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ON THE DEGREE OF APPROXIMATION

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Abstract. In the present paper we obtain the degree of approximation using the Euler's means, of functions belonging to Lip $(\psi(t), p)$ class. It is also proved that the order of approximation arrived at is best possible and is free from the means generating sequences.

1. Introduction and Results

Let f be periodic with period 2π, and integrable in the sense of Lebesgue. The Fourier series associated with f at the point x is given by

$$f(x) \sim \frac{1}{2}\, a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

Let

$$S_n(x) = \frac{1}{2}\, a_0 + \sum_{v=1}^{n} (a_v \cos vx + b_v \sin vx)$$

denote the n-th partial sum of the Fourier series (1.1).

A 2π periodic function $f(x)$ is said to belong to the class Lip $(\psi(t), p)$, $p > 1$, if

$$\mid f(x+t) - f(x) \mid \leq M(\psi(t) t^{-1/p}), \quad 0 < t < \pi,$$

where $\psi(t)$ is a positive increasing function and M is a positive number independent of x and t.

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Let $\{S_n\}$ be the sequence of partial sums of the given series $\sum_{n=0}^{+\infty} U_n$. Then, for $q > 0$, the Euler (E, q) means of $\{S_n\}$ are defined to be

$$W_n = (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k.$$

The series is said to be Euler (E, q) summable to S provided that the sequence $\{W_n\}$ converges to S as $n \to +\infty$.

We write

$$\phi(t) = \frac{f(x+t) + f(x-t) - 2f(x)}{2},$$

$$\sigma_n(f, x) = (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k,$$

$$S(t) = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin\Big(k + \frac{1}{2}\Big) t,$$

$$R(t) = \sin\left\{\frac{t}{2} + n \tan^{-1}\Big(\frac{\sin t}{q + \cos t}\Big)\right\}.$$

G. Alexits [1] proved the following theorem concerning the degree of approximation of a function $f \in \operatorname{Lip} \alpha$ by the (C, δ) means of its Fourier series.

Theorem A. *If a periodic function $f \in \operatorname{Lip} \alpha$, $0 < \alpha \leq 1$, then the degree of approximation of the (C, δ) means of its Fourier series for $0 < \alpha < \delta \leq 1$ is given by*

$$\max_{0 \leq x \leq 2\pi} | f(x) - \sigma_n^{\delta}(x) | = O\Big(\frac{1}{n^{\alpha}}\Big)$$

and for $0 < \alpha \leq \delta \leq 1$ is given by

$$\max_{0 \leq x \leq 2\pi} | f(x) - \sigma_n^{\delta}(x) | = O\Big(\frac{\log n}{n^{\alpha}}\Big),$$

where $\sigma_n^{\delta}(x)$ are the (C, δ) means of the partial sums of (1.1).

Later on Hölland, Sahney and Tzimbalario [2] extended Thereom A to functions belonging to $C^*[0, 2\pi]$, the class of 2π-periodic continuous functions on $[0, 2\pi]$, using Nörlund means of Fourier series. Their theorem is as follows:

Theorem B. *If $w(t)$ is the modulus of continuity of $f \in C^*[0, 2\pi]$ then the degree of approximation of f by the Nörlund means of the Fourier series for f is given by*

$$E_n = \max_{0 \le t \le 2\pi} \mid f(t) - T_n(t) \mid = O\left(\frac{1}{p_n} \sum_{k=1}^{n} \frac{P_k w\left(1/k\right)}{k}\right),$$

where T_n are the (N, p_n) means of Fourier series of f.

Hölland, Sahney and Tzimbalario [2] have shown that the Theorem B reduces to the Theorem A if we deal with Cesáro means of order δ and consider a function $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$.

H. H. Khan and A. Wafi [3] have given the answer to open problem (i) imposed by Hölland, Sahney and Tzimbalario [2] by using a more general operator (matrix means) of which (N, p_n) is a special case for the Fourier series. Their theorem may be stated as follows:

Theorem C. *If $\{\triangle\lambda_{n,k}\}_{k=0}^{n}$ is a non-negative and non-decreasing sequence with respect to k and if $w(t)$ is the modulus of continuity of $f \in C^*[0, 2\pi]$, then the degree of approximation of f by matrix means of the Fourier series of f is given by*

$$\max_{0 \le t \le 2\pi} \mid f(x) - \sigma_n(f, x) \mid = O\left(\sum_{k=1}^{n} \frac{\triangle\lambda_{n,n-k} w\left(1/k\right)}{k}\right),$$

where $\sigma_n(f, x)$ are the matrix means of the Fourier series (1.1).

In this paper we have considered the problem of determining the degree of approximation for yet another class, the so called $\mathrm{Lip}\,(\psi(t), p)$ class which does include the $\mathrm{Lip}\,\alpha$ class discussed by G. Alexits [1]. We use Euler's means instead of triangular means. It may be remarked that the order of approximation arrived at is best possible and is free from the means generating sequences. Our theorem is as follows:

Theorem 1.1. *If $f(x)$ is periodic with period 2π and belongs to the class $\mathrm{Lip}\,(\psi(t), p)$ for $p > 1$, and if*

$$\left\{\int_0^{1/\sqrt{n}} \left(\frac{\psi(t)}{t^{1/p}}\right)^p dt\right\}^{1/p} = O\left(\psi\left(\frac{1}{\sqrt{n}}\right)\right)$$

and

$$\left\{\int_{1/\sqrt{n}}^{\pi}\left(\frac{\psi(t)}{t^{1/p+2}}\right)^{p}dt\right\}^{1/p} = O\left(\psi\left(\frac{1}{\sqrt{n}}\right)n\right),$$

then

$$\max_{0\le x\le 2\pi} \mid f(x)-\sigma_n(f,x)\mid = O\left(\psi\left(\frac{1}{\sqrt{n}}\right)(n)^{1/2p}\right). \tag{1.2}$$

The inequality (1.2) *is best possible in the sense that there exists a positive constant* K *such that*

$$\max_{0\le x\le 2\pi} \mid f(x)-\sigma_n(f,x)\mid \ge K\left(\psi\left(\frac{1}{\sqrt{n}}\right)n^{1/2p}\right). \tag{1.3}$$

In order to prove the theorem we need the following lemma:

Lemma 1.1. *If* $0<t\le\pi$, *then*

$$(1+q)^{-n}(1+q^2+2q\cos t)^{n/2} \le e^{-2qt^2n/\{\pi(1+q)\}^2}, \quad 0<t\le\pi.$$

Proof. We have

$$\begin{aligned}(1+q)^{-2}(1+q^2+2q\cos t) &= 1-\frac{4q\sin^2(t/2)}{(1+q)^2}\\ &\le 1-\frac{4qt^2}{\pi^2(1+q)^2}\\ &\le e^{-4qt^2/\{\pi(1+q)\}^2},\end{aligned}$$

since $e^x(1-x)<1$ when $0<x<1$. Therefore

$$(1+q)^{-n}(1+q^2+2q\cos t)^{n/2} \le e^{-2qt^2n/\{\pi(1+q)\}^2},$$

which completes the proof of this lemma.

2. Proof of the Theorem

Since

$$S_k(x)-f(x) = \frac{1}{\pi}\int_0^{\pi}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}\phi(t)dt,$$

we get

$$
\begin{aligned}
&\left[(1+q)^{-n}\ \sum_{k=0}^{n}\binom{n}{k}q^{n-k}\Big\{S_k(x)-f(x)\Big\}\right] \\
(2.1)\qquad &= \frac{1}{\pi(1+q)^n}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{1}{2}t}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin\left(k+\frac{1}{2}\right)t\right\}dt \\
&= \frac{1}{\pi(1+q)^n}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{1}{2}t}S(t)\,dt \\
&= \frac{1}{\pi(1+q)^n}\left[\int_0^{1/\sqrt{n}}+\int_{1/\sqrt{n}}^{\pi}\right]\frac{\phi(t)}{\sin\frac{1}{2}t}S(t)dt \\
&= J_1(x)+J_2(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mid S(t)\mid}{(1+q)^n} &\leq \frac{1}{(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}e^{i(k+\frac{1}{2})t}\right| = \frac{\mid q+e^{it}\mid^n}{(1+q)^n} \\
&= \frac{(1+q^2+2q\cos t)^{n/2}}{(1+q)^n}\leq e^{-2qt^2n/\{\pi(1+q)\}^2},
\end{aligned}
$$

using Lemma 1.1.

Applying Hölder's inequality and the fact that $\phi(t)\in \mathrm{Lip}\,(\psi(t),p)$, we have

$$
|J_1(x)| = O\left\{\int_0^{1/\sqrt{n}}|\phi(t)|^p dt\right\}^{1/p}\left\{\int_0^{1/\sqrt{n}}\left|\ \left[(1+q)^{-n}\frac{S(t)}{\sin\frac{t}{2}}\right]\ \right|^{p'}dt\right\}^{1/p'},
$$

where $p^{'}=p/(p-1)$. Further,

$$
\begin{aligned}
|J_1(x)| &= O\left\{\int_0^{1/\sqrt{n}}\left(\frac{\psi(t)}{t^{1/p}}\right)^p dt\right\}^{1/p} O\left\{\int_0^{1/\sqrt{n}}\left(\frac{e^{-2qt^2n/\{\pi(1+q)\}^2}}{\left|\ \sin\frac{t}{2}\ \right|}\right)^{p'}dt\right\}^{1/p'} \\
&= O\left(\frac{1}{\sqrt{n}}\right)O\left\{\int_0^{1/\sqrt{n}}t^{-p'}dt\right\}^{1/p'}
\end{aligned}
$$

$$= O\left[\psi\left(\frac{1}{\sqrt{n}}\right)\cdot(\frac{1}{\sqrt{n}})^{1/p'-1}\right]$$

$$= O\left(\psi\left(\frac{1}{\sqrt{n}}\right)\cdot(\sqrt{n})^{1/p}\right).$$

For evaluating $J_2(x)$, we have

$$\begin{aligned}
|J_2(x)| &= O\left\{\int_{1/\sqrt{n}}^{\pi}\frac{\mid\phi(t)\mid}{\sin\frac{1}{2}t}[(1+q)^{-n}(1+q^2+2q\cos t)^{n/2}]\cdot\mid R(t)\mid dt\right\}\\
&= O\left\{\int_{1/\sqrt{n}}^{\pi}\frac{\mid\phi(t)\mid}{\sin\frac{1}{2}t}(1+q)^{-n}(1+q^2+2q\cos t)^{n/2}dt\right\}\\
&= O\left\{\int_{1/\sqrt{n}}^{\pi}\frac{\mid\phi(t)\mid}{\sin\frac{1}{2}t}e^{(-2nqt^2)/(\pi(1+q))^2}dt\right\} \qquad \text{(by Lemma 1.1)}\\
&= O\left[\frac{1}{n}\int_{1/\sqrt{n}}^{\pi}\frac{\psi(t)}{t^{1+1/p}\cdot\sin\frac{1}{2}t}\left\{\frac{\partial}{\partial t}\left(-e^{(-2qnt^2)/(\pi(1+q))^2}\right)\right\}\right]dt.
\end{aligned}$$

Applying Hölder's inequality and the fact that $\phi(t)\in\mathrm{Lip}\,(\psi(t),p)$, we have

$$\begin{aligned}
|J_2(x)| &= O\left[\frac{1}{n}\int_{1/\sqrt{n}}^{\pi}\left(\frac{\psi(t)}{t^{1/p+2}}\right)^p dt\right]^{1/p}\times\\
&\qquad\times\left[\int_{1/\sqrt{n}}^{\pi}\left\{\frac{\partial}{\partial t}\left(-e^{(-2qnt^2)/(\pi(1+q))^2}\right)\right\}^{p'}dt\right]^{1/p'}\\
&= O\left(\psi\left(\frac{1}{\sqrt{n}}\right)(\sqrt{n})^{1/p}\right).
\end{aligned}$$

Therefore,

$$\sigma_n(f,x)-f(x)=O\left(\psi\left(\frac{1}{\sqrt{n}}\right)(\sqrt{n})^{1/p}\right),$$

which completes the proof of the first part of the theorem.

To prove the second part, that (1.2) is the best possible, we suppose that δ is a small positive number less than $\pi/4$.

Following condition (2.1), we have

$$| \sigma_n(f,x) - f(x) | = \left| \int_0^{\pi} \sum_{k=0}^{n} \frac{\phi(t)}{\sin\frac{t}{2}} (1+q)^{-n} \binom{n}{k} q^{n-k} \sin\left(k+\frac{1}{2}\right) t\, dt \right|$$

$$= \left| \left\{ \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right\} \frac{\phi(t)}{\sin\frac{t}{2}} \sum_{k=0}^{n} (1+q)^{-n} \binom{n}{k} q^{n-k} \sin\left(k+\frac{1}{2}\right) t\, dt \right|,$$

i.e.,

$$\begin{aligned} |\sigma_n(f,x) - f(x)| &= |I_1(x) + I_2(x) + I_3(x)| \\ &\geq |I_2(x)| - |I_1(x)| - |I_3(x)|. \end{aligned} \tag{2.2}$$

Applying Hölder's inequality and the fact that $\phi(t) \in \mathrm{Lip}\,(\psi(t), p)$, $p > 1$, and proceeding in the same way as in $J_1(x)$, we have

$$\max_{0 \leq x \leq 2\pi} |I_1(x)| \leq K_1 \psi\left(\frac{1}{n}\right) (n)^{1/p},$$

where K_1 is some constant.

Similarly for $I_3(x)$, we have

$$\max_{0 \leq x \leq 2\pi} |I_3(x)| = O\left[\frac{1}{n} \int_{\delta}^{\pi} \frac{\psi(t)}{t^{2+1/p}} \left\{ \frac{\partial}{\partial t} \left(-e^{(-2qnt^2)/(\pi(1+q))^2} \right) \right\} dt \right].$$

Applying Hölder's inequality and the fact that $\phi(t) \in \mathrm{Lip}\,(\psi(t), p)$, $p > 1$, and proceeding in the same way as in $J_2(x)$, we have

$$\max_{0 \leq x \leq 2\pi} |I_3(x)| = O\left(\frac{1}{n}\right).$$

Therefore,

$$\max_{0 \leq x \leq 2\pi} |I_3(x)| < \frac{K_3}{n},$$

where K_3 is a constant, different from the constant K_1.

Now for evaluating $I_2(x)$, we have

$$\begin{aligned} |I_2(x)| \geq & \left| 2 \int_{1/n}^{\delta} \frac{\phi(t)}{t} (1+q)^{-n} (1+q^2+2q\cos t)^{n/2} R(t)\, dt \right| \\ & - \left| \int_{1/n}^{\delta} \phi(t) \left\{ \operatorname{cosec}\frac{t}{2} - \frac{2}{t} \right\} (1+q)^{-n} (1+q^2+2q\cos t)^{n/2} R(t) \} dt \right| \end{aligned}$$

$$= 2|I_{2,1}(x)| - |I_{2,2}(x)| \qquad \text{(say)}.$$

Now

$$\max_{0\le x\le 2\pi} |I_{2,2}(x)| = O\left(\int_{1/n}^{\delta} \frac{\psi(t)}{t^{1/p-1}}(1+q)^{-n}(1+q^2+2q\cos t)^{n/2}dt\right)$$

$$= O\left(\frac{1}{n}\int_{1/n}^{\delta} \frac{\psi(t)}{t^{1/p}}\left\{\frac{\partial}{\partial t}\left(-e^{(-2qnt^2)/(\pi(1+q))^2}\right)\right\}dt\right) \qquad \text{(using Lemma 1.1)}.$$

Applying Hölder's inequality and the fact that $\phi(t) \in \mathrm{Lip}\,(\ \psi(t), p)$, $p > 1$, we have

$$\max_{0\le x\le 2\pi} |I_{2,2}(x)| = O\left[\frac{1}{n}\int_{1/n}^{\delta} \left(\frac{\psi(t)}{t^{1/p}}\right)^{p} dt\right]^{1/p} \times$$

$$\times \left[\int_{1/n}^{\delta} \left\{\frac{\partial}{\partial t}\left(-e^{(-2qnt^2)/(\pi(1+q))^2}\right)\right\}^{p'} dt\right]^{1/p'} = O\left(\frac{1}{n}\right),$$

and therefore

$$\max_{0\le x\le 2\pi} |I_{2,2}(x)| < \frac{K_{2,2}}{n},$$

where $K_{2,2}$ is any constant.

Now by (1.2), there exists a constant $K_4 > 0$ such that

$$-K_4\frac{\psi(t)}{t^{1/p}} < \phi(t) < K_4\frac{\psi(t)}{t^{1/p}}.$$

Therefore,

$$K_4\frac{\psi(t)}{t^{1/p}} < \phi(t) + 2K_4\frac{\psi(t)}{t^{1/p}}$$

and hence

$$|I_{2,1}(x)| > \left|\int_{1/n}^{\delta} \frac{\phi(t) + 2K_4\left(\frac{\psi(t)}{t^{1/p}}\right)}{t}(1+q)^{-n}(1+q^2+2q\cos t)^{n/2}R(t)\,dt\right|$$

$$-2K_4\left|\int_{1/n}^{\delta} \frac{\psi(t)}{t^{1/p+1}}(1+q)^{-n}(1+q^2+2q\cos t)^{n/2}R(t)\,dt\right|$$

$$= |I_{2,1,1}(x)| - 2K_4|I_{2,1,2}(x)| \qquad \text{(say)}.$$

Since $\frac{\psi(t)}{t^{1/p+1}}(1+q)^{-n}(1+q^2+2q\cos t)^{n/2}$ is a positive non-increasing function in $[1/n, \delta]$, therefore by the Second Mean value theorem, we get

$$I_{2,1,2}(x) = \frac{\psi(1/n)}{(1/n)^{1+1/p}}(1+q)^{-n}\left(1+q^2+2q\cos\left(\frac{1}{n}\right)\right)^{n/2}\int_{1/n}^{\theta} R(t)\,dt$$

for $1/n \le \theta \le \delta$, i.e.,

$$I_{2,1,2}(x) = O\left(\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-p}\right).$$

Therefore,

$$\max_{0\le x\le 2\pi} 2K_4|I_{2,1,2}(x)| < K_{2,1,2}\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-p},$$

where $K_{2,1,2}$ is any constant.

Now by the First mean value theorem, we get

$$|I_{2,1,1}(x)| = \left| R(t')\int_{1/n}^{\delta}\frac{\phi(t)+2K_4\left(\frac{\psi(t)}{t^{1/p}}\right)}{t}(1+q)^{-n}(1+q^2+2q\cos t)^{n/2}\,dt\right|$$

for $1/n \le t' \le \delta$, i.e.,

$$\begin{aligned}|I_{2,1,1}(x)| &\ge |R(t')|\left|\int_{1/\sqrt{n}}^{\delta}\frac{\phi(t)+2K_4\left(\frac{\psi(t)}{t^{1/p}}\right)}{t}(1+q)^{-n}(1+q^2+2q\cos t)^{n/2}dt\right| \\ &\ge |R(t')|\left|\int_{1/\sqrt{n}}^{\delta}\left(\frac{\psi(t)}{t^{1/p+1}}\right)(1+q)^{-n}(1+q^2+2q\cos t)^{n/2}dt\right|.\end{aligned}$$

Now since,

$$\begin{aligned}\sin^{-1}\left(\frac{2\sqrt{q}}{1+q}\sin\frac{t}{2}\right) &= \sin^{-1}\left(\frac{2}{\pi}\frac{\pi\sqrt{q}\sin t/2}{(1+q)}\right) \le \frac{\pi\sqrt{q}}{1+q}\sin\frac{t}{2} \\ &\le \frac{\pi}{2}\frac{\sqrt{q}t}{1+q} \le \beta \qquad \left(0 < t \le \frac{4\beta}{\pi}\right),\end{aligned}$$

where β is strictly less than $\pi/2$, and

$$\cos\left(\frac{\pi\sqrt{q}t}{2(1+q)}\right) > \exp\left[-\pi^2/2\left(\frac{qt^2\sec^2\beta}{4(1+q)^2}\right)\right]$$

and so,

$$\begin{aligned}|I_{2,1,1}(x)| &> K_4\,|R(t')| \int_{1/\sqrt{n}}^{\delta} \left(\frac{\psi(t)}{t^{1+1/p}}\right) e^{[-(qn\pi^2t^2)/(2(1+q))^2]}\,dt,\\ &> K_{2,1,1}\psi\left(\frac{1}{\sqrt{n}}\right)\left(\frac{1}{\sqrt{n}}\right)^{-1/p},\end{aligned}$$

where the constant $K_{2,1,1}$ depends upon $|R(t')|$ and other parameters. However the integral $I_{2,1,1}$ is not zero, therefore, the constant $K_{2,1,1}$ is positive.

Now (2.2) will be

$$\begin{aligned}\max_{0\le x\le 2\pi} \pi \mid f(x) \;-\; \sigma_n(f,x) \mid &\ge K_{2,1,1}\psi\left(\frac{1}{\sqrt{n}}\right)\left(\frac{1}{\sqrt{n}}\right)^{-1/p}\\ &\quad -(K_1+K_{2,1,2})\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1/p} - \frac{K_3+K_{2,2}}{n}\\ &\ge K_{2,1,1}\,\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1/p} - \frac{K_1+K_3+K_{2,2}+K_{2,1,2}}{n}\\ &= \psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1/p}\left[K_{2,1,1} - \frac{K_1+K_3+K_{2,2}+K_{2,1,2}}{\psi\left(1/\sqrt{n}\right)(n)^{1+1/(2p)}}\right].\end{aligned}$$

And for any given constant K' such that

$$K_{2,1,1} - K' > 0,$$

we can find a positive number $n_0 = n_0(K')$ such that

$$(K_1+K_3+K_{2,2}+K_{2,1,2})\,n^{1-1/p}\,\psi\left(\frac{1}{\sqrt{n}}\right) < K' \quad \text{for} \quad n > n_0.$$

Therefore, there exists a positive constant K, depending on $K_{2,1,1}$ and K' such that

$$\max_{0\le x\le 2\pi} |f(x) - \sigma_n(f,x)| \ge K\psi\left(\frac{1}{\sqrt{n}}\right)(n)^{1/2p},$$

and hence the inequality (1.3) is the best possible.

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