_n^\omega \varphi^{-1} I$, where L_n^ω is the usual Lagrange interpolation operator with respect to the nodes x_{nk}^ω, $k = 1, \ldots, n$, the collocation method (2.3) can be written as operator equation $A_n^\omega u_n = M_n^\omega f$, where $A_n^\omega = M_n^\omega A L_n$ and $L_n : \boldsymbol{L}_\sigma^2 \longrightarrow \boldsymbol{L}_\sigma^2$ is the projection

$$L_n u = \sum_{j=0}^{n-1} \langle u, \widetilde{u}_j \rangle_\sigma \, \widetilde{u}_j \,.$$

Remark that the system $(\widetilde{u}_n)_{n=0}^{+\infty}$ forms an orthonormal basis in $\boldsymbol{L}_\sigma^2$. We call a sequence (A_n) of operators $A_n : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$ stable in $\boldsymbol{L}_\sigma^2$ if, for all sufficiently large n, these operators are invertible and the norms of their inverses are uniformly bounded.

Theorem 2.1. ([13], Thm. 2.1) *For piecewise continuous functions $a, b : [-1,1] \longrightarrow \mathbb{C}$, the sequence $(M_n^\omega(aI + bS)L_n)$ is stable in $\boldsymbol{L}_\sigma^2$ if and only if,*

(a) *in the case $\omega = \sigma$, the operator $aI + bS : \boldsymbol{L}_\sigma^2 \longrightarrow \boldsymbol{L}_\sigma^2$ is invertible,*

(b) *in the case $\omega = \varphi$, the operators $aI \pm bS : \boldsymbol{L}_\sigma^2 \longrightarrow \boldsymbol{L}_\sigma^2$ are both invertible.*

We remark that the collocation methods under consideration in [13] as well as in the present paper are special cases of the collocation methods investigated in [11] and [12].

Let λ_{nk}^ω, $k = 1, \ldots, n$, denote the Christoffel numbers with respect to the weight $\omega(x)$, i.e. $\lambda_{nk}^\sigma = \pi/n$ and $\lambda_{nk}^\varphi = \pi[\varphi(x_{nk}^\varphi)]^2/(n+1)$, and let $\psi_{n,\varphi} = \sqrt{(n+1)/\pi}$. By means of the Gaussian rule one can easily prove that the system $\left\{ \psi_{n,\varphi} \widetilde{\ell}_{nk}^\varphi : k = 1, \ldots, n \right\}$ forms an orthonormal basis in the space $\operatorname{im} L_n$. The matrix representation of the operator A_n^ω with respect to this basis is equal to $\boldsymbol{A}_n^\varphi = \left[\; \alpha_{jk}^\varphi \; \right]_{j,k=1}^n$ with $\alpha_{jk}^\varphi = \left(A \widetilde{\ell}_{nk}^\varphi \right) (x_{nj}^\varphi)$. Hence, the spectral norm $\|\boldsymbol{A}_n^\varphi\|$ of the matrix $\boldsymbol{A}_n^\varphi$ is equal to the norm $\|A_n^\varphi\|_{\boldsymbol{L}_\sigma^2 \to \boldsymbol{L}_\sigma^2}$ of the operator $A_n^\varphi : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$. Let ℓ_n^2 denote the n-dimensional complex vector space equipped with the norm

$$\|(\xi_k)_{k=1}^n\|_{\ell_n^2} = \left(\sum_{k=1}^n |\xi_k|^2 \right)^{1/2} .$$

As a consequence of Theorem 2.1 we get the following corollary.

Corollary 2.2. *Let $a, b : [-1,1] \longrightarrow \mathbb{C}$ be piecewise continuous functions. The sequence $(\boldsymbol{A}_n^\varphi)$ of matrices considered as operators $\boldsymbol{A}_n^\varphi : \ell_n^2 \longrightarrow \ell_n^2$ is stable if and only if both operators $aI \pm bS : \boldsymbol{L}_\sigma^2 \longrightarrow \boldsymbol{L}_\sigma^2$ are invertible.*

Let us mention that the operators $\boldsymbol{A}_n^\varphi$ (as well as the operators $\boldsymbol{A}_n^\sigma$) are strongly convergent (see [13]), and consequently, their norms are uniformly bounded. Thus, Corollary 2.2 implies that stability of the collocation method w.r.t. the Chebyshev nodes of second kind is necessary and sufficient for the boundedness of the sequence of the spectral condition numbers of $\boldsymbol{A}_n^\varphi$.

In order to get an analogous result in case of $\omega = \sigma$, let us denote by $\boldsymbol{U}_n^\omega = \left[\ \tau_{jk}\ \right]_{j,k=1}^n$ the matrix of the basis transformation (in $\operatorname{im} L_n$) from $\{\widetilde{u}_0, \ldots, \widetilde{u}_{n-1}\}$ to $\{\psi_{\omega,n}\widetilde{\ell}_{n1}^\omega, \ldots, \psi_{\omega,n}\widetilde{\ell}_{nn}^\omega\}$, where $\psi_{\sigma,n} = \sqrt{n/\pi}$. Moreover, let $T_n(x)$ denote the normalized Chebyshev polynomials of first kind and of degree n,

$$T_0(x) = \sqrt{\frac{1}{\pi}}\,, \quad T_n(\cos s) = \sqrt{\frac{2}{\pi}}\cos ns\,, \quad n = 1, 2, \ldots\ .$$

In the case $\omega = \varphi$, we have $\tau_{jk} = \langle \psi_{\varphi,n}\widetilde{\ell}_{nk}^\varphi, \widetilde{u}_{j-1}\rangle_\sigma$. Thus,

$$\tau_{jk} = \sqrt{\frac{n+1}{\pi}}\,\frac{1}{\varphi(x_{nk}^\varphi)}\left\langle \ell_{nk}^\varphi, U_{j-1}\right\rangle_\varphi = \sqrt{\frac{2}{n+1}}\,\sin\frac{jk\pi}{n+1}\,, \tag{2.4}$$

i.e., $\boldsymbol{U}_n^\varphi = \sqrt{\frac{2}{n+1}}\,\boldsymbol{S}_n^1$, where $\boldsymbol{S}_n^1 = \left[\ \sin\frac{jk\pi}{n+1}\ \right]_{j,k=1}^n$ is the first discrete sine transform (DST-1) of order n (cf. [7]). Of course, due to the fact, that both bases are orthonormal ones, the matrix $\boldsymbol{U}_n^\varphi$ is unitary. Using the relation

$$(1-x^2)U_{n-1}(x) = \frac{1}{2}\left[T_{n-1}(x) - T_{n+1}(x)\right]\,, \quad n = 2, 3, \ldots\,,$$

in case $\omega = \sigma$ we get

$$\begin{aligned}
\tau_{jk} &= \left\langle \psi_{\sigma,n}\widetilde{\ell}_{nk}^\sigma, \widetilde{u}_{j-1}\right\rangle_\sigma = \sqrt{\frac{n}{\pi}}\,\frac{1}{\varphi(x_{nk}^\sigma)}\left\langle \ell_{nk}^\sigma, \varphi^2 U_{j-1}\right\rangle_\sigma \\
&= \begin{cases} \sqrt{\frac{\pi}{n}}\,\varphi(x_{nk}^\sigma)U_{j-1}(x_{nk}^\sigma) & : \quad j = 1, 2, \ldots, n-1\,, \\ \sqrt{\frac{\pi}{n}}\,\frac{1}{\varphi(x_{nk}^\sigma)}\,\frac{1}{2}\,T_{n-1}(x_{nk}^\sigma) & : \quad j = n\,, \end{cases}
\end{aligned}$$

$$= \begin{cases} \sqrt{\frac{2}{n}}\,\sin\frac{j(2k-1)\pi}{2n} & : \quad j = 1, 2, \ldots, n-1\,, \\ \frac{1}{2}\sqrt{\frac{2}{n}}\,\sin\frac{n(2k-1)\pi}{2n} & : \quad j = n\,. \end{cases} \tag{2.5}$$

Hence, $\boldsymbol{U}_n^\sigma = \sqrt{\frac{2}{n}}\,\boldsymbol{D}_n \boldsymbol{S}_n^2$, where $\boldsymbol{S}_n^2 = \left[\ \sin\frac{j(2k-1)\pi}{2n}\ \right]_{j,k=1}^n$ is the second discrete sine transform (DST-2) of order n and $\boldsymbol{D}_n = \operatorname{diag}\left[1,\ldots,1,\frac{1}{2}\right] \in \mathbb{R}^{n\times n}$. Since, for $j = 1,\ldots,n$,

$$\widetilde{u}_{j-1}(x) = \frac{\psi_{\sigma,n}}{\psi_{\sigma,n}}\sum_{k=1}^n \widetilde{u}_{j-1}(x_{nk}^\sigma)\widetilde{\ell}_{nk}^\sigma(x) = \sqrt{\frac{2}{n}}\sum_{k=1}^n \sin\frac{j(2k-1)\pi}{2n}\,\psi_{\sigma,n}\widetilde{\ell}_{nk}^\sigma(x)\,,$$

the inverse matrix of $\boldsymbol{U}_n^\sigma$ is given by

$$(\boldsymbol{U}_n^\sigma)^{-1} = \left[\ \sqrt{\frac{2}{n}}\sin\frac{j(2k-1)\pi}{2n}\ \right]_{k,j=1}^n = \sqrt{\frac{2}{n}}\left(\boldsymbol{S}_n^2\right)^T = (\boldsymbol{U}_n^\sigma)^T \boldsymbol{D}_n^{-1}\,.$$

The matrix representation $\boldsymbol{A}_n^\sigma$ of the operator $A_n^\sigma : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$ with respect to the basis $\left\{\psi_{\sigma,n}\widetilde{\ell}_{nk}^\sigma : k = 1,\ldots,n\right\}$ is equal to $\boldsymbol{A}_n^\sigma = \left[\ \alpha_{jk}^\sigma\ \right]_{j,k=1}^n$ with $\alpha_{jk}^\sigma = \left(A\widetilde{\ell}_{nk}^\sigma\right)(x_{nj}^\sigma)$. If $\boldsymbol{A}_n$ denotes the matrix representation of $A_n : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$ with respect to the orthonormal basis $\{\widetilde{u}_0,\ldots,\widetilde{u}_{n-1}\}$, then

$$\boldsymbol{A}_n^\sigma = (\boldsymbol{U}_n^\sigma)^T D_n^{-1}\boldsymbol{A}_n\boldsymbol{U}_n^\sigma\,. \tag{2.6}$$

If, for some nonnegative constants c_n and d_n,

$$c_n\|\xi_n\|_{\ell_n^2} \le \|\boldsymbol{A}_n\xi_n\|_{\ell_n^2} \le d_n\|\xi_n\|_{\ell_n^2} \qquad \forall\,\xi_n \in \ell_n^2\,,$$

then, by means of (2.6), one can easily prove that

$$\frac{c_n}{2}\|\xi_n\|_{\ell_n^2} \le \|\boldsymbol{A}_n^\sigma\xi_n\|_{\ell_n^2} \le 2\,d_n\|\xi_n\|_{\ell_n^2} \qquad \forall\,\xi_n \in \ell_n^2\,.$$

Consequently, we have the following corollary of Theorem 2.1.

Corollary 2.3. *Let $a, b : [-1,1] \longrightarrow \mathbb{C}$ be piecewise continuous functions. The sequence $(\boldsymbol{A}_n^\sigma)$ of matrices considered as operators $\boldsymbol{A}_n^\sigma : \ell_n^2 \longrightarrow \ell_n^2$ is stable if and only if the operator $aI + bS : \boldsymbol{L}_\sigma^2 \longrightarrow \boldsymbol{L}_\sigma^2$ is invertible.*

3. The Structure of the Discretized Equations

With the help of well known relations between the Chebyshev polynomials of first and second kind and the Cauchy singular integral operator S one can show that $\boldsymbol{A}_n^\omega$ can be written in the form (see [10, Section 5])

$$\boldsymbol{A}_n^\omega = \boldsymbol{a}_n^\omega + \boldsymbol{b}_n^\omega\boldsymbol{S}_n\boldsymbol{D}_n(\beta)\,,$$

where

$$\boldsymbol{a}_n^\sigma = \operatorname{diag}\left[a(x_{nk}^\omega) - \frac{x_{nk}^\sigma b(x_{nk}^\sigma)}{n\,\varphi(x_{nk}^\sigma)} \right]_{k=1}^n , \quad \boldsymbol{a}_n^\varphi = \operatorname{diag}\left[\, a(x_{nk}^\varphi)\,\right]_{k=1}^n ,$$

$$\boldsymbol{b}_n^\omega = \operatorname{diag}\left[\, b(x_{nk}^\omega)\,\right]_{k=1}^n, \ \boldsymbol{S}_n = \left[\frac{\alpha_k^\omega - \alpha_j^\omega}{x_{nk}^\omega - x_{nj}^\omega} \right]_{j,k=1}^n , \ \boldsymbol{D}_n(\beta) = \operatorname{diag}\left[\, \beta_k^\omega \,\right]_{k=1}^n,$$

and

$$\alpha_k^\sigma = (-1)^k \varphi(x_{nk}^\sigma) , \quad \beta_k^\sigma = \frac{(-1)^k}{n\mathrm{i}} , \quad \alpha_k^\varphi = (-1)^k , \quad \beta_k^\varphi = \frac{(-1)^k \varphi(x_{nk}^\varphi)}{(n+1)\mathrm{i}} .$$

Since matrices of the form $\boldsymbol{S}_n$ can be represented with the help of fast discrete trigonometric transforms, in [10] the authors propose to apply iterative methods to solve the systems $\boldsymbol{A}_n^\omega \xi_n = \eta_n$ of linear equations effectively. This means the following. Let

$$\boldsymbol{C}_n^\omega = \left[\frac{1}{x_{nk}^\omega - x_{nj}^\omega} \right]_{j,k=1}^n \qquad \text{(zeros on the main diagonal).}$$

Then $\boldsymbol{S}_n = \boldsymbol{C}_n^\omega \boldsymbol{D}_n(\alpha) - \boldsymbol{D}_n(\alpha)\boldsymbol{C}_n^\omega$ and ([7, Theorems 5.2, 5.5])

$$\boldsymbol{C}_n^\omega = \begin{cases} -\dfrac{1}{n}\, \boldsymbol{D}_n^\sigma \boldsymbol{C}_n^3 \boldsymbol{\Gamma}_n^\sigma \boldsymbol{C}_n^2 \boldsymbol{D}_n^\sigma & : \ \omega = \sigma , \\[2ex] -\dfrac{1}{n+1} \boldsymbol{D}_n^\varphi \boldsymbol{S}_n^1 \boldsymbol{\Gamma}_n^\varphi \boldsymbol{S}_n^1 \boldsymbol{D}_n^\varphi & : \ \omega = \varphi , \end{cases}$$

where

$$\boldsymbol{D}_n^\omega = \operatorname{diag}\left[\frac{(-1)^k}{\varphi(x_{nk}^\omega)} \right]_{k=1}^n , \quad \boldsymbol{\Gamma}_n^\omega = \operatorname{tridiag}\left[\, -\boldsymbol{c}_\omega \quad \boldsymbol{0} \quad \boldsymbol{c}_\omega \,\right] ,$$

$$\boldsymbol{c}_\sigma = \frac{1}{2}\left[\, 1 \quad 3 \quad \cdots \quad 2n-3 \,\right] , \quad \boldsymbol{c}_\varphi = \frac{1}{2}\left[\, 3 \quad 5 \quad \cdots \quad 2n-1 \,\right] ,$$

and $\boldsymbol{C}_n^2$ and $\boldsymbol{C}_n^3$ denote the second and third discrete cosine transform of order n, respectively,

$$\boldsymbol{C}_n^2 = \left[\cos \frac{(j-1)(2k-1)\pi}{2n} \right]_{j,k=1}^n , \quad \boldsymbol{C}_n^3 = \left(\boldsymbol{C}_n^2\right)^T .$$

Consequently, using these representations four trigonometric transforms are needed to apply the matrix $\boldsymbol{A}_n^\omega$ to a vector.

In what follows let us find other representations of the matrices $\boldsymbol{A}_n^\sigma$ and $\boldsymbol{A}_n^\varphi$, which allow the matrix vector multiplication with the help of only two trigonometric transforms. For this, recall the well known relation

$$S\widetilde{u}_m = S\varphi U_m = \mathrm{i}\, T_{m+1}\,, \quad m = 0, 1, 2, \dots\,,$$

and use (2.5) and (2.4) to compute

$$\begin{aligned}\left(S\widetilde{\ell}^\omega_{nk}\right)(x^\omega_{nj}) &= \mathrm{i}\sum_{m=1}^{n}\left\langle \widetilde{\ell}^\omega_{nk}, \widetilde{u}_{m-1}\right\rangle_\sigma T_m(x^\omega_{nj}) \\ &= \begin{cases} \dfrac{2\mathrm{i}}{n}\displaystyle\sum_{m=1}^{n-1}\cos\frac{m(2j-1)\pi}{2n}\,\sin\frac{m(2k-1)\pi}{2n} & : \ \omega = \sigma\,, \\ \dfrac{2\mathrm{i}}{n+1}\displaystyle\sum_{m=1}^{n}\cos\frac{mj\pi}{n+1}\,\sin\frac{mk\pi}{n+1} & : \ \omega = \varphi\,, \end{cases}\end{aligned}$$

where, in the case $\omega = \sigma$, we took into account that $T_n(x^\sigma_{nj}) = 0$. Hence we have

$$\text{(3.1)} \quad \boldsymbol{A}_n^\sigma = \widetilde{\boldsymbol{a}}_n^\sigma - \frac{2}{n\mathrm{i}}\,\boldsymbol{b}_n^\sigma \boldsymbol{C}_n^3 \boldsymbol{V}_n \boldsymbol{S}_n^2 \quad \text{and} \quad \boldsymbol{A}_n^\varphi = \boldsymbol{a}_n^\varphi - \frac{2}{(n+1)\mathrm{i}}\,\boldsymbol{b}_n^\varphi \widetilde{\boldsymbol{C}}^1_n \boldsymbol{S}_n^1$$

with $\boldsymbol{V}_n = \left[\,\delta_{j-1,k}\,\right]_{j,k=1}^n$, $\widetilde{\boldsymbol{a}}_n^\sigma = \operatorname{diag}\left[\,a(x^\sigma_{nk})\,\right]_{k=1}^n$, and a submatrix $\widetilde{\boldsymbol{C}}^1_n = \left[\,\cos\dfrac{jk\pi}{n+1}\,\right]_{j,k=1}^n$ of the first discrete cosine transform

$$\boldsymbol{C}^1_{n+2} = \left[\,\cos\frac{jk\pi}{n+1}\,\right]_{j,k=0}^{n+1}$$

of order $n+2$.

4. The Application of Krylov Subspace Methods

Given a regular matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and a vector $\eta \in \mathbb{C}^n$ we have to solve the system

$$\text{(4.1)} \qquad \boldsymbol{A}\xi = \eta\,.$$

The basic idea of the Krylov subspace methods consists in minimizing the residual $\eta - \boldsymbol{A}\xi$ for $\xi = \xi_0 + \zeta$, where

$$\zeta \in \mathbb{K}_k(\boldsymbol{A}, \rho_0) = \operatorname{span}\left\{\rho_0, \boldsymbol{A}\rho_0, \dots, \boldsymbol{A}^{k-1}\rho_0\right\}$$

and $\rho_0 = \eta - \boldsymbol{A}\xi_0$ (see, for example, [16]). Remark that in our situation the matrices $\boldsymbol{A}_n^\omega$ are not symmetric as well as not sparse. But, due to the representations (3.1) these matrices have a nice structure which enables us to apply them to a vector of length n with $O(n \log n)$ computational complexity (cf. [19]).

In the present paper we compare the numerical results (i.e., the number of iteration steps necessary to obtain a prescribed accuracy) for three Krylov subspace methods (GMRES, FOM, and CGNR) described shortly in the following:

• The **GMRES** (**g**eneralized **m**inimal **r**esidual) algorithm minimizes the norm $\|\eta - \boldsymbol{A}\xi\|$ ($\|.\| = \|.\|_{\ell^2_n}$), which is equivalent to finding $\xi_k = \xi_0 + \zeta_k$, $\zeta_k \in \mathbb{K}_k(\boldsymbol{A}, \rho_0)$, such that

$$\eta - \boldsymbol{A}\xi_k \perp \boldsymbol{A}\mathbb{K}_k(\boldsymbol{A}, \rho_0)\,.$$

Using Arnoldi's procedure with the modified Gram-Schmidt algorithm to compute an orthonormal basis in $\mathbb{K}_k(\boldsymbol{A}, \rho_0)$ leads to the following algorithm, in which $\varepsilon > 0$ defines the prescribed accuracy:

1° $\xi_0\,,\quad \rho_0 := \eta - \boldsymbol{A}\,\xi_0\,,\quad \boldsymbol{v}_1 := \|\rho_0\|^{-1}\rho_0\,,\quad k := 0$

2° (Arnoldi's procedure with modified Gram-Schmidt algorithm)

- $k := k+1\,,\quad \widehat{\boldsymbol{v}} := \boldsymbol{A}\boldsymbol{v}_k$
- $i = 1, 2, \ldots, k:\quad h_{ik} := \langle \widehat{\boldsymbol{v}}, \boldsymbol{v}_i \rangle\,,\quad \widehat{\boldsymbol{v}} := \widehat{\boldsymbol{v}} - h_{ik}\boldsymbol{v}_i$
- $h_{k+1,k} := \|\widehat{\boldsymbol{v}}\|\,,\quad \boldsymbol{v}_{k+1} := h_{k+1,k}^{-1}\widehat{\boldsymbol{v}}$

3° $\xi_k := \xi_0 + \boldsymbol{V}_k\zeta_k$, where ζ_k is the solution of the minimization problem

$$\mathcal{J}(\zeta) := \|\gamma \boldsymbol{e}_1 - \widetilde{\boldsymbol{H}}_k\zeta\| \longrightarrow \min \quad (\zeta \in \mathbb{C}^k,\ \gamma = \|\rho_0\|) \tag{4.2}$$

with

$$\widetilde{\boldsymbol{H}}_k = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1k} \\ h_{21} & h_{22} & & h_{2k} \\ & \ddots & \ddots & \vdots \\ & & \ddots & h_{kk} \\ 0 & & & h_{k+1,k} \end{bmatrix}, \quad \boldsymbol{V}_k = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{bmatrix}$$

4° if $\|\boldsymbol{A}\xi_k - \eta\| > \varepsilon\|\rho_0\|$ goto 2.

To solve the least squares problem (4.2) one uses plane rotations to get a QR-factorization of $\widetilde{\boldsymbol{H}}_k$, say $\boldsymbol{Q}_k\widetilde{\boldsymbol{H}}_k = \boldsymbol{R}_k$, $\boldsymbol{Q}_k = \left(\boldsymbol{Q}_k^{-1}\right)^T \in \mathbb{C}^{(k+1)\times(k+1)}$. This factorization can be updated at each iteration step and the norm of the minimal residual is computed by $\|\boldsymbol{A}\xi_k - \eta\| = (\gamma\boldsymbol{Q}_k\boldsymbol{e}_1)_{k+1}$ (the last component of $\gamma\boldsymbol{Q}_k\boldsymbol{e}_1$ in the kth iteration step) without computing the optimal solution ξ_k (see [17]). If $n_{\boldsymbol{A}}$ is the number of multiplications, which are necessary to apply the matrix $\boldsymbol{A}$ (of order n) to a vector of length n, then m (= dimension of the Krylov space) steps of GMRES require $O(m^2n) + mn_{\boldsymbol{A}}$ multiplications (cf. [17]).

• The **FOM** (**f**ull **o**rthogonalization **m**ethod) determines $\xi_k = \xi_0 + \zeta_k$, $\zeta_k \in \mathbb{K}_k(\boldsymbol{A}, \rho_0)$, by the condition

$$\eta - \boldsymbol{A}\xi_k \perp \mathbb{K}_k(\boldsymbol{A}, \rho_0)\,.$$

The algorithm can be shortly described as follows:

1° $\xi_0\,,\quad \rho_0 := \eta - \boldsymbol{A}\,\xi_0\,,\quad \boldsymbol{v}_1 := \|\rho_0\|^{-1}\rho_0\,,\quad k := 0$

2° (Arnoldi's procedure with modified Gram-Schmidt algorithm)

– $k := k+1\,,\quad \widehat{\boldsymbol{v}} := \boldsymbol{A}\boldsymbol{v}_k$
– $i = 1, 2, \ldots, k:\quad h_{ik} := \langle \widehat{\boldsymbol{v}}, \boldsymbol{v}_i\rangle\,,\quad \widehat{\boldsymbol{v}} := \widehat{\boldsymbol{v}} - h_{ik}\boldsymbol{v}_i$
– $h_{k+1,k} := \|\widehat{\boldsymbol{v}}\|\,,\quad \boldsymbol{v}_{k+1} := h_{k+1,k}^{-1}\widehat{\boldsymbol{v}}$

3° $\xi_k := \xi_0 + \boldsymbol{V}_k\zeta_k$, where $\zeta_k = \gamma\boldsymbol{H}_k^{-1}\boldsymbol{e}_1$, $\gamma = \|\rho_0\|$, and

$$\boldsymbol{H}_k = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1k} \\ h_{21} & h_{22} & & h_{2k} \\ & \ddots & \ddots & \vdots \\ 0 & & \ddots & h_{kk} \end{bmatrix}, \quad \boldsymbol{V}_k = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{bmatrix}$$

4° if $\|\boldsymbol{A}\xi_k - \eta\| > \varepsilon\|\rho_0\|$ goto 2.

The residual norm after the kth step is equal to $\|\boldsymbol{A}\xi_k - \eta\| = h_{k+1,k}\left|\boldsymbol{e}_k^T\zeta_k\right|$. Here m steps of the algorithm need $O(m^2n + m^3) + m\,n_{\boldsymbol{A}}$ multiplications.

• The **CGNR** method (**c**onjugate **g**radient algorithm for the **n**ormal equation) is the application of the CG-algorithm to the system $\boldsymbol{A}^T\boldsymbol{A}\xi = \eta$, which corresponds to the least squares problem of minimizing the **r**esidual $\|\eta - \boldsymbol{A}\xi\|$. Consequently, ξ_k is determined by the condition

$$\boldsymbol{A}^T(\eta - \boldsymbol{A}\xi_k) \perp \mathbb{K}_k(\boldsymbol{A}^T\boldsymbol{A}, \boldsymbol{A}^T\rho_0)\,.$$

The algorithm can be organized in the following way (cf. [16, Sect. 8.3]):

1° $\xi_0\,,\quad \rho_0 := \eta - \boldsymbol{A}\xi_0\,,\ \ \zeta_0 := \boldsymbol{A}^T\rho_0\,,\quad \boldsymbol{p}_0 := \zeta_0\,,\,k := 0$

2° $\widehat{\boldsymbol{v}} := \boldsymbol{A}\boldsymbol{p}_k$

3° $\alpha := \|\zeta_k\|^2/\|\widehat{\boldsymbol{v}}\|^2$

4° $\xi_{k+1} := \xi_k + \alpha\boldsymbol{p}_k\,,\quad \rho_{k+1} := \rho_k - \alpha\,\widehat{\boldsymbol{v}}\,,$
if $\|\rho_{k+1}\| < \varepsilon\|\rho_0\|$ then stop

5° $\zeta_{k+1} := \boldsymbol{A}^T\rho_{k+1}$

6° $\beta := \|\zeta_{k+1}\|^2/\|\zeta_k\|^2\,,\quad \boldsymbol{p}_{k+1} := \zeta_{k+1} + \beta\,\boldsymbol{p}_k$

7° $k := k+1$ goto 2.

For m steps of CGNR one needs $O(m\,n)+m(n_{\boldsymbol{A}}+n_{\boldsymbol{A}^T})$ multiplications. If we denote by $\kappa = \operatorname{cond}(\boldsymbol{A})$ the spectral condition number of the matrix $\boldsymbol{A}$, i.e., $\operatorname{cond}(\boldsymbol{A}) = \|\boldsymbol{A}\|\|\boldsymbol{A}^{-1}\|$, where $\|\boldsymbol{A}\|$ denotes the norm of $\boldsymbol{A} : \ell^2_n \longrightarrow \ell^2_n$, then the ℓ^2_n-norm of the residual ρ_m after m steps of CGNR can be estimated by

$$\|\rho_m\| \le 2\left(\frac{\kappa-1}{\kappa+1}\right)^m \|\rho_0\|$$

(see [15, Section 2]). Thus, taking into account the representations (3.1), in the case of stable sequences $(\boldsymbol{A}^\omega_n)$ (cf. Corollaries 2.2 and 2.3), we get that the complexity of CGNR can be estimated by $O\Big((\log\varepsilon)(n\log n)\Big)$ if $\varepsilon > 0$ is the prescribed accuracy.

5. Numerical Examples

Let us consider three examples of Cauchy singular integral equations of the form (2.1) together with the collocation methods (2.3) and the Krylov subspace methods for the solution of the respective systems of linear equations considered in the previous section. In the first example both operators $aI + bS$ and $aI - bS$ are invertible in $\boldsymbol{L}^2_\sigma$, while in the second and third example only the operator $aI + bS$ is invertible. Due to the Corollaries 2.2 and 2.3 this means that in the last two examples the sequence of the condition numbers of the respective matrices are bounded only for collocation with respect to the Chebyshev nodes of first kind. In each example we choose ε equal to 10^{-10}. In the tables we collect the number of iterations (equal to the dimension of the Krylov space in the last step) necessary to achieve this accuracy by the different methods. For all computations the initial guess is equal to the vector $\xi_0 = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$.

Example 5.1. Let us consider an example with the piecewise continuous coefficients

$$a(x)=\left\{\begin{array}{ccc} 2 & : & x\le -0.5 \\ 3+x & : & x>-0.5 \end{array}\right\}, \quad b(x)=\mathrm{i}\left\{\begin{array}{ccc} x & : & x\le 0.5 \\ x-1 & : & x>0.5 \end{array}\right\}$$

and the right hand side

$$f(x)=\left\{\begin{array}{ccc} 1 & : & x\le 0.5 \\ x^2-1 & : & x>0.5 \end{array}\right\}.$$

Collocation w.r.t. x^{σ}_{nj} , $k=1,\dots,n$

	number of iterations		
n	GMRES	FOM	CGNR
512	24	24	26
1024	25	25	27
16384	26	26	28
32768	26	26	29
65536	26	26	30
131072	27	27	30

Collocation w.r.t. x^{φ}_{nj} , $k=1,\dots,n$

	number of iterations		
n	GMRES	FOM	CGNR
511	21	22	23
1023	22	22	23
16383	22	22	24
32767	22	22	24
65535	23	23	25
131071	23	23	25

We see that the number of iterations does not essentially increase with the size n of the matrices.

Example 5.2. In this example we have $a(x)=\sqrt{1-x}$, $b(x)=-\mathrm{i}\,x$, and $f(x)=|x|$.

Collocation w.r.t. x^{σ}_{nj} , $k=1,\dots,n$

	number of iterations		
n	GMRES	FOM	CGNR
512	73	74	26
1024	73	75	26
16384	74	76	27
32768	74	76	27
65536	75	76	27
131072	75	76	28

Collocation w.r.t. x^{φ}_{nj} , $k=1,\dots,n$

	number of iterations		
n	GMRES	FOM	CGNR
511	70	71	32
1023	70	71	32
16383	69	71	36
32767	69	71	37
65535	69	70	38
131071	69	70	39

Example 5.3. Let $a(x)=\sqrt{1.01-x^2}$, $b(x)=-\mathrm{i}\,x$, and $f(x)=|x|$.

Collocation w.r.t. x_{nj}^{σ}, $k = 1, \ldots, n$

	number of iterations		
n	GMRES	FOM	CGNR
512	56	57	8
1024	56	57	8
16384	56	57	9
32768	56	57	9
65536	56	57	9
131072	56	57	9

Collocation w.r.t. x_{nj}^{φ}, $k = 1, \ldots, n$

	number of iterations		
n	GMRES	FOM	CGNR
511	53	54	9
1023	53	53	9
16383	51	52	12
32767	51	52	12
65535	50	51	12
131071	50	51	13

We see that also in the case of increasing condition numbers the number of iterations behaves well. For this, one should take into consideration that, in case of the collocation points x_{nj}^{φ}, only one (Example 5.2) or two (Example 5.3) singular values of the matrices $\boldsymbol{A}_n^{\varphi}$ tend to zero and the others stay away from zero (see [13, Section 5]).

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Technical University of Chemnitz
Faculty of Mathematics
Department of Mathematics
D-09107 Chemnitz, Germany
e-mails: `peter.junghanns@mathematik.tu-chemnitz.de`
`karla.rost@mathematik.tu-chemnitz.de`