FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 19 (2004), 37–51

ENHANCED ASYMPTOTIC APPROXIMATION BY LINEAR APPROXIMATION OPERATORS

Ulrich Abel and Biancamaria Della Vecchia

Dedicated to Prof. G. Mastroianni for his 65th birthday

Abstract. The concern of this paper is to continue the investigation of local convergence properties of linear approximation operators published by Kirov and Popova. Given a sequence of linear operators L_n new operators $L_{n,r}$ can be constructed by application of L_n to the r-th partial sum of the Taylor series of the approximated function. In the first part of the paper we derive the complete asymptotic expansion for the operators $L_{n,r}$ as n tends to infinity, provided that the underlying operators L_n possess such a property. As an application we obtain the complete asymptotic expansions for the enhanced variant of some special approximation operators such as Bernstein and Bernstein-Durrmeyer operators. In the second part we study the operators which arise by replacing the derivatives in the Taylor series by certain differences of the function.

1. Introduction

In his paper [17] Kirov introduced, for functions $f \in C^r[0,1]$ $(r = 0, 1, 2, \ldots)$, the polynomials

$$(1.1)\quad B_{n,r}(f;x) = \sum_{\nu=0}^{n} \sum_{j=0}^{r} \frac{1}{j!} f^{(j)}\left(\frac{\nu}{n}\right)\left(x - \frac{\nu}{n}\right)^j \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \quad (n \in \mathbb{N}).$$

For $r = 0$, they coincide with the classical Bernstein polynomials B_n. For $r \geq 1$, in contrast with the last ones, they are sensitive to the degree of smoothness of the function f as approximations to f.

Received November 22, 2004.

2000 *Mathematics Subject Classification.* Primary 41A36; Secondary 41A25, 41A28, 41A60.

Kirov proved for the operators (1.1) natural generalizations of the classical theorems of Popoviciu and Voronovskaja. The latter result asserts that, for $f \in C^{r+2}[0,1]$ and each $x \in [0,1]$, there holds the asymptotic relation ([17, Theorem 2])

$$\begin{aligned}(1.2)\quad B_{n,r}(f;x) &= f(x) + (-1)^r T_{n,r+1}(x)\frac{f^{(r+1)}(x)}{(r+1)!\, n^{r+1}} \\ &\quad +(-1)^r T_{n,r+2}(x)\frac{(r+1)f^{(r+2)}(x)}{(r+2)!\, n^{r+2}} + o(n^{r/2+1})\end{aligned}$$

as n tends to infinity, where

$$T_{n,s}(x) = \sum_{\nu=0}^{n} (\nu - nx)^s \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

(cf. [14, Eq. (1.2), p. 303]).

In the subsequent paper [18] Kirov and Popova associated, in a more general setting, to each linear operator $L_n : C[a,b] \to C[a,b]$ a new operator $L_{n,r}$ of r-th order ($r = 0, 1, 2, \ldots$) defined by

$$(1.3)\qquad L_{n,r}(f;x) = L_n(P_{x,r}f;x),$$

where $P_{x,r}f$ is the r-th Taylor polynomial

$$(1.4)\qquad P_{x,r}(f;t) = \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j$$

of the function $f \in C^r[a,b]$ in a neighbourhood of the point $t \in [a,b]$. For $r = 0$, we have $L_{n,0} \equiv L_n$. Kirov and Popova studied the properties of the operators (1.3) and proved a Korovkin-type theorem.

Instead of $[a,b]$ we could, of course, consider an arbitrary finite or infinite interval I, where in the latter case the most operators L_n require that f is bounded on I or that f satisfies a certain growth condition.

The operators (1.1) appear as a special case of the operators (1.3) if $L_n \equiv B_n$ are the Bernstein polynomials and $I = [0,1]$.

The purpose of this paper is the investigation of the asymptotic behaviour of sequences $L_{n,r}$ of operators (1.3) originating from approximation properties of the operators L_n as n tends to infinity.

Throughout the paper let $(\varphi_k)_{k=1}^{+\infty}$ be a sequence of functions defined on $\mathbb{N}$, such that for each $k \in \mathbb{N}$

$$\begin{aligned} \lim_{n\to+\infty} \varphi_k(n) &= 0 \qquad \text{and} \\ \varphi_{k+1}(n) &= o(\varphi_k(n)) \qquad (n \to +\infty). \end{aligned}$$

We consider operators L_n satisfying an asymptotic relation and derive for the corresponding operators $L_{n,r}$ a complete asymptotic expansion of the form

$$L_{n,r}(f;x) \sim f(x) + \sum_{k=1}^{+\infty} \varphi_k(n)\, c_k^{[r]}(f;x) \qquad (n \to +\infty) \tag{1.5}$$

with certain coefficients $c_k^{[r]}(f;x) \quad (k = 1, 2, \ldots)$ independent of n. Formula (1.5) means that, for all $q \in \mathbb{N}$,

$$L_{n,r}(f;x) = f(x) + \sum_{k=1}^{q} \varphi_k(n)\, c_k^{[r]}(f;x) + o(n^{-q})$$

as $n \to +\infty$.

In particular, we obtain a complete asymptotic expansion for the operators $B_{n,r}$ in the form

$$B_{n,r}(f;x) \sim f(x) + \sum_{k=1}^{+\infty} \frac{c_k(f;x)}{n^k} \qquad (n \to +\infty), \tag{1.6}$$

provided that f is bounded in $[0,1]$ and possesses derivatives of sufficiently high order at x. For all coefficients $c_k(f;x) \quad (k = 1, 2, \ldots)$ we determine explicit expressions.

The asymptotic relation (1.6) gives much more insight in the asymptotic behaviour of the operators (1.1) than Eq. (1.2).

Since the operators $L_{n,r}$ have the lack that they require the existence of all derivatives $f', f'', \ldots, f^{(r)}$ on the whole interval $[a,b]$, we establish operators similar to $L_{n,r}$ which work without any derivative of f. We construct such operators by replacing the derivatives $f^{(j)}$ in the Taylor polynomial $P_{x,r}$ by suitable differences of the function f and prove that they possess the same properties as the $L_{n,r}$ concerning asymptotic approximation in the most common case $\varphi_k(n) = n^{-k}$ $(k = 1, 2, \ldots)$.

We shall make use of the *Stirling numbers* of the first and second kind, denoted by S_m^k and σ_m^k, respectively. Recall that the Stirling numbers are defined by the equations

$$(1.7)\quad x^{\underline{m}} = \sum_{k=0}^{m} S_m^k \, x^k \quad \text{resp.} \quad x^m = \sum_{k=0}^{m} \sigma_m^k \, x^{\underline{k}} \qquad (m = 0, 1, \ldots).$$

We mention that there are several results on asymptotic expansions of special approximation operators such as Bernstein-Kantorovich operators [4], the operators of Balázs and Szabados [7], the Meyer-König and Zeller operators [1, 3], the operators of Butzer, Bleimann and Hahn [2, 5], and the Gamma operators [12]. The Jakimovski-Leviatan operators and their Kantorovich variant were studied by Abel and Ivan [9, 10]. Similar results on a certain positive linear operator can be found in [15, 11].

2. Asymptotic Approximation by Operators $L_{n,r}$

2.1. The result. As first main result we obtain the complete asymptotic expansion of the operators $L_{n,r}$, provided the operators L_n possess a complete asymptotic expansion. Throughout the paper we assume that the functions f under consideration admit derivatives of sufficiently high orders.

Theorem 2.1. *Let $q, r \in \mathbb{N}$. Suppose the linear operators $L_n : C[a,b] \to C[a,b]$ satisfy, for $x \in [a,b]$, an asymptotic expansion*

$$(2.1)\quad L_n(f;x) = f(x) + \sum_{k=1}^{q} \varphi_k(n) \sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) f^{(\ell)}(x) + o(\varphi_q(n)) \quad (n \to +\infty)$$

with integers $L_k \geq \ell_k \geq 1$ and certain values $g_{k,\ell}(x)$ independent of n. Then, the operators $L_{n,r}$, as defined in Eq. (1.3), possess the asymptotic expansion

$$(2.2)\quad L_{n,r}(f;x) = f(x) + (-1)^r \sum_{k=1}^{q} \varphi_k(n) \sum_{\ell=\max\{\ell_k, r+1\}}^{L_k} \binom{\ell-1}{r} g_{k,\ell}(x) f^{(\ell)}(x) + o(\varphi_q(n))$$

as $n \to +\infty$.

Several known linear approximation operators such as Bernstein polynomials, Durrmeyer operators, Kantorovich polynomials, Baskakov operators, Szász-Mirakjan operators and many others, satisfy an asymptotic expansion of the form (2.1) with the special sequence

$$L_k = 2k \qquad (k \in \mathbb{N}).$$

For such operators, we have the following

Corollary 2.1. *Under the assumptions of Theorem* 2.1 *and the additional condition* $L_k = 2k$ $(k \in \mathbb{N})$, *there holds*

$$\text{(2.3)}\quad L_{n,r}(f;x) = f(x) + (-1)^r \sum_{k=\lfloor r/2\rfloor+1}^{q} \varphi_k(n) \times$$
$$\times \sum_{\ell=\max\{\ell_k, r+1\}}^{2k} \binom{\ell-1}{r} g_{k,\ell}(x) f^{(\ell)}(x) + o(\varphi_q(n)) \quad (n \to +\infty).$$

Remark 2.1. In Eq. (2.3) we use the convention that a sum is to be read as 0 if the lower index is greater than the upper index. Note that in the case $r \le 2q-1$, Corollary 2.1 states that

$$\text{(2.4)}\qquad L_{n,r}(f;x) = f(x) + O\left(\varphi_{\lfloor r/2\rfloor+1}(n)\right) \qquad (n \to +\infty).$$

Proof of Theorem 2.1. By definitions (1.3) and (1.4), there holds

$$L_{n,r}(f;x) = \sum_{j=0}^{r} \frac{1}{j!} L_n((x-t)^j f^{(j)}(t); x)$$

and assumption (2.1) yields

$$\begin{aligned} L_{n,r}(f;x) &= f(x) + \sum_{j=0}^{r} \frac{1}{j!} \sum_{k=1}^{q} \varphi_k(n) \sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) \\ &\times \left(\frac{d}{dt}\right)^{\ell} \left((x-t)^j f^{(j)}(t)\right)\Big|_{t=x} + o(\varphi_q(n)) \qquad (n \to +\infty). \end{aligned}$$

Using Leibniz rule, we obtain

$$\left(\frac{d}{dt}\right)^{\ell} \left((x-t)^j f^{(j)}(t)\right)\Big|_{t=x} = (-1)^j j! \binom{\ell}{j} f^{(\ell)}(x),$$

and therefore

$$L_{n,r}(f;x) = f(x) + \sum_{k=1}^{q} \varphi_k(n) \sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) f^{(\ell)}(x) \sum_{j=0}^{r} (-1)^j \binom{\ell}{j} + o(\varphi_q(n))$$

as $n \to +\infty$, so that Theorem 2.1 follows by the well-known identity

$$\sum_{j=0}^{r} (-1)^j \binom{\ell}{j} = (-1)^r \binom{\ell-1}{r} \qquad (\ell, r \in \mathbb{N}). \quad \square$$

2.2. Bernstein Polynomials. In order to illustrate Theorem 2.1 we apply it to the generalized Bernstein polynomials (1.1). As in many cases here we have that $\varphi_k(n) = n^{-k}$. The complete asymptotic expansion for the Bernstein polynomials

$$B_n(f;x) \sim f(x) + \sum_{k=1}^{+\infty} n^{-k} \sum_{\ell=k+1}^{2k} \frac{f^{(\ell)}(x)}{\ell!} \sum_{j=0}^{k} x^{\ell-j} \sum_{m=k}^{\ell} (-1)^{\ell-m} \binom{\ell}{m} S_{m-j}^{m-k} \sigma_m^{m-j}$$

as $n \to +\infty$ is known (cf. [2, Eq. (4) and Lemma 1]). Thus, the assumptions of Corollary 2.1 are satisfied with $\ell_k = k+1$, $L_k = 2k$ and

$$g_{k,\ell}(x) = \frac{1}{\ell!} \sum_{j=0}^{k} x^{\ell-j} \sum_{m=k}^{\ell} (-1)^{\ell-m} \binom{\ell}{m} S_{m-j}^{m-k} \sigma_m^{m-j}.$$

As usual, we put $X = x(1-x)$ (cf. [14, Theorem 1.1, p. 303]) and $X' = 1-2x$. Then, there holds

$$\begin{aligned}
B_{n,0}(f;x) &\equiv B_n(f;x) = f(x) + \frac{1}{2n} X f''(x) \\
&+ n^{-2} \left(\frac{1}{6} X X' f^{(3)}(x) + \frac{1}{8} X^2 f^{(4)}(x) \right) + o(n^{-2}), \\
B_{n,1}(f;x) &= f(x) - \frac{1}{2n} X f''(x) \\
&- n^{-2} \left(\frac{1}{3} X X' f^{(3)}(x) + \frac{3}{8} X^2 f^{(4)}(x) \right) + o(n^{-2}), \\
B_{n,2}(f;x) &= f(x) + n^{-2} \left(\frac{1}{6} X X' f^{(3)}(x) + \frac{3}{8} X^2 f^{(4)}(x) \right) \\
&+ n^{-3} \left(\frac{1}{8} X(1-6X) f^{(4)}(x) + \frac{1}{2} X^2 X' f^{(5)}(x) + \frac{5}{24} X^3 f^{(6)}(x) \right) \\
&+ o(n^{-3}), \\
B_{n,3}(f;x) &= f(x) - \frac{1}{8n^2} X^2 f^{(4)}(x) \\
&- n^{-3} \left(\frac{1}{24} X(1-6X) f^{(4)}(x) + \frac{1}{3} X^2 X' f^{(5)}(x) + \frac{5}{24} X^3 f^{(6)}(x) \right) \\
&+ o(n^{-3}).
\end{aligned}$$

2.3. Bernstein-Durrmeyer operators. As a further example we consider the Bernstein-Durrmeyer operators M_n. In this case we have $\varphi_k(n) = 1/(k!(n+2)^{\overline{k}})$, where $z^{\overline{k}}$ denotes the rising factorial $z^{\overline{k}} = z(z+1)\cdots(z+k-1)$,

$z^{\overline{0}} = 1$. The complete asymptotic expansion for the Bernstein-Durrmeyer operators M_n is given by the concise formula (see [6])

$$M_n(f;x) \sim f(x) + \sum_{k=1}^{+\infty} \frac{1}{k!(n+2)^{\overline{k}}} \left(X^k f^{(k)}(x)\right)^{(k)},$$

where we again put $X = x(1-x)$. Hence, the assumptions of Corollary 2.1 are satisfied with $\ell_k = k$, $L_k = 2k$ and

$$g_{k,\ell}(x) = \binom{k}{\ell - k} \left(X^k\right)^{(2k-\ell)}$$

Thus, we conclude that

$$M_{n,r}(f;x) \sim f(x) + \sum_{k=\lfloor r/2\rfloor+1}^{+\infty} \frac{(-1)^r}{k!(n+2)^{\overline{k}}} \sum_{\ell=0}^{k} \binom{\ell+k-1}{r}\binom{k}{\ell} \left(X^k\right)^{(k-\ell)} f^{(k+\ell)}(x).$$

Application of the Vandermonde convolution

$$\binom{\ell+k-1}{r} = \sum_{j=0}^{r} \binom{\ell}{j}\binom{k-1}{r-j}$$

and the identity $\binom{k}{\ell}\binom{\ell}{j} = \binom{k}{j}\binom{k-j}{\ell-j}$ yields

$$\begin{aligned}
&\sum_{\ell=0}^{k} \binom{\ell+k-1}{r}\binom{k}{\ell} \left(X^k\right)^{(k-\ell)} f^{(k+\ell)}(x) \\
&\qquad = \sum_{j=0}^{r} \binom{k-1}{r-j}\binom{k}{j} \sum_{\ell=0}^{k-j} \binom{k-j}{\ell} \left(X^k\right)^{(k-j-\ell)} f^{(k+j+\ell)}(x).
\end{aligned}$$

Finally, by Leibniz rule it follows

$$M_{n,r}(f;x) \sim f(x) + \sum_{k=\lfloor r/2\rfloor+1}^{+\infty} \frac{(-1)^r}{k!(n+2)^{\overline{k}}} \sum_{j=0}^{r} \binom{k-1}{r-j}\binom{k}{j} \left(X^k f^{(k+j)}(x)\right)^{(k-j)}.$$

As an immediate consequence we obtain the following Voronovskaja type result:

$$\begin{aligned}
&\lim_{n\to+\infty} n^r \left(M_{n,2r-1}(f;x) - f(x)\right) = -\frac{X^r}{r!} f^{(2r)}(x), \\
&\lim_{n\to+\infty} n^{r+1} \left(M_{n,2r}(f;x) - f(x)\right) \\
&\qquad = \frac{X^r}{(r+1)!} \left[(r+1)^2 X' f^{(2r+1)}(x) + (2r+1)\, X f^{(2r+2)}(x)\right].
\end{aligned}$$

2.4. Szász-Mirakjan-Durrmeyer operators. The last example is the Durrmeyer variant S_n of the Szász-Mirakjan operators defined by

$$(2.5)\ S_n(f;x) = ne^{-nx} \sum_{\nu=0}^{+\infty} \frac{(nx)^\nu}{\nu!} \int_0^{+\infty} e^{-nt} \frac{(nt)^\nu}{\nu!} f(t)\ dt \qquad (x \geq 0).$$

They are a special case by the more general Jakimovski-Leviatan-Durrmeyer operators studied in [8] and possess the complete asymptotic expansion

$$S_n(f;x) \sim f(x) + \sum_{k=1}^{+\infty} \frac{1}{k!n^k} \left(x^k f^{(k)}(x) \right)^{(k)} \qquad (n \to +\infty).$$

Hence, the assumptions of Corollary 2.1 are satisfied with $\varphi_k(n) = n^{-k}$, $\ell_k = k$, $L_k = 2k$ and

$$g_{k,\ell}(x) = \frac{1}{(\ell-k)!} \binom{k}{\ell-k} x^{\ell-k}.$$

Thus, we conclude that

$$S_{n,r}(f;x) \sim f(x) + \sum_{k=\lfloor r/2 \rfloor+1}^{+\infty} \frac{(-1)^r}{n^k} \sum_{\ell=0}^{k} \binom{\ell+k-1}{r} \binom{k}{\ell} \frac{x^\ell}{\ell!} f^{(k+\ell)}(x).$$

As in the preceding example we obtain

$$\sum_{\ell=0}^{k} \binom{\ell+k-1}{r} \binom{k}{\ell} \frac{x^\ell}{\ell!} f^{(k+\ell)}(x)$$
$$= \sum_{j=0}^{r} \binom{k-1}{r-j} \binom{k}{j} \sum_{\ell=0}^{k-j} \binom{k-j}{\ell} \frac{x^{j+\ell}}{(j+\ell)!} f^{(k+j+\ell)}(x).$$

Since $x^{j+\ell}/(j+\ell)! = \left(x^k\right)^{(k-j-\ell)}/k!$, the Leibniz rule implies

$$S_{n,r}(f;x) \sim f(x) + \sum_{k=\lfloor r/2 \rfloor+1}^{+\infty} \frac{(-1)^r}{k!n^k} \sum_{j=0}^{r} \binom{k-1}{r-j} \binom{k}{j} \left(x^k f^{(k+j)}(x) \right)^{(k-j)}.$$

As an immediate consequence we obtain the following Voronovskaja type result:

$$\lim_{n\to+\infty} n^r \left(S_{n,2r-1}(f;x) - f(x) \right) = -\frac{x^r}{r!} f^{(2r)}(x),$$

$$\lim_{n\to+\infty} n^{r+1} \left(S_{n,2r}(f;x) - f(x) \right)$$
$$= \frac{x^r}{(r+1)!} \left[(r+1)^2 f^{(2r+1)}(x) + (2r+1)\, x f^{(2r+2)}(x) \right].$$

3. Asymptotic approximation by operators $L_{n,r,\alpha}^{\Delta}$

For practical use the operators $L_{n,r}$ are not easy to handle, since they require the existence of all derivatives $f', f'', \ldots, f^{(r)}$. For certain functions f, their computation may demand great effort. Moreover, the operators $L_{n,r}$ have the lack, that the derivatives must exist on the whole interval $[a, b]$.

It would be desirable to establish operators similar to $L_{n,r}$ which improve the order of convergence (locally) even if f possesses only local smoothness properties, but work without any derivative of f. To overcome this difficulty we construct such operators by replacing the derivatives $f^{(j)}$ in the Taylor polynomial $P_{x,r}$ by suitable differences of the function f. It turns out that these new operators improve the degree of approximation in the same way as the operators $L_{n,r}$. In this section we consider only the most common case $\varphi_k(n) = n^{-k}$ $(k = 1, 2, \ldots)$.

We mention before some preliminaries. As usual, we define for a function f, and $h \in \mathbb{R}$ the (forward) differences $\Delta_h f(x) = f(x+h) - f(x)$ and differences of higher order $\Delta_h^m f(x) = \Delta_h(\Delta_h^{m-1} f(x))$. Furthermore, put, for each $h \in \mathbb{R}$, $\Delta_h^0 f(x) = f(x)$. Analogously to the Taylor polynomial $P_{x,r}$ in Eq. (1.4) we consider the truncated Newton series $P_{x,r}^{[h]}$ defined by

$$P_{x,r}^{[h]}(f;t) = \sum_{j=0}^{r} \frac{1}{j!} \left(\frac{x-t}{h}\right)^{\underline{j}} \Delta_h^j f(t). \tag{3.1}$$

Let $\alpha : [a, b] \to \mathbb{R}$ be a function with $\alpha(x) \neq 0$ $(x \in [a, b])$. For a given linear operator $L_n : C[a, b] \to C[a, b]$, we define the new operator $L_{n,r,\alpha}^{\Delta}$ of r-th order $(r = 0, 1, 2, \ldots)$ by

$$L_{n,r,\alpha}^{\Delta}(f;x) = L_n(P_{x,r}^{[\alpha(x)/n]} f; x). \tag{3.2}$$

or, in a more explicit form,

$$L_{n,r,\alpha}^{\Delta}(f;x) = \sum_{j=0}^{r} \frac{1}{j!} \sum_{\ell=0}^{j} S_j^{\ell} \left(\frac{n}{\alpha(x)}\right)^{\ell} L_n((x-t)^{\ell} \Delta_{\alpha(x)/n}^{j} f(t); x), \tag{3.3}$$

where we made use of Eq. (1.7).

However, there arises the problem, that the computation of the operators $L_{n,r,\alpha}^{\Delta}(f;x)$ may require the evaluation of $f(t)$ for some $t \notin [a, b]$. One way to overcome this difficulty is a continuous continuation of f to $\mathbb{R}$ by the definition $f(x) = f(a)$ $(x < a)$ and $f(x) = f(b)$ $(x > b)$, which we shall assume in the following.

Usually this does not influence the asymptotic properties of $L_{n,r,\alpha}^{\Delta}(f;x)$, since, for the most approximation operators, $L_n(f;x)$ is essentially given by values of $f(t)$ for arguments t close to x.

However, it can affect the good approximation properties of $L_{n,r,\alpha}^{\Delta}(f;x)$, for moderate values of n. Therefore, we choose the function α in a proper manner, for example,

$$\alpha(x) = \begin{cases} b-x & \text{if } a \le x \le (a+b)/2, \\ a-x & \text{if } (a+b)/2 < x \le b. \end{cases}$$

The operators $L_{n,r,\alpha}^{\Delta}$ improve the degree of approximation of the operators L_n in a completely analogous fashion as it do the operators $L_{n,r}$. To be precise, we have the following result.

Theorem 3.1. *Let $q,r \in \mathbb{N}$. Suppose the linear operators $L_n : C[a,b] \to C[a,b]$ satisfy, for $x \in (a,b)$, an asymptotic expansion*

$$(3.4)\quad L_n(f;x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) f^{(\ell)}(x) + o\left(n^{-q}\right) \quad (n \to +\infty)$$

with integers $L_k \ge \ell_k \ge 1$ and certain values $g_{k,\ell}(x)$ independent of n. Then, the operators $L_{n,r,\alpha}^{\Delta}$, as defined in Eq. (3.2)*, possess the asymptotic expansion*

$$(3.5)\quad L_{n,r,\alpha}^{\Delta}(f;x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{m=0}^{k-1} \alpha^m(x) \sum_{\ell=\ell_{k-m}}^{L_{k-m}} g_{k-m,\ell}(x) f^{(\ell+m)}(x) \times$$
$$\times \sum_{j=0}^{r} (-1)^j \binom{\ell}{j} \frac{j!}{(m+j)!} \sum_{\mu=0}^{r-j} S_{j+\mu}^{j} \sigma_{j+m}^{j+\mu} + o\left(n^{-q}\right) \quad (n \to +\infty).$$

As analogous result to Corollary 2.1 for the operators $L_{n,r}$ in the case $L_k = 2k$ $(k \in \mathbb{N})$ we have for the operators $L_{n,r,\alpha}^{\Delta}$ the following result.

Corollary 3.1. *Under the assumptions of Theorem* 3.1 *and the additional condition $L_k = 2k$ $(k \in \mathbb{N})$, there holds*

$$(3.6)\quad L_{n,r,\alpha}^{\Delta}(f;x) = f(x) + \sum_{k=\lfloor r/2 \rfloor+1}^{q} n^{-k} \sum_{m=0}^{k-1} \alpha^m(x) \sum_{\ell=\ell_{k-m}}^{L_{k-m}} g_{k-m,\ell}(x) f^{(\ell+m)}(x)$$
$$\times \sum_{j=0}^{r} (-1)^j \binom{\ell}{j} \frac{j!}{(m+j)!} \sum_{\mu=0}^{r-j} S_{j+\mu}^{j} \sigma_{j+m}^{j+\mu} + o\left(n^{-q}\right) \quad (n \to +\infty).$$

Remark 3.1. As in Remark 2.1 we have in the case $r \leq 2q-1$

$$L^{\Delta}_{n,r,\alpha}(f;x) = f(x) + O\left(n^{-(\lfloor r/2 \rfloor+1)}\right) \qquad (n \to +\infty). \tag{3.7}$$

For the proof of Theorem 3.1, we need some auxiliary results.

Lemma 3.1. *Let $m, q \in \mathbb{N}$ with $q \geq m$ and $x \in \mathbb{R}$. Furthermore, let $f \in C^q(I)$, where I is an interval containing x. Then, we have*

$$\Delta_h^m f(x) = m! \sum_{\nu=m}^{q} \frac{f^{(\nu)}(x)}{\nu!} \sigma_\nu^m h^\nu + o(h^q) \qquad (h \to 0).$$

Remark 3.2. The formula

$$\Delta_h^m f(x) = m! \sum_{\nu=m}^{+\infty} \frac{f^{(\nu)}(x)}{\nu!} \sigma_\nu^m h^\nu$$

for f analytic in $x \in \mathbb{C}$ is well-known. Since Lemma 3.1 seems not to appear in the literature, we sketch a short proof.

Proof of Lemma 3.1. By Taylor's formula, we have

$$f(t) = \sum_{\nu=0}^{q} \frac{f^{(\nu)}(x)}{\nu!} (t-x)^\nu + \frac{(t-x)^q}{q!} \left(f^{(q)}(\xi_t) - f^{(q)}(x) \right)$$

with ξ_t between x and t. Let $h \in \mathbb{R}$ be so small that $x + mh \in I$. Applying Δ_h^m on both sides of the latter formula and taking the limit $t \to x$ we obtain

$$\Delta_h^m f(x) = \sum_{\nu=0}^{q} \frac{f^{(\nu)}(x)}{\nu!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (jh)^\nu + R,$$

where for the remainder R there holds

$$\begin{aligned} |R| &= \left| \frac{h^q}{q!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^q \left(f^{(q)}(\xi_{x+jh}) - f^{(q)}(x+jh) \right) \right| \\ &\leq 2^m m^q \frac{|h|^q}{q!} \omega(f^{(q)}; mh) \end{aligned}$$

with the ordinary modulus of continuity ω. Lemma 3.1 now follows by the well-known formula

$$\sigma_\nu^m = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^\nu$$

(see, e.g., [16, Eq. (3), p. 176]). $\square$

Lemma 3.2. *Let $r \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $x \in \mathbb{R}$. Furthermore, let $f \in C^{r+q}(I)$, where I is an interval containing x. Then, the derivatives of the truncated Newton series $P_{x,r}^{[h]}$, as defined by Eq.* (3.1), *satisfy the asymptotic relation*

$$\begin{aligned}(3.8)\quad & \left[\left(\frac{d}{dt}\right)^{\ell} P_{x,r}^{[h]}(f;t)\right]_{t=x} \\ & = \sum_{k=0}^{q} h^k f^{(\ell+k)}(x) \sum_{j=0}^{r} (-1)^j \binom{\ell}{j} \frac{j!}{(k+j)!} \sum_{\mu=0}^{r-j} S_{j+\mu}^{j} \sigma_{j+k}^{j+\mu} + o(h^q),\end{aligned}$$

when $h \to 0$ $(\ell = 0, 1, 2, \ldots)$.

Remark 3.3. Note that, in the special case $\ell = 0$, we have, for each $h \in \mathbb{R}$, $h \neq 0$,

$$P_{x,r}^{[h]}(f;x) = f(x). \tag{3.9}$$

Proof of Lemma 3.2. Applying Leibniz rule to definition (3.1) yields

$$\left(\frac{d}{dt}\right)^{\ell} P_{x,r}^{[h]}(f;t) = \sum_{j=0}^{r} \frac{1}{j!} \sum_{\mu=0}^{\ell} \binom{\ell}{\mu} \left[\left(\frac{d}{dt}\right)^{\mu} \left(\frac{x-t}{h}\right)^{\underline{j}}\right] \Delta_h^j f^{(\ell-\mu)}(t).$$

Taking advantage of the identity

$$\left(\frac{x-t}{h}\right)^{\underline{j}} = \sum_{i=0}^{j} S_j^i \left(\frac{x-t}{h}\right)^i$$

we obtain

$$\left[\left(\frac{d}{dt}\right)^{\ell} P_{x,r}^{[h]}(f;t)\right]_{t=x} = \sum_{j=0}^{r} \frac{1}{j!} \sum_{\mu=0}^{j} (-1)^{\mu} \mu! \binom{\ell}{\mu} S_j^{\mu} h^{-\mu} \Delta_h^j f^{(\ell-\mu)}(x).$$

Now we insert the asymptotic expansion of Lemma 3.1 in order to get

$$\begin{aligned}& \left[\left(\frac{d}{dt}\right)^{\ell} P_{x,r}^{[h]}(f;t)\right]_{t=x} \\ & = \sum_{j=0}^{r} \sum_{\mu=0}^{j} (-1)^{\mu} \mu! \binom{\ell}{\mu} S_j^{\mu} h^{-\mu} \left[\sum_{\nu=j}^{r+q} \frac{f^{(\ell-\mu+\nu)}(x)}{\nu!} \sigma_{\nu}^{j} h^{\nu} + o(h^{r+q})\right] \\ & = \sum_{j=0}^{r} \sum_{\mu=0}^{j} (-1)^{j-\mu} \ell^{\underline{j-\mu}} S_j^{j-\mu} h^{-j+\mu} \sum_{\nu=0}^{r+q-j} \frac{f^{(\ell+\mu+\nu)}(x)}{(\nu+j)!} \sigma_{\nu+j}^{j} h^{\nu+j} + o(h^q)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{q} h^k \sum_{j=0}^{r} \sum_{\mu=0}^{j} [\mu \le k][k-\mu \le r+q-j](-1)^{j-\mu} \ell^{\underline{j-\mu}} S_j^{j-\mu} \sigma_{j+k-\mu}^{j} \\
&\qquad \times \frac{f^{(\ell+k)}(x)}{(k-\mu+j)!} + o(h^q) \\
&= \sum_{k=0}^{q} h^k \sum_{\mu=0}^{r} \sum_{j=0}^{r-\mu} (-1)^j \ell^{\underline{j}} S_{j+\mu}^{j} \sigma_{j+k}^{j+\mu} \frac{f^{(\ell+k)}(x)}{(k+j)!} + o(h^q)
\end{aligned}
$$

as $h \to 0$, which implies Lemma 3.2. $\square$

Proof of Theorem 3.1. By definition (3.2) and assumption (3.4), there holds

$$
\begin{aligned}
L_{n,r,\alpha}^{\Delta}(f;x) &= P_{x,r}^{[\alpha(x)/n]}(f;x) + \sum_{k=1}^{q} n^{-k} \sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) \\
&\qquad \times \left[\left(\frac{d}{dt}\right)^{\ell} P_{x,r}^{[\alpha(x)/n]}(f;t) \right]_{t=x} + o\left(n^{-q}\right) \quad (n \to +\infty)
\end{aligned}
$$

and, by Lemma 3.2 and Remark 3.3, we obtain

$$
\begin{aligned}
(3.10)\quad L_{n,r,\alpha}^{\Delta}(f;x) &= f(x) + \sum_{k=1}^{q} n^{-k} \sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) \sum_{m=0}^{q} \left(\frac{\alpha(x)}{n}\right)^{m} f^{(\ell+m)}(x) \\
&\times \sum_{j=0}^{r} (-1)^j \binom{\ell}{j} \frac{j!}{(m+j)!} \sum_{\mu=0}^{r-j} S_{j+\mu}^{j} \sigma_{j+m}^{j+\mu} + o\left(n^{-q}\right) \quad (n \to +\infty),
\end{aligned}
$$

which implies Theorem 3.1. $\square$

Proof of Corollary 3.1. In view of Eq. (3.4) we have to show that

$$
\begin{aligned}
(3.11)\qquad & \sum_{m=0}^{k-1} \alpha^m(x) \sum_{\ell=\ell_{k-m}}^{L_{k-m}} g_{k-m,\ell}(x) f^{(\ell+m)}(x) \times \\
&\qquad \times \sum_{j=0}^{r} (-1)^j \binom{\ell}{j} \frac{j!}{(m+j)!} \sum_{\mu=0}^{r-j} S_{j+\mu}^{j} \sigma_{j+m}^{j+\mu} = 0
\end{aligned}
$$

if $1 \le k \le \lfloor r/2 \rfloor$. For $m = 0$, the summand in Eq. (3.11) becomes

$$
\sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) f^{(\ell)}(x) \sum_{j=0}^{r} (-1)^j \binom{\ell}{j} = \sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) f^{(\ell)}(x) (-1)^r \binom{\ell-1}{r},
$$

which vanishes if $L_k = 2k \le r$.

For $m \ge 1$, the well-known "orthogonality"-relation for the Stirling-numbers (see, e.g., [16, p. 183, Eq. (2)]), implies

$$\sum_{\mu=0}^{r-j} S_{j+\mu}^{j} \sigma_{j+m}^{j+\mu} = 0$$

if $r - j \ge m$. Furthermore, $\binom{\ell}{j} = 0$ if $\ell \le L_{k-m} = 2(k-m) < j$. Thus, the summands in Eq. (3.11) vanish if $j \le r - m$ or $2(k-m) < j$. This is surely the case if $r - m \ge 2(k-m)$, i.e., if $2k \le r + m$. Therefore, in Eq. (3.11) only summands with $2k > r$ occur. This completes the proof of Corollary 3.1. □

REFERENCES

1. U. Abel: *The moments for the Meyer-König and Zeller operators.* J. Approx. Theory **82** (1995), 352–361.
2. U. Abel: *On the asymptotic approximation with operators of Bleimann, Butzer and Hahn.* Indag. Math. (N.S.) **7**(1) (1996), 1–9.
3. U. Abel: *The complete asymptotic expansion for Meyer-König and Zeller operators.* J. Math. Anal. Appl. **208** (1997), 109–119.
4. U. Abel: *Asymptotic approximation with Kantorovich polynomials.* Approx. Theory and Appl. **14:3** (1998), 106–116.
5. U. Abel: *On the asymptotic approximation with bivariate operators of Bleimann, Butzer and Hahn.* J. Approx. Theory **97** (1999), 181–198.
6. U. Abel: *Asymptotic approximation with Bernstein-Durrmeyer operators and their derivatives.* Approx. Theory and Appl. **16:2** (2000), 1–12.
7. U. Abel and B. Della Vecchia: *Asymptotic approximation by the operators of Balázs and Szabados.* Acta Sci. Math. (Szeged) **66** (2000), 137–145.
8. U. Abel, B. Della Vecchia and M. Ivan: *Simultaneous asymptotic approximation by Jakimovski-Leviatan-Durrmeyer operators and their linear combinations.* (submitted).
9. U. Abel and M. Ivan: *Asymptotic expansion of the Jakimovski-Leviatan operators and their derivatives.* In: Functions, Series, Operators (L. Leindler, F. Schipp, J. Szabados, eds.), Budapest, 2002, pp. 103–119.
10. U. Abel and M. Ivan: *The asymptotic expansion for approximation operators of Favard-Szász type.* Friedberger Hochschulschriften **2** (1999).
11. U. Abel and M. Ivan: *Asymptotic approximation with a sequence of positive linear operators.* J. Comput. Anal. Appl. **3** (2001), 331–341.

12. U. Abel and M. Ivan: *Asymptotic approximation of functions and their derivatives by Müller's Gamma operators.* Result. Math. **43** (2003), 1–12.

13. S. N. Bernstein: *Complément à l'article de E. Voronowskaja.* Dokl. Akad. Nauk USSR **4** (1932), 86–92.

14. R. A. DeVore and G. G. Lorentz: *Constructive Approximation.* Springer, Berlin – Heidelberg, 1993.

15. M. Ivan and I. Raşa: *A sequence of positive linear operators.* Rev. Anal. Numér. Théor. Approx. **24** (1–2) (1995), 159–164.

16. C. Jordan: *Calculus of Finite Differences.* Chelsea, New York, 1965.

17. G. H. Kirov: *A generalization of the Bernstein polynomials.* Math. Balkanica (N.S.) **6** (1992), 147–153.

18. G. H. Kirov and L. Popova: *A generalization of the linear positive operators.* Math. Balkanica (N.S.) **7** (1993), 149–162.

19. G. G. Lorentz: *Bernstein Polynomials*, University of Toronto Press, Toronto, 1953.

20. E. V. Voronovskaja: *Détermination de la forme asymptotique de l'approximation des fonctions par les polynômes de S. Bernstein.* Dokl. Akad. Nauk. SSSR, A (1932), 79–85.

Fachhochschule Giessen–Friedberg
University of Applied Sciences
Fachbereich MND
Wilhelm-Leuschner-Strasse 13
D-61169 Friedberg, Germany
e-mail: `Ulrich.Abel@mnd.fh-friedberg.de`

Dipartimento di Matematica
Università degli Studi di Roma "La Sapienza"
Piazzale Aldo Moro 2
00185 Roma, Italy
e-mail: `dellavecchia@iamna.iam.na.cnr.it`