FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 19 (2004), 65–72

DURRMEYER-SCHURER TYPE OPERATORS

Dan Bărbosu

Dedicated to Prof. G. Mastroianni for his 65th birthday

Abstract. Starting with the Schurer operators ([5]) some Durrmeyer type operators are constructed. A convergence theorem is established and some estimations for the rate of convergence are given.

1. Preliminaries

It is well known that the classical Bernstein operators $B_m : C([0,1]) \mapsto C([0,1])$ are defined for any $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f(k/m), \tag{1.1}$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}\,, \tag{1.2}$$

are the Bernstein fundamental polynomials.

Starting with the operators (1.1), J. L. Durrmeyer (see [3]) introduced in 1967 the operators $D_m : L_1([0,1]) \mapsto C([0,1])$, defined by

$$(D_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t)\, dt. \tag{1.3}$$

Considering a given non-negative integer p, F. Schurer (see [5]) in 1962 introduced and studied the operators $\widetilde{B}_{m,p} : C([0,1+p]) \mapsto C([0,1])$, defined

Received April 2, 2003.
2000 *Mathematics Subject Classification.* Primary 41A36; Secondary 41A25, 41A63.

by

$$(\widetilde{B}_{m,p}f)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f(k/m), \tag{1.4}$$

where

$$(\widetilde{p}_{m,k})(x) = \binom{m}{k} x^k (1-x)^{m+p-k}, \tag{1.5}$$

are the Schurer fundamental polynomials.

In the present paper we modify the operators (1.4) in Durrmeyer sense see also G. G. Lorentz ([4]).

Actually, we replace $f(k/m)$ by an integral mean of $f(x)$ on $[0,1]$ as follows

$$(\widetilde{D}_{m,p}f)(x) = (m+p+1) \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) \int_0^1 \widetilde{p}_{m,k}(t) f(t)\, dt, \tag{1.6}$$

where f belongs to the space $L_1([0,1])$.

The focus of the paper is to investigate the operators (1.6). Section 2 provided a convergence theorem for the sequence $\left\{\widetilde{D}_{m,p}f\right\}_{m\geq 1}$. In Section 3 we prove some results in connection with the rate of convergence for $\widetilde{D}_{m,p}f$ under different assumptions of the function f.

2. Convergence Theorem for the Sequence $\{\widetilde{D}_{m,p}f\}_{m\geq 1}$

We shall use the well known Bohman-Korovkin theorem (see [1]). In this sense, we need some auxiliary results.

Lemma 2.1. *The Durrmeyer-Schurer operators* (1.6) *are linear and positive.*

Proof. The assertion follows from the definition (1.6). □

Lemma 2.2. *The operator* (3) *transform any polynomial of degree* $s \leq m+p$ *into a polynomial of degree* s.

Proof. From Lemma 2.1 follows that is sufficient to prove the assertion for the test functions $e_s(t) = t^s$, where s is a non-negative integer with the property $s \leq m+p$.

Taking into account of (1.6), we get

$$\int_0^1 \widetilde{p}_{m,k}(t)t^s dt = \binom{m+p}{k}\int_0^1 t^{k+s}(1-t)^{m+p-k}\,dt = \binom{m+p}{k}B(k+s+1,\ m+p-k+1). \tag{2.1}$$

Note that in the right side of (2.1), $B(k+s+1,\ m+p-k+1)$ denotes the Beta function, i.e.,

$$B(k+s+1,\ m+p-k+1) = \frac{(k+s)!(m+p-k)!}{(m+p+s+1)!}\,. \tag{2.2}$$

Using (2.1) and (2.2), we can write

$$(\widetilde{D}_{m,p}e_s)(x) = \frac{(m+p+1)!}{(m+p+s+1)!}\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\cdot\frac{(k+s)!}{k!}\,. \tag{2.3}$$

On the other hand, for any $x, y \in \mathbb{R}$ and any $s, m, p \in \mathbb{N}$ satisfying the inequality $s \le m+p$ we have

$$\frac{\partial^s}{\partial x^s}\left(x^s(x+y)^{m+p}\right) = \sum_{k=0}^{m+p}\binom{m+p}{k}x^k\cdot y^{m+p-k}\cdot\frac{(k+s)!}{k!}\,. \tag{2.4}$$

Using the well-known Leibniz formula, the left side of (2.4) can be expressed in the form

$$\frac{\partial^s}{\partial x^s}\left(x^s(x+y)^{m+p}\right) = \sum_{r=0}^{s}\binom{s}{r}\frac{d^{s-r}(x^s)}{dx^{s-r}}\ \frac{d^r}{dx^r}\left((x+y)^{m+p}\right) = \sum_{r=0}^{s}\binom{s}{r}\frac{s!}{r!}\frac{(m+p)!}{(m+p-r)!}x^r(x+y)^{m+p-r}\,. \tag{2.5}$$

From (2.4) and (2.5), with $y := 1-x$, follows the identity

$$\sum_{r=0}^{s}\binom{s}{r}\frac{s!}{r!}\cdot\frac{(m+p)!}{(m+p-r)!}\ x^r = \sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\frac{(k+s)!}{k!}\,. \tag{2.6}$$

Taking into account of (2.6), from (2.3) we get

$$(\widetilde{D}_{m,p}e_s)(x) = \frac{(m+p+1)!}{(m+p+s+1)!}\sum_{r=0}^{s}\binom{s}{r}\frac{s!}{r!}\frac{(m+p)!}{(m+p-r)!}\ x^r, \tag{2.7}$$

which proves that $\widetilde{D}_{m,p}e_s$ is a polynomial of degree $s \le m+p$. $\square$

Like usually, for any integer $s \ge 0$ we denote by $e_s(x) = x^s$, $x \in [0, 1+p]$ the test functions.

Lemma 2.3. *The Durrmeyer-Schurer operators* (1.6) *verify*

$$(\widetilde{D}_{m,p}e_0)(x) = 1, \tag{2.8}$$

$$(\widetilde{D}_{m,p}e_1)(x) = \frac{(m+p)x+1}{m+p}, \tag{2.9}$$

$$(\widetilde{D}_{m,p}e_2)(x) = \frac{(m+p-1)(m+p)x^2 + 4(m+p)x + 2}{(m+p+2)(m+p+3)}. \tag{2.10}$$

Proof. The assertions follows from (2.7), for $s \in \{0, 1, 2\}$. $\square$

Theorem 2.1. *The sequence $\{\widetilde{D}_{m,p}f\}_{m\ge 1}$ converges to f, uniformly on $[0,1]$, for any $f \in C([0,1])$.*

Proof. Using Lemma 2.3 follows that $\lim\limits_{m\to\infty} \widetilde{D}_{m,p}e_s = e_s$ uniformly on $[0,1]$ for $s \in \{0,1,2\}$. Applying then the well known Bohman-Korovkin theorem (see [1]), we arrive to the desired result. $\square$

3. Estimations of the Rate of Convergence in Terms of First Order Modulus of Smoothness

We shall use the first order modulus of smoothness $\omega_1 : [0, +\infty) \mapsto \mathbb{R}$, defined for any real functions f, bounded on the interval $I \subset \mathbb{R}$, by

$$\omega_1(f;\delta) = \sup\Big\{|f(x') - f(x'')| : x', x'' \in I,\ |x' - x''| \le \delta\Big\}. \tag{3.1}$$

It is well known (see [6]) the following result, obtained by O. Shisha and B. Mond in 1968.

Theorem 3.1. *Let $L : C(I) \mapsto B(I)$ be a linear and positive operator and let φ_x be the function defined by $\varphi_x(t) = |t - x|$, $(x,t) \in I \times I$.*

(i) *If $f \in C_B(I)$, for any $x \in I$ and any $\delta > 0$, the operator L verify*

$$\begin{aligned}|(Lf)(x) - f(x)| \le{}& |f(x)||(Le_0)(x) - 1| \\ &+ \Big\{(Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\Big\}\,\omega_1(f;\delta).\end{aligned}$$

(ii) *If $f \in C_B^1(I)$, for any $x \in I$ and any $\delta > 0$, the operator L verifies*

$$\begin{aligned} |(Lf)(x) - f(x)| \leq\; & |f(x)||(Le_0)(x) - 1| + |f'(x)||(Le_1)(x) - x(Le_0)(x)| \\ & + \sqrt{(L\varphi_x^2)(x)}\left\{\sqrt{(Le_0)(x)} + \delta^{-1}\sqrt{(L\varphi_x^2)(x)}\right\}\omega_1(f';\delta). \end{aligned}$$

For applying Theorem 3.1, we need

Lemma 3.1. *The operator* (1.6) *verifies*

$$(\widetilde{D}_{m,p}\varphi_x^2)(x) = \frac{2(m+p-3)x(1-x)+2}{(m+p+2)(m+p+3)}\,, \tag{3.2}$$

where $\varphi_x(t) = |t-x|,\ (t,x) \in [0,1+p] \times [0,1+p]$.

Proof. Because $\widetilde{D}_{m,p}$ is linear, it follows

$$(\widetilde{D}_{m,p}\varphi_x^2)(x) = (\widetilde{D}_{m,p}e_2)(x) - 2x(\widetilde{D}_{m,p}e_1)(x) + x^2(\widetilde{D}_{m,p}e_0)(x).$$

Next, we apply Lemma 2.3. □

Theorem 3.2. *For any $f \in C([0,1+p])$, any $x \in [0,1]$ and any non-negative integers m,p satisfying $m+p \geq 3$, the Durrmeyer-Schurer operators* (1.6) *verify*

$$\left|(\widetilde{D}_{m,p}f)(x) - f(x)\right| \leq 2\omega_1(f;\delta(m,p))\,, \tag{3.3}$$

where

$$\delta(m,p) = \sqrt{\frac{m+p+1}{2(m+p+2)(m+p+3)}}\,. \tag{3.4}$$

Proof. Taking into account of Theorem 3.1, we get

$$\left|(\widetilde{D}_{m,p}f)(x) - f(x)\right| \leq \left(1 + \frac{1}{\delta}\sqrt{\frac{2(m+p-3)x(1-x)+2}{(m+p+2)(m+p+3)}}\right)\omega_1(f;\delta) \tag{3.5}$$

for any $f \in C([0,1+p])$, any $x \in [0,1]$, any non-negative integers m,p satisfying $m+p \geq 3$ and any $\delta > 0$.

But, if $m+p \geq 3$ and $x \in [0,1]$, the inequality

$$2(m+p-3)x(1-x)+2 \leq \frac{m+p+1}{2} \tag{3.6}$$

holds. From (3.5) and (3.6) it follows

$$\left|(\widetilde{D}_{m,p}f)(x)-f(x)\right| \le \left(1+\frac{1}{\delta}\sqrt{\frac{m+p+1}{2(m+p+2)(m+p+3)}}\right)\omega_1(f;\delta).$$

Taking $\delta=\delta(m,p)$ (defined in (3.4)), this inequality becomes (3.3). $\square$

An extension of Theorem 3.2 is the following result:

Theorem 3.3. *For any $f\in L_1([0,1+p])$, any $x\in[0,1]$ and any non-negative integers m,p satisfying $m+p\ge 3$ the inequality* (3.3) *holds.*

Proof. Applying Lemma 2.3 (the identity (2.8)), it follows

$$\left|(\widetilde{D}_{m,p}f)(x)-f(x)\right| \le (m+p+1)\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\int_0^1 \widetilde{p}_{m,k}(t)|f(t)-f(x)|dt. \tag{3.7}$$

On the other hand $|f(t)-f(x)|\le \omega_1(f;|t-x|)\le \left(1+\delta^{-2}(t-x)^2\right)\omega_1(f;\delta)$. For $|t-x|<\delta$ the last increase is clear. For $|t-x|\ge\delta$ we use the following properties

$$\omega_1(f;\lambda\delta)\le(1+\lambda)\omega_1(f;\delta)\le(1+\lambda^2)\omega_1(f;\delta),$$

where we choose $\lambda=\delta^{-1}|t-x|$. In this way, the relation (3.7) implies

$$\begin{aligned}
&\left|(\widetilde{D}_{m,p}f)(x)-f(x)\right| \\
&\le (m+p+1)\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\int_0^1 \widetilde{p}_{m,k}(t)\left(1+\delta^{-2}(x-t)^2\right)\omega_1(f;\delta) \\
&\le (m+p+1)\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\int_0^1 \widetilde{p}_{m,k}(t)\left(1+\delta^{-2}(x-t)^2\right)\omega_1(f;\delta)dt \\
&= \left\{\left(\widetilde{D}_{m,p}e_0\right)(x)+\delta^{-2}\left(\widetilde{D}_{m,p}\varphi_x^2\right)(x)\right\}\omega_1(f;\delta).
\end{aligned}\tag{3.8}$$

Taking into account of Lemma 2.3 and Lemma 3.1, from (3.8) it follows

$$\left|(\widetilde{D}_{m,p}f)(x)-f(x)\right| \le \left(1+\frac{2(m+p-3)x(1-x)+2}{(m+p+2)(m+p+3)}\frac{1}{\delta^2}\right)\omega_1(f;\delta).$$

Using (3.6), this inequality becomes

$$\left|(\widetilde{D}_{m,p}f)(x)-f(x)\right| \le \left(1+\frac{m+p+1}{2(m+p+2)(m+p+3)}\frac{1}{\delta^2}\right)\omega_1(f;\delta),$$

wherefrom, choosing $\delta = \delta(m,p)$ (as in Theorem 3.2), we get the desired result. $\square$

Further, we estimate the rate of convergence for smooth functions.

Theorem 3.4. *For any $f \in C^1([0,1+p])$, any $x \in [0,1]$ and any non negative integers satisfying $m+p \geq 3$, the operator* (1.6) *verify*

$$|(\widetilde{D}_{m,p}f)(x) - f(x)| \leq \frac{|1-2x|}{m+p+2}|f'(x)| + 2\delta(m,p)\,\omega_1\big(f';\delta(m,p)\big)\,, \tag{3.9}$$

where $\delta(m,p)$ is given by (3.4).

Proof. Lemma 2.3 (the equality (2.9)) leads us to

$$(\widetilde{D}_{m,p}e_1)(x) = \frac{1-2x}{m+p+2}\,.$$

Let $\delta(m,p)$ be given by (3.4). Applying Theorem 3.1 we arrive to

$$|(\widetilde{D}_{m,p}f)(x)| \leq \frac{|1-2x|}{m+p+2}\,|f'(x)| + \delta(m,p)\left(1 + \frac{\delta(m,p)}{\delta}\right)\omega_1\left(f';\delta(m,p)\right),$$

i.e., (3.9), if we put $\delta = \delta(m,p)$. $\square$

Corollary 3.1. *Under conditions of Theorem* 3.4*, the following inequality*

$$|(\widetilde{D}_{m,p}f)(x,y) - f(x)| \leq \frac{1}{m+p+2}\|f'\| + \sqrt{2}\,\delta(m,p)\,\omega_1\left(f';\delta(m,p)\right).$$

holds, for any $x \in [0,1]$.

Proof. We apply Theorem 3.4, taking into account that $|1-2x| \leq 1$, for each $x \in [0,1]$, and $|f'(x)| \leq \max\limits_{x\in[0,1+p]} |f'(x)|$, for each $x \in [0,1+p]$. $\square$

Corollary 3.2. *For any $f \in C^1([0,1])$, any $x \in [0,1]$ and any non-negative integer $m \geq 3$, the Durrmeyer operators* (1.3) *satisfy*

$$\begin{aligned} |(D_m f)(x) - f(x)| &\leq \frac{|1-2x|}{m+2}\,|f'(x)| + 2\delta_m\,\omega_1\left(f';\delta_m\right) \\ &\leq \frac{1}{m+2}\,\|f'\| + 2\delta_m\,\omega_1\left(f';\delta_m\right), \end{aligned}$$

where $\delta_m = \sqrt{(m+1)/(2(m+2)(m+3))}$.

Proof. The assertion follows from Theorem 3.4, for $p = 0$. $\square$

REFERENCES

1. O. Agratini: *Aproximare prin operatori liniari.* Presa universitară Clujeană, Cluj-Napoca 2000 (in Romanian).
2. F. Altomare and M. Campiti: *Korovkin-type Approximation Theory and its Applications.* de Gruyter Series Studies in Mathematics, vol. 17, Walter de Gruyter & Co., Berlin, New York, 1994.
3. J.L. Durrmeyer: *Une formule d'inversion de la transformée de Laplace: Application à la theorie des moments.* Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.
4. G.G. Lorentz: *Bernstein Polynomials.* Toronto, Univ. of Toronto Press, 1953.
5. F. Schurer: *Linear positive operators in approximation theory.* Math. Inst. Tech. Univ Delft Report, 1962.
6. O. Shisha and B. Mond: *The degree of convergence of linear positive operators.* Proc. Nat. Acad. Sci. USA **60** (1968), 1196–1200.
7. D.D. Stancu, Gh. Coman, O. Agratini, R. Trâmbiţaş: *Analiză numerică şi teoria aproximării.* vol. I. Presa Universitară Clujeană, Cluj-Napoca, 2001 (in Romanian).
8. D.D. Stancu: *Curs şi culegere de probleme de analiză numerică.* Vol. 1, Univ. "Babeş-Bolyai" Cluj-Napoca, Facultatea de Matematică, Cluj-Napoca, 1977 (in Romanian).

North University of Baia Mare
Faculty of Science
Department of Mathematics and Computer Science
Str. Victoriei 76, 4800 Baia Mare, ROMANIA

e-mail: dbarbosu@ubm.ro; danbarbosu@yahoo.com