DURRMEYER-SCHURER TYPE OPERATORS

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Dedicated to Prof. G. Mastroianni for his 65th birthday

Abstract. Starting with the Schurer operators ([5]) some Durrmeyer type operators are constructed. A convergence theorem is established and some estimations for the rate of convergence are given.

1. Preliminaries

It is well known that the classical Bernstein operators $B_m : C([0,1]) \mapsto C([0,1])$ are defined for any $f \in C([0,1])$ by

$$ (B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f(k/m), $$

where

$$ p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, $$

are the Bernstein fundamental polynomials.

Starting with the operators (1.1), J. L. Durrmeyer (see [3]) introduced in 1967 the operators $D_m : L_1([0,1]) \mapsto C([0,1])$, defined by

$$ (D_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) \, dt. $$

Considering a given non-negative integer $p$, F. Schurer (see [5]) in 1962 introduced and studied the operators $\tilde{B}_{m,p} : C([0, 1 + p]) \mapsto C([0,1])$, defined

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by

\[(\tilde{B}_{m,p}f)(x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x)f(k/m),\]

where

\[(\tilde{p}_{m,k})(x) = \binom{m}{k} x^k (1-x)^{m+p-k},\]

are the Schurer fundamental polynomials.

In the present paper we modify the operators (1.4) in Durrmeyer sense see also G. G. Lorentz ([4]).

Actually, we replace \(f(k/m)\) by an integral mean of \(f(x)\) on \([0,1]\) as follows

\[(\tilde{D}_{m,p}f)(x) = (m + p + 1) \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \int_0^1 \tilde{p}_{m,k}(t)f(t)\,dt,\]

where \(f\) belongs to the space \(L_1([0,1])\).

The focus of the paper is to investigate the operators (1.6). Section 2 provided a convergence theorem for the sequence \(\{\tilde{D}_{m,p}f\}_{m \geq 1}\). In Section 3 we prove some results in connection with the rate of convergence for \(\tilde{D}_{m,p}f\) under different assumptions of the function \(f\).

2. Convergence Theorem for the Sequence \(\{\tilde{D}_{m,p}f\}_{m \geq 1}\)

We shall use the well known Bohman-Korovkin theorem (see [1]). In this sense, we need some auxiliary results.

Lemma 2.1. The Durrmeyer-Schurer operators (1.6) are linear and positive.

Proof. The assertion follows from the definition (1.6). □

Lemma 2.2. The operator (3) transform any polynomial of degree \(s \leq m + p\) into a polynomial of degree \(s\).

Proof. From Lemma 2.1 follows that is sufficient to prove the assertion for the test functions \(e_s(t) = t^s\), where \(s\) is a non-negative integer with the property \(s \leq m + p\).
Taking into account of (1.6), we get

\[
\int_0^1 \tilde{p}_{m,k}(t)t^s \, dt = \left( \frac{m+p}{k} \right) \int_0^1 t^{k+s}(1-t)^{m+p-k} \, dt \\
= \left( \frac{m+p}{k} \right) B(k+s+1, m+p-k+1).
\]

Note that in the right side of (2.1), \( B(k+s+1, m+p-k+1) \) denotes the Beta function, i.e.,

\[
B(k+s+1, m+p-k+1) = \frac{(k+s)!(m+p-k)!}{(m+p+s+1)!}.
\]

Using (2.1) and (2.2), we can write

\[
(\tilde{D}_{m,p}e_s)(x) = \left( \frac{m+p+1}{m+p+s+1} \right) \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \cdot \frac{(k+s)!}{k!}.
\]

On the other hand, for any \( x, y \in \mathbb{R} \) and any \( s, m, p \in \mathbb{N} \) satisfying the inequality \( s \leq m + p \) we have

\[
\frac{\partial^s}{\partial x^s} (x^s(x+y)^{m+p}) = \sum_{k=0}^{m+p} \left( \frac{m+p}{k} \right) x^k \cdot y^{m+p-k} \cdot \frac{(k+s)!}{k!}.
\]

Using the well-known Leibniz formula, the left side of (2.4) can be expressed in the form

\[
\frac{\partial^s}{\partial x^s} (x^s(x+y)^{m+p}) = \sum_{r=0}^{s} \binom{s}{r} \frac{d^{s-r}(x^s)}{dx^{s-r}} \left( (x+y)^{m+p} \right) \\
= \sum_{r=0}^{s} \binom{s}{r} \frac{s!}{r!} \frac{(m+p)!}{(m+p-r)!} x^r (x+y)^{m+p-r}.
\]

From (2.4) and (2.5), with \( y := 1 - x \), follows the identity

\[
\sum_{r=0}^{s} \binom{s}{r} \frac{s!}{r!} \frac{(m+p)!}{(m+p-r)!} x^r = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \frac{(k+s)!}{k!}.
\]

Taking into account of (2.6), from (2.3) we get

\[
(\tilde{D}_{m,p}e_s)(x) = \frac{(m+p+1)!}{(m+p+s+1)!} \sum_{r=0}^{s} \binom{s}{r} \frac{s!}{r!} \frac{(m+p)!}{(m+p-r)!} x^r.
\]
which proves that $\tilde{D}_{m,p}e_s$ is a polynomial of degree $s \leq m + p$. □

Like usually, for any integer $s \geq 0$ we denote by $e_s(x) = x^s$, $x \in [0,1+p]$ the test functions.

**Lemma 2.3.** The Durrmeyer-Schurer operators (1.6) verify

\[
(\tilde{D}_{m,p}e_0)(x) = 1,
\]
\[
(\tilde{D}_{m,p}e_1)(x) = \frac{(m + p)x + 1}{m + p},
\]
\[
(\tilde{D}_{m,p}e_2)(x) = \frac{(m + p - 1)(m + p)x^2 + 4(m + p)x + 2}{(m + p + 2)(m + p + 3)}.
\]

**Proof.** The assertions follows from (2.7), for $s \in \{0,1,2\}$. □

**Theorem 2.1.** The sequence $\{\tilde{D}_{m,p}f\}_{m \geq 1}$ converges to $f$, uniformly on $[0,1]$, for any $f \in C([0,1])$.

**Proof.** Using Lemma 2.3 follows that $\lim_{m \to \infty} \tilde{D}_{m,p}e_s = e_s$ uniformly on $[0,1]$ for $s \in \{0,1,2\}$. Applying then the well known Bohman-Korovkin theorem (see [1]), we arrive to the desired result. □

3. **Estimations of the Rate of Convergence in Terms of First Order Modulus of Smoothness**

We shall use the first order modulus of smoothness $\omega_1 : [0, +\infty) \mapsto \mathbb{R}$, defined for any real functions $f$, bounded on the interval $I \subset \mathbb{R}$, by

\[
\omega_1(f; \delta) = \sup \left\{ |f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta \right\}.
\]

It is well known (see [6]) the following result, obtained by O. Shisha and B. Mond in 1968.

**Theorem 3.1.** Let $L : C(I) \mapsto B(I)$ be a linear and positive operator and let $\varphi_x$ be the function defined by $\varphi_x(t) = |t - x|$, $(x,t) \in I \times I$.

(i) If $f \in C_B(I)$, for any $x \in I$ and any $\delta > 0$, the operator $L$ verify

\[
|(Lf)(x) - f(x)| \leq |f(x)||\langle L_0 \rangle(x) - 1|
\]
\[
+ \left\{ \langle L_0 \rangle(x) + \delta^{-1} \sqrt{\langle L_0 \rangle(x) \langle L_\varphi^2 \rangle(x)} \right\} \omega_1(f; \delta).
\]
(ii) If \( f \in \mathcal{C}^1_B(I) \), for any \( x \in I \) and any \( \delta > 0 \), the operator \( L \) verifies
\[
| (Lf)(x) - f(x) | \leq |f(x)|| (Le_0)(x) - 1| + |f'(x)|| (Le_1)(x) - x(Le_0)(x) |
+ \sqrt{(L\varphi^2_x)(x)} \left\{ \sqrt{(Le_0)(x)} + \delta^{-1}\sqrt{(L\varphi^2_x)(x)} \right\} \omega_1(f'; \delta).
\]

For applying Theorem 3.1, we need

**Lemma 3.1.** The operator (1.6) verifies
\[
\tilde{D}_{m,p}\varphi^2_x(x) = \frac{2(m + p - 3)x(1 - x) + 2}{(m + p + 2)(m + p + 3)},
\]
where \( \varphi_x(t) = |t - x|, \ (t, x) \in [0, 1 + p] \times [0, 1 + p] \).

**Proof.** Because \( \tilde{D}_{m,p} \) is linear, it follows
\[
\tilde{D}_{m,p}\varphi^2_x(x) = (\tilde{D}_{m,p}\varphi_2)(x) - 2x(\tilde{D}_{m,p}\varphi_1)(x) + x^2(\tilde{D}_{m,p}\varphi_0)(x).
\]

Next, we apply Lemma 2.3.  

**Theorem 3.2.** For any \( f \in \mathcal{C}([0, 1 + p]), \ any \ x \in [0, 1] \) and any non-negative integers \( m, p \) satisfying \( m + p \geq 3 \), the Durrmeyer-Schurer operators (1.6) verify
\[
\left| (\tilde{D}_{m,p}f)(x) - f(x) \right| \leq 2\omega_1(f; \delta(m, p)),
\]
where
\[
\delta(m, p) = \sqrt{\frac{m + p + 1}{2(m + p + 2)(m + p + 3)}}.
\]

**Proof.** Taking into account of Theorem 3.1, we get
\[
\left| (\tilde{D}_{m,p}f)(x) - f(x) \right| \leq \left( 1 + \frac{1}{\delta} \sqrt{\frac{2(m + p - 3)x(1 - x) + 2}{(m + p + 2)(m + p + 3)}} \right) \omega_1(f'; \delta)
\]
for any \( f \in \mathcal{C}([0, 1 + p]), \ any \ x \in [0, 1], \ any \ non-negative \ integers \ m, p \)
satisfying \( m + p \geq 3 \) and any \( \delta > 0 \).

But, if \( m + p \geq 3 \) and \( x \in [0, 1] \), the inequality
\[
2(m + p - 3)x(1 - x) + 2 \leq \frac{m + p + 1}{2}
\]
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holds. From (3.5) and (3.6) it follows

\[ |(\tilde{D}_{m,p}f)(x) - f(x)| \leq \left( 1 + \frac{1}{\delta} \sqrt{\frac{m + p + 1}{2(m + p + 2)(m + p + 3)}} \right) \omega_1(f; \delta). \]

Taking \( \delta = \delta(m, p) \) (defined in (3.4)), this inequality becomes (3.3). \( \square \)

An extension of Theorem 3.2 is the following result:

**Theorem 3.3.** For any \( f \in L_1([0, 1 + p]) \), any \( x \in [0, 1] \) and any non-negative integers \( m, p \) satisfying \( m + p \geq 3 \) the inequality (3.3) holds.

**Proof.** Applying Lemma 2.3 (the identity (2.8)), it follows

\[ (\tilde{D}_{m,p}f)(x) - f(x) \leq (m + p + 1) \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \int_0^1 \tilde{p}_{m,k}(t) |f(t) - f(x)| dt. \quad (3.7) \]

On the other hand \( |f(t) - f(x)| \leq \omega_1(f; |t - x|) \leq (1 + \delta^{-2} |t - x|^2) \omega_1(f; \delta). \)

For \( |t - x| < \delta \) the last increase is clear. For \( |t - x| \geq \delta \) we use the following properties

\( \omega_1(f; \lambda \delta) \leq (1 + \lambda) \omega_1(f; \delta) \leq (1 + \lambda^2) \omega_1(f; \delta), \)

where we choose \( \lambda = \delta^{-1} |t - x| \). In this way, the relation (3.7) implies

\[ |(\tilde{D}_{m,p}f)(x) - f(x)| \]

\[ \leq (m + p + 1) \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \int_0^1 \tilde{p}_{m,k}(t) (1 + \delta^{-2} |x - t|^2) \omega_1(f; \delta) \]

\[ \leq (m + p + 1) \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \int_0^1 \tilde{p}_{m,k}(t) (1 + \delta^{-2} |x - t|^2) \omega_1(f; \delta) dt \]

\[ = \left\{ (\tilde{D}_{m,p}e_0)(x) + \delta^{-2} (\tilde{D}_{m,p}f^2)(x) \right\} \omega_1(f; \delta). \]

Taking into account of Lemma 2.3 and Lemma 3.1, from (3.8) it follows

\[ |(\tilde{D}_{m,p}f)(x) - f(x)| \leq \left( 1 + \frac{2(m + p - 3) x (1 - x) + 2}{(m + p + 2)(m + p + 3)} \right) \omega_1(f; \delta). \]

Using (3.6), this inequality becomes

\[ |(\tilde{D}_{m,p}f)(x) - f(x)| \leq \left( 1 + \frac{m + p + 1}{2(m + p + 2)(m + p + 3)} \right) \omega_1(f; \delta), \]
wherefrom, choosing $\delta = \delta(m,p)$ (as in Theorem 3.2), we get the desired result. \qed

Further, we estimate the rate of convergence for smooth functions.

**Theorem 3.4.** For any $f \in C^1([0,1+p])$, any $x \in [0,1]$ and any non negative integers satisfying $m + p \geq 3$, the operator (1.6) verify

$$
(3.9) \quad |(\tilde{D}_{m,p}f)(x) - f(x)| \leq \frac{|1-2x|}{m+p+2} |f'(x)| + 2\delta(m,p) \omega_1(f';\delta(m,p)),
$$

where $\delta(m,p)$ is given by (3.4).

**Proof.** Lemma 2.3 (the equality (2.9)) leads us to

$$
(\tilde{D}_{m,p}e_1)(x) = \frac{1-2x}{m+p+2}.
$$

Let $\delta(m,p)$ be given by (3.4). Applying Theorem 3.1 we arrive to

$$
|(\tilde{D}_{m,p}f)(x)| \leq \frac{|1-2x|}{m+p+2} |f'(x)| + \delta(m,p) \left(1 + \frac{\delta(m,p)}{\delta}\right) \omega_1(f';\delta(m,p)),
$$

i.e., (3.9), if we put $\delta = \delta(m,p)$. \qed

**Corollary 3.1.** Under conditions of Theorem 3.4, the following inequality

$$
|(\tilde{D}_{m,p}f)(x,y) - f(x)| \leq \frac{1}{m+p+2} \|f'\| + \sqrt{2} \delta(m,p) \omega_1(f';\delta(m,p)).
$$

holds, for any $x \in [0,1]$.

**Proof.** We apply Theorem 3.4, taking into account that $|1-2x| \leq 1$, for each $x \in [0,1]$, and $|f'(x)| \leq \max_{x \in [0,1+p]} |f'(x)|$, for each $x \in [0,1+p]$. \qed

**Corollary 3.2.** For any $f \in C^1([0,1])$, any $x \in [0,1]$ and any non-negative integer $m \geq 3$, the Durrmeyer operators (1.3) satisfy

$$
|(D_m f)(x) - f(x)| \leq \frac{|1-2x|}{m+2} |f'(x)| + 2\delta_m \omega_1(f';\delta_m)
$$

$$
\leq \frac{1}{m+2} \|f'\| + 2\delta_m \omega_1(f';\delta_m),
$$

where $\delta_m = \sqrt{(m+1)/(2(m+2)(m+3))}$.

**Proof.** The assertion follows from Theorem 3.4, for $p = 0$. \qed
REFERENCES


