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ON THE SCHECHTER ESSENTIAL SPECTRUM ON BANACH SPACES AND APPLICATION

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Abstract. This paper is devoted to the investigation of the stability of the Schechter essential spectrum of closed densely defined linear operators A subjected to additive perturbations K such that $(\lambda - A - K)^{-1}K$ or $K(\lambda - A - K)^{-1}$ belonging to arbitrary subsets of $\mathcal{L}(X)$ (where X denotes a Banach spaces) contained in the set $\mathcal{J}(X)$. Our approach consists principally in considering the class of A-closable (not necessarily bounded) which contained in the set of A-resolvent Fredholm perturbations which zero index (see Definition 3.5). They are used to describe the Schechter essential spectrum of singular neutron transport equations in bounded geometries.

1. Introduction

Let X and Y be two Banach spaces. By an operator A from X into Y we mean a linear operator with domain $D(A) \subset X$ and range $R(A) \subset Y$. We denote by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$) the set of all closed, densely defined linear operators (resp. the Banach algebra of all bounded linear operators) from X into Y. The subset of all compact operators of $\mathcal{L}(X,Y)$ is designated by $\mathcal{K}(X,Y)$. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$ and $\mathcal{C}(X,Y)$ are replaced, respectively, by $\mathcal{L}(X)$, $\mathcal{K}(X)$ and $\mathcal{C}(X)$.

Definition 1.1. An operator $A \in \mathcal{L}(X,Y)$ is said to be weakly compact if $A(B)$ is relatively weakly compact in Y for every bounded subset $B \subset X$.

The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$ the family of weakly compact operators on X, $\mathcal{W}(X) := \mathcal{W}(X,X)$, is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (cf. [4, 6]).

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Definition 1.2. Let X and Y be two Banach spaces. An operator $A \in \mathcal{L}(X,Y)$ is called strictly singular if, for every infinite-dimensional subspace M, the restriction of A to M is not a homeomorphism.

Let $\mathcal{S}(X,Y)$ denote the set of strictly singular operators from X into Y.

The concept of strictly singular operators was introduced in the pioneering paper by Kato [21] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators we refer to [6, 21]. For our own use, let us recall the following four facts. The set $\mathcal{S}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$, if $X = Y$, $\mathcal{S}(X) := \mathcal{S}(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$, if X is a Hilbert space then $\mathcal{K}(X) = \mathcal{S}(X)$ and the class of weakly compact operators on L_1-spaces (resp. $C(K)$-spaces with K a compact Haussdorff space) is nothing else but the family of strictly singular operators on L_1-spaces (resp. $C(K)$-spaces) (see [36, Theorem 1]).

Let X be a Banach space. If N is a closed subspace of X, we denote by π_N^X the quotient map $X \to X/N$. The codimension of N, codim(N), is defined to be the dimension of the vector space X/N.

Definition 1.3. Let X and Y be two Banach spaces and $S \in \mathcal{L}(X,Y)$. S is said to be strictly cosingular from X into Y, if there exists no closed subspace N of Y with codim(N)=$+\infty$ such that $\pi_N^Y S : X \to Y/N$ is surjective.

Let $C\mathcal{S}(X,Y)$ denote the set of strictly cosingular operators from X into Y. This class of operators was introduced by Pelczynski [36]. It forms a closed subspace of $\mathcal{L}(X,Y)$ which is, $C\mathcal{S}(X) := C\mathcal{S}(X,X)$, a closed two-sided ideal of $\mathcal{L}(X)$ if $X = Y$ (cf. [41]).

For $A \in \mathcal{C}(X,Y)$, we let $\sigma(A)$, $\rho(A)$ and $N(A)$ denote respectively the spectrum, the resolvent set and the null space of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y. The set of Fredholm operators from X into Y is defined by

$$\begin{aligned}\Phi(X,Y) \;&=\; \Big\{A \in \mathcal{C}(X,Y) \text{ such that } \alpha(A) < +\infty,\\ &\qquad\qquad R(A) \text{ is closed in } Y \text{ and } \beta(A) < +\infty\Big\},\end{aligned}$$

the set of bounded Fredholm operators from X into Y is defined by

$$\Phi^b(X,Y) = \Phi(X,Y) \cap \mathcal{L}(X,Y),$$

and the set Φ_A is defined by

$$\Phi_A = \{\lambda \in \mathbf{C} \text{ such that } \lambda - A \in \Phi(X, Y)\}.$$

If $A \in \Phi(X, Y)$, the number $i(A) = \alpha(A) - \beta(A)$ is called the index of A. If $X = Y$ then $\Phi(X, Y)$ and $\Phi^b(X, Y)$ are replaced, respectively, by $\Phi(X)$ and $\Phi^b(X)$.

Definition 1.4. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$. F is called a Fredholm perturbation if $U + F \in \Phi^b(X, Y)$ whenever $U \in \Phi^b(X, Y)$.

The set of Fredholm perturbations is denoted by $\mathcal{F}^b(X, Y)$. This class of operators is introduced and investigated in [5]. In particular, it is shown that $\mathcal{F}^b(X, Y)$ is closed subset of $\mathcal{L}(X, Y)$, and if $X = Y$, then $\mathcal{F}^b(X) := \mathcal{F}^b(X, X)$ is closed two-sided ideal of $\mathcal{L}(X)$. The following result was established in [5, pp. 69–70].

Proposition 1.2 ([5, pp. 69–70]). *Let X, Y and Z be three Banach spaces. If at least one of the sets $\Phi^b(X, Y)$ and $\Phi^b(Y, Z)$ is not empty, then*

(i) *$F \in \mathcal{F}^b(X, Y)$, $A \in \mathcal{L}(Y, Z)$ imply $AF \in \mathcal{F}^b(X, Z)$;*

(ii) *$F \in \mathcal{F}^b(Y, Z)$, $A \in \mathcal{L}(X, Y)$ imply $FA \in \mathcal{F}^b(X, Z)$.*

Definition 1.5. Let X be a Banach space and $R \in \mathcal{L}(X)$. R is said to be a Riesz operator if $\Phi_R = \mathbf{C} \setminus \{0\}$.

For further information on the family of Riesz operators we refer to [1, 19] and the references therein.

Remark 1.1. a) The family of Riesz operators is not an ideal of $\mathcal{L}(X)$ (see [1]).

b) In [38], it is proved that $\mathcal{F}^b(X)$ is the largest ideal of $\mathcal{L}(X)$ contained in the family of Riesz operators.

Remark 1.2. Let X and Y be two Banach spaces. If in Definition 1.4 we replace $\Phi^b(X, Y)$ by $\Phi(X, Y)$ we obtain the set $\mathcal{F}(X, Y)$.

Definition 1.6. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$. F is called a upper (resp. lower) Fredholm perturbation if $U + F \in \Phi^b_+(X, Y)$ (resp. $\Phi^b_-(X, Y)$) whenever $U \in \Phi^b_+(X, Y)$ (resp. $\Phi^b_-(X, Y)$).

The sets of upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}^b_+(X, Y)$ and $\mathcal{F}^b_-(X, Y)$, respectively. In [5], it is shown that $\mathcal{F}^b_+(X, Y)$ and $\mathcal{F}^b_-(X, Y)$ are closed subset of $\mathcal{L}(X, Y)$, and if $X = Y$, then $\mathcal{F}^b_+(X) := \mathcal{F}^b_+(X, X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

The following identity was established in [26, Lemma 2.3(ii)].

Lemma 1.1 ([26]). *Let X be an arbitrary Banach space. Then $\mathcal{F}(X) = \mathcal{F}^b(X)$.*

An immediate consequence of this result is that $\mathcal{F}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Remark 1.3. Let X and Y be two Banach spaces. In contrast to the result of Lemma 1.1, whether or not $\mathcal{F}(X,Y)$ is equal to $\mathcal{F}^b(X,Y)$ seems to be unknown.

In general, we have the following inclusions:

$$\mathcal{K}(X) \subset \mathcal{S}(X) \subset \mathcal{F}_+^b(X) \subset \mathcal{F}(X) \subset \mathcal{J}(X),$$

$$\mathcal{K}(X) \subset C\mathcal{S}(X) \subset \mathcal{F}_-^b(X) \subset \mathcal{F}(X) \subset \mathcal{J}(X),$$

where $\mathcal{F}_-^b(X) := \mathcal{F}_-^b(X,X)$ and $\mathcal{J}(X)$ denotes the set

$$\mathcal{J}(X) = \left\{ F \in \mathcal{L}(X) \ \text{ such that } \ -1 \in \Phi_F^0 \right\},$$

where $\Phi_F^0 := \{\lambda \in \Phi_F \text{ such that } i(\lambda - F) = 0\}$.

Remark 1.4. $\mathcal{J}(X)$ is not an ideal of $\mathcal{L}(X)$ (since $I \in \mathcal{J}(X)$).

Definition 1.7. A Banach space X is said to have the Dunford-Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T : X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets of Y.

It is well known that any L_1 space has the property DP [3]. Also, if Ω is a compact Hausdorff space $C(\Omega)$ has the property DP [7]. For further examples we refer to [2] or [4, p. 494, 497, 508, and 511]. Note that the property DP is not preserved under conjugation. However, if X is a Banach space whose dual has the property DP then X has the property DP (see, e.g., [7]). For more information we refer to the paper by Diestel [2] which contains a survey and exposition of the Dunford-Pettis property and related topics.

There are many ways to define the essential spectrum of a closed, densely defined linear operator on a Banach space. Hence several definitions of the essential spectrum may be found in the literature see, for example, [8, 37] or the comments in [39, Chapter 11, p. 283], which coincide for self-adjoint operators on Hilbert spaces. Throughout this paper we are concerned with the so-called Schechter essential spectrum.

Definition 1.8. Let X and Y be two Banach spaces and let $A \in \mathcal{C}(X,Y)$. We define the Schechter essential spectrum of the operator A by

$$\sigma_{\text{ess}}(A) = \bigcap_{C \in \mathcal{K}(X,Y)} \sigma(A+C).$$

The following proposition gives a characterization of the Schechter essential spectrum by means of Fredholm operators:

Proposition 1.1 ([39, Theorem 5.4, p. 180])**.** *Let X and Y be two Banach spaces and let $A \in \mathcal{C}(X,Y)$. Then*

$$\lambda \notin \sigma_{\text{ess}}(A) \quad \textit{if and only if} \quad \lambda \in \Phi_A^0.$$

One of the central questions in the study of the Schechter essential spectrum of closed densely defined operators on Banach spaces X consists of showing what are the conditions that we must impose on $K \in \mathcal{C}(X)$ in order that $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$. Let $A \in \mathcal{C}(X)$. If K is a compact operator on Banach spaces then $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$ (see Definition 1.8). If K is a strictly singular on L_p-spaces then $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$ (see [29, Theorem 3.2]). If K is a weakly compact on Banach spaces which possess the Dunford-Pettis property then $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$ (see [24, Theorem 3.2]). If $K \in \mathcal{L}(X)$ and $(\lambda - A)^{-1}K$ is strictly singular (resp. weakly compact) on L_p spaces $p > 1$ (resp. on Banach spaces which possess the Dunford-Pettis property), then $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$ (see [12, 14]). In [15], Jeribi extended this analysis of the Schechter essential spectrum to the case of general Banach spaces. In fact, let $\mathcal{I}(X)$ be an arbitrary two-sided ideal of $\mathcal{L}(X)$. If $\mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{F}(X)$, then $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$ for all $K \in \mathcal{L}(X)$ such that $(\lambda - A)^{-1}K \in \mathcal{I}(X)$ or for all $K \in \mathcal{L}(X)$ such that $K(\lambda - A)^{-1} \in \mathcal{I}(X)$. Recently, Jeribi [17] gives an extension of the work [15] where a detailed treatment of the Schechter essential spectrum of a closed densely defined linear operators A subjected to additive perturbations K such that $(\lambda - A)^{-1}K$ or $K(\lambda - A)^{-1}$ belonging to arbitrary subsets of $\mathcal{L}(X)$ (where X denotes a Banach spaces) contained in the ideal of Fredholm perturbations. Our approach consists principally in considering the class of A-closable (not necessarily bounded) which contained in the set of A-resolvent Fredholm perturbations (see Definition 3.4). The aim of this paper consists principally of considering the class of A-closable operator K (not necessarily bounded) which is contained in the set $A\mathcal{J}(X)$, and of proving that $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$ for all K in any subset of operators in $A\mathcal{J}(X)$. More precisely, let $A \in \mathcal{C}(X)$. Then $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$ for all $K \in \mathcal{C}(X)$

such that K is A-bounded and $K(\lambda - A - K)^{-1} \in \mathcal{J}(X)$ for all $\lambda \in \rho(A+K)$. Our results extend and improve many known ones in the literature (see [10], [12], [14], [15], [16], [17], [18], [24], [25], [27] [29] and [31]).

This work is inspired by [15, 17, 29], where a detailed treatment of the Schechter essential spectrum of a closed densely defined linear operators A subjected to additive perturbations contained in the set of A-Fredholm perturbations.

The purpose of the first part of this paper is to point out how, by means of the concept of A-resolvent Fredholm perturbation which zero index and the technique developed in [29] it is possible to improve the definition of the Schechter essential spectrum, in the same way as in Theorem 2.1 of [15] and Theorem 2.1 of [29]. Our results generalize many known ones in the literature and, in particular, extend and unify those obtained in [10], [12], [14], [15], [16], [17], [18], [24], [25], [27] [29] and [31].

In the second part of the paper we study the Schechter essential spectrum of the following singular neutron transport operator

$$A\psi(x,v) = -v\frac{\partial \psi}{\partial x}(x,v) - \sigma(v)\psi(x,v) + \int_{\mathbb{R}^n} \kappa(v,v')\psi(x,v')d\mu(v')$$

with vacuum boundary conditions, i.e., $\psi_{|\Gamma_-} = 0$ with

$$\Gamma_- = \{(x,v) \in \partial D \times \mathbb{R}^n \text{ such that } v.\nu_x < 0\},$$

where $(x,v) \in D \times \mathbb{R}^n$ and ν_x stands for the outer unit normal vector at $x \in \partial D$. Here D is an open bounded subset of $\mathbb{R}^n$ and $d\mu(.)$ is a positive Radon measure on $\mathbb{R}^n$. This operator describes the transport of particles (neutrons, photons, molecules of gas, etc.) in the domain D. For the neutrons, the function $\psi(x,v)$ represents the number (or probability) density of gas particles having the position x and the velocity v. For the photons, ψ describes the specific intensity of the light. For the molecules of gas, ψ describes the deviation of the number density of the gas molecules from their equilibrium number density. For gas molecules, the transport equation is obtained by linearization of the nonlinear Boltzmann equation or some nonlinear simplification of it (such as the Enskog equation or the BGK model) about the equilibrium distribution. The functions $\sigma(.)$ and $\kappa(.,.)$ are called, respectively, the collision frequency and the scattering kernel and will be assumed to be unbounded. More precisely, we will assume that there exist a closed subset $\mathcal{O} \subset \mathbb{R}^n$ with zero $d\mu$ measure and a constant $\sigma_0 > 0$ such that

$$\sigma(.) \in L^{\infty}_{\text{loc}}(\mathbb{R}^n \setminus \mathcal{O}), \quad \sigma(v) > \sigma_0 \ \ a.e. \tag{1.1}$$

$$\left[\int_{\mathbb{R}^n} \left(\frac{\kappa(., v')}{\sigma(v')^{1/p}}\right)^q d\mu(v')\right]^{1/q} \in L_p(\mathbb{R}^n), \tag{1.2}$$

where q denotes the conjugate exponent of p. These assumptions were motivated by free gas models (cf. [35, 40]) and were already used in [33] in L_p spaces (see [34, Chapter 9] or [42]). The first part of the condition (1.1) means that the singularities of the collision frequency are contained in a set of zero $d\mu$ measure. In fact, unbounded and nonnegative collision frequencies act as strong absorptions which might lead to unbounded collision operators.

We organize the paper in the following way: The next section is devoted to the Schechter essential spectrum of closed densely defined linear operators on a Banach space. The main results of this section is Theorem 2.1. In Section 3 we extend a part of the results obtained in Section 2 to a large class of perturbating operators containing $\mathcal{J}(X)$ which we denote by $A\mathcal{J}(X)$. In particular, it is proved that if $A \in \mathcal{C}(X)$ then $\sigma_{\text{ess}}(A + K) = \sigma_{\text{ess}}(A)$ for all $K \in A\mathcal{J}(X)$. In Section 4 we apply the results obtained in the second section to investigate the Schechter essential spectrum of singular neutron transport equations in bounded geometries.

2. Invariance of Schechter Essential Spectrum

Let X be a fixed Banach space and let $\mathcal{I}(X)$ an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the condition

$$\mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{F}(X). \tag{$H1$}$$

Remark 2.1. It should be observed that if $\mathcal{I}(X)$ is a nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the condition $\mathcal{I}(X) \subset \mathcal{F}(X)$, then $\mathcal{F}_0(X) \subset \mathcal{I}(X) \subset \mathcal{F}(X)$, where $\mathcal{F}_0(X)$ stands for the ideal of finite rank operators. This follows from Lemma 1.1 and [5, Proposition 4 p. 70].

If $A \in \mathcal{C}(X)$ we define the sets:

$$\mathcal{G}_A(X) = \{K \in \mathcal{L}(X) \text{ such that } (\lambda - A)^{-1}K \in \mathcal{I}(X) \text{ for some } \lambda \in \rho(A)\},$$

$$\mathcal{D}_A(X) = \{K \in \mathcal{L}(X) \text{ such that } K(\lambda - A)^{-1} \in \mathcal{I}(X) \text{ for some } \lambda \in \rho(A)\},$$

$$\begin{aligned}\mathcal{O}_A(X) &= \{K \in \mathcal{L}(X) \text{ such that } (\lambda - A - K)^{-1}K \in \mathcal{J}(X) \\ &\qquad \text{for all } \lambda \in \rho(A+K)\}, \\ \mathcal{V}_A(X) &= \{K \in \mathcal{L}(X) \text{ such that } K(\lambda - A - K)^{-1} \in \mathcal{J}(X) \\ &\qquad \text{for all } \lambda \in \rho(A+K)\}.\end{aligned}$$

Remark 2.2. (i) Observe that, in the definition of the sets $\mathcal{G}_A(X)$ and $\mathcal{D}_A(X)$, if an operator satisfies the required condition for a fixed $\lambda \in \rho(A)$, then it satisfies it for every $\lambda \in \rho(A)$.

(ii) We have the following inclusion:

$$\mathcal{K}(X) \subset \mathcal{G}_A(X) \subset \mathcal{O}_A(X), \qquad \mathcal{K}(X) \subset \mathcal{D}_A(X) \subset \mathcal{V}_A(X).$$

In fact, let $K \in \mathcal{G}_A(X)$ (resp. $K \in \mathcal{D}_A(X)$) then $(\lambda - A)^{-1}K \in \mathcal{I}(X)$ (resp. $K(\lambda - A)^{-1} \in \mathcal{I}(X)$) where $\mathcal{I}(X)$ is an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the condition $(H1)$.

Let $\mu \in \rho(A)$, we have

$$(\lambda - A - K)^{-1}K = [I + (\lambda - A - K)^{-1}(\mu - \lambda + K)](\mu - A)^{-1}K \tag{2.1}$$

and

$$K(\lambda - A - K)^{-1} = K(\mu - A)^{-1}[I + (\mu - \lambda + K)(\lambda - A - K)^{-1}]. \tag{2.2}$$

Using (2.1) (resp. (2.2)), and the fact that $\mathcal{I}(X)$ is a two-sided ideal of $\mathcal{L}(X)$, we infer that $(\lambda - A - K)^{-1}K \in \mathcal{I}(X)$ (resp. $K(\lambda - A - K)^{-1} \in \mathcal{I}(X)$). So, $K \in \mathcal{O}_A(X)$ (resp. $K \in \mathcal{V}_A(X)$).

(iii) If we take $A := I$ and $K := I$ then $(\lambda - A)^{-1}K$ is not in an ideal $\mathcal{I}(X)$. But $(\lambda - A - K)^{-1}K$ is in $\mathcal{J}(X)$. So, the set $\mathcal{G}_A(X)$ (resp. $\mathcal{D}_A(X)$) is strictly included in $\mathcal{O}_A(X)$ (resp. $\mathcal{V}_A(X)$).

We define the right spectrum of A by

$$\sigma_r(A) = \bigcap_{K \in \mathcal{O}_A(X)} \sigma(A+K).$$

Similarly, we define the left spectrum of A by

$$\sigma_l(A) = \bigcap_{K \in \mathcal{V}_A(X)} \sigma(A+K).$$

The main result of this section is the following:

Theorem 2.1. *Let X be a Banach space and let $A \in \mathcal{C}(X)$. Then*

$$\sigma_{\text{ess}}(A) = \sigma_r(A) = \sigma_l(A).$$

Remark 2.3. a) Note that the sets $\mathcal{O}_A(X)$ and $\mathcal{V}_A(X)$ may characterize the Schechter essential spectrum. Since $\mathcal{K}(X) \subset \mathcal{O}_A(X)$ and $\mathcal{K}(X) \subset \mathcal{V}_A(X)$, so $\mathcal{K}(X)$ is then the minimal subset of $\mathcal{L}(X)$ (in the sense of inclusion) for which Theorem 2.1 holds true. Hence Theorem 2.1 provides an improvement of the definition of $\sigma_{\text{ess}}(.)$ valid for a somewhat large variety of subsets of $\mathcal{L}(X)$. Also, it may be viewed as an extension of, [12] Theorem 3.1, [18] Theorem 1, [14] Theorem 3.2, [9] Theorem 1, [10] Theorem 2.1, [26] Theorem 3.4 and [15] Theorem 2.1 to general Banach spaces.

b) Note that in applications (transport operators, operators arising in dynamic populations, etc. (see [14, 28, 29, 30, 11, 13])), the operator B is, in general, a bounded perturbation of $A \in \mathcal{C}(L_p)$ by an integral operator on L_p-spaces $p > 1$. The integral operator $K := B - A$ is not compact. For some physical conditions on K, the operator $(\lambda - A)^{-1}K$ or $K(\lambda - A)^{-1}$ are compact on L_p-spaces $p > 1$. This implies that $(\lambda - A - K)^{-1}K$ or $K(\lambda - A - K)^{-1}$are compact on L_p-spaces $p > 1$. So, $\mathcal{K}(X) \subset_{\neq} \mathcal{O}_A(X)$ and $\mathcal{K}(X) \subset_{\neq} \mathcal{V}_A(X)$.

c) For all $K \in \mathcal{O}_A(X)$, $\sigma_{\text{ess}}(A + K) = \sigma_{\text{ess}}(A)$.

d) For all $K \in \mathcal{V}_A(X)$, $\sigma_{\text{ess}}(A + K) = \sigma_{\text{ess}}(A)$.

Proof of Theorem 2.1. We first claim that $\sigma_{\text{ess}}(A) \subset \sigma_r(A)$ (resp. $\sigma_{\text{ess}}(A) \subset \sigma_l(A)$). Indeed, if $\lambda \notin \sigma_r(A)$ (resp. $\lambda \notin \sigma_l(A)$) then there exists $K \in \mathcal{O}_A(X)$ (resp. $K \in \mathcal{V}_A(X)$) such that $\lambda \in \rho(A + K)$, hence $\lambda \in \Phi_{(A+K)}$ and $i(\lambda - A - K) = 0$. Using the equality $\lambda - A = (\lambda - A - K)(I + (\lambda - A - K)^{-1}K)$ (resp. $\lambda - A = (I + K(\lambda - A - K)^{-1})(\lambda - A - K)$) together with Atkinson's theorem ([32], Proposition 2.c.7.(ii), p. 77) one gets $\lambda \in \Phi_A$ and $i(\lambda - A) = 0$. Finally, the use of Proposition 1.1 shows that $\lambda \notin \sigma_{\text{ess}}(A)$ which proves the claim.

On the other hand, since $\mathcal{K}(X) \subset \mathcal{O}_A(X)$ (resp. $\mathcal{K}(X) \subset \mathcal{V}_A(X)$) we infer that $\sigma_r(A) \subset \sigma_{\text{ess}}(A)$ (resp. $\sigma_l(A) \subset \sigma_{\text{ess}}(A)$) which completes the proof of theorem. $\square$

Corollary 2.1. *Let X be a Banach space, $A \in \mathcal{C}(X)$ and let $\mathcal{T}(X)$ and $\mathcal{U}(X)$ be any subset of $\mathcal{L}(X)$ (not necessarily an ideal) satifying the condition*

$$\mathcal{K}(X) \subset \mathcal{T}(X) \subset \mathcal{O}_A(X), \tag{2.3}$$

$$\mathcal{K}(X) \subset \mathcal{U}(X) \subset \mathcal{V}_A(X). \tag{2.4}$$

Then

$$\sigma_{\text{ess}}(A) = \bigcap_{K \in \mathcal{T}_A(X)} \sigma(A+K) = \bigcap_{K \in \mathcal{U}_A(X)} \sigma(A+K).$$

Remark 2.4. a) Note that any subset $\mathcal{T}(X)$ and $\mathcal{U}(X)$ of $\mathcal{L}(X)$ (not necessarily an ideal) satisfying the condition (2.3) and (2.4) may characterize the Schechter essential spectrum.

b) For all $K \in \mathcal{T}_A(X)$, $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$.

c) For all $K \in \mathcal{U}_A(X)$, $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$.

Proof of Corollary 2.1. Set

$$\mathcal{Z} := \bigcap_{K \in \mathcal{T}_A(X)} \sigma(A+K) \quad \text{and} \quad \mathcal{Z}' := \bigcap_{K \in \mathcal{U}_A(X)} \sigma(A+K).$$

We already know from (2.2) and (2.3) that $\mathcal{K}(X) \subset \mathcal{T}_A(X) \subset \mathcal{O}_A(X)$ and $\mathcal{K}(X) \subset \mathcal{U}_A(X) \subset \mathcal{V}_A(X)$. From this we infer that $\sigma_r(A) \subset \mathcal{Z} \subset \sigma_{\text{ess}}(A)$ and $\sigma_l(A) \subset \mathcal{Z}' \subset \sigma_{\text{ess}}(A)$. Now, the result follows from Theorem 2.1. $\square$

By Lemma 4.1 of [23] and Theorem 2.1 we have:

Corollary 2.2. *Let X be a Banach space, $\mathcal{I}(X)$ an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the hypothesis $(H1)$ and let $A \in \mathcal{C}(X)$. Then*

$$\sigma C(A) \subset \sigma_r(A) \quad \textit{and} \quad \sigma R(A) \subset \sigma_r(A),$$
$$\sigma C(A) \subset \sigma_l(A) \quad \textit{and} \quad \sigma R(A) \subset \sigma_l(A),$$

where $\sigma C(A)$ (resp. $\sigma R(A)$) denotes the continuous spectrum (resp. the residual spectrum) of A.

The following result provides a characterization of the right and left spectrums on a Banach space X.

Corollary 2.3. *Let X be a Banach space and let $A \in \mathcal{C}(X)$. Then*

$$\lambda \notin \sigma_r(A) \quad \textit{if and only if} \quad \lambda \in \Phi_A \quad \textit{and} \quad i(\lambda - A) = 0,$$

$$\lambda \notin \sigma_l(A) \quad \textit{if and only if} \quad \lambda \in \Phi_A \quad \textit{and} \quad i(\lambda - A) = 0.$$

Proof. This corollary immediately follows from Theorem 2.1 *and Proposition* 1.1. $\square$

Corollary 2.4. *Let X be a Banach space and let $A \in \mathcal{C}(X)$ such that $\sigma_{\text{ess}}(A) = \varnothing$ (i.e., $\sigma(A) = \sigma P(A)$ where $\sigma P(A)$ denotes the point spectrum of A). If $K \in \mathcal{O}_A(X)$ or $K \in \mathcal{V}_A(X)$ then $\sigma(A+K) = \sigma P(A+K)$.*

Proof. This corollary immediately follows from Corollary 2.2 *and Remark* 2.3 c) d). □

3. Extention to Unbounded Perturbations

Let X be a fixed Banach space. Unless otherwise stated in all that follows $\mathcal{I}(X)$ will denote an arbitrary nonzero two-sided ideal of $\mathcal{L}(X)$ satisfying the condition

$$(H2) \qquad \mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{J}(X).$$

Let $A \in \mathcal{C}(X)$. Then it follows from the closedness of A that $D(A)$ endowed with the graph norm $\|.\|_A$ (i.e., $\|x\|_A := \|x\| + \|Ax\|$) is a Banach space. Let X_A denote $(D(A), \|.\|_A)$, in this new space the operator A satisfies $\|Ax\| \leq \|x\|_A$, and this prove that A is a bounded operator from X_A into X (i.e., $A \in \mathcal{L}(X_A, X)$). Let $J \in \mathcal{L}(X)$. If $D(A) \subset D(J)$, then J will be called A-defined. If J is A-defined, we will denote by $\hat{J}$ its restriction to $D(A)$. Moreover, if $\hat{J} \in \mathcal{L}(X_A, X)$, we say that J is A-bounded. One checks easily that if J is closed (or closable) (cf. [22, Remark 1.5, p. 191]) then J is A-bounded.

Definition 3.1. An operator J is called A-closed if $x_n \to x$, $Ax_n \to y$, $Jx_n \to z$ for $(x_n)_n \in D(A)$ implies that $x \in J$ and $Jx = z$. It will be called A-closable if $x_n \to 0$, $Ax_n \to 0$, $Jx_n \to z$ implies $z = 0$.

Remark 3.1. (i) If J is bounded, then J is A-bounded.

(ii) If J is closed then J is A-closed.

(iii) If J is closable then J is A-closable.

(iv) If A is closed then by [37, Lemma 2.1] we have J is A-closed if and only if J is A-closable if and only if J is A-bounded.

Let $A \in \mathcal{C}(X)$ and let J be an A-bounded operator on X. Let $\lambda \in \rho(A)$. Since J is A-bounded, according to Lemma 2.1 in [37], $J(\lambda - A)^{-1}$ is a closed linear operator defined on all of X and therefore bounded, by the closed graph theorem. We define the set $\sigma_a(A)$ by

$$\sigma_a(A) = \bigcap_{K \in \mathcal{N}_A(X)} \sigma(A+K),$$

where $\mathcal{N}_A(X) = \{J \in \mathcal{C}(X)$ such that J is A-bounded and $J(\lambda - A - J)^{-1} \in \mathcal{J}(X)$ for all $\lambda \in \rho(A+J)\}$.

Remark 3.2. Note that any arbitrary nonzero two-sided ideal $\mathcal{I}(X)$ of $\mathcal{L}(X)$ realizes the condition $\mathcal{I}(X) \subset \mathcal{N}_A(X)$.

We have the following result:

Theorem 3.1. *Let X be a Banach space and let $A \in \mathcal{C}(X)$. Then*

$$\sigma_{\text{ess}}(A) = \sigma_a(A).$$

Remark 3.3. a) Note that the set $\mathcal{N}_A(X)$ may characterize the Schechter essential spectrum. Since $\mathcal{K}(X) \subset \mathcal{N}_A(X)$, $\mathcal{K}(X)$ is the minimal subset of $\mathcal{C}(X)$ (in the sense of inclusion) for which Theorem 3.1 holds true. Hence Theorem 3.1 provides an improvement of the definition of $\sigma_{\text{ess}}(.)$ valid for a somewhat large variety of subsets of $\mathcal{C}(X)$. Also, it may be viewed as an extension of [12, Theorem 3.1], [18, Theorem 1], [14, Theorem 3.2], [9, Theorem 1], [10, Theorem 2.1], [26, Theorem 3.4], [15] and [17, Theorem 2.1] to unbounded linear operators.

b) For all $K \in \mathcal{N}_A(X)$, $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$.

Proof of Theorem 3.1. By hypothesis $(H2)$ and Remark 3.2 we have $\mathcal{K}(X) \subset \mathcal{N}_A(X)$. So, $\sigma_a(A) \subset \sigma_{\text{ess}}(A)$. Conversely, let $\lambda \notin \sigma_a(A)$ then there exists $J \in \mathcal{N}_A(X)$ such that $\lambda \in \rho(A+J)$, hence $\lambda \in \Phi_{(A+J)}$ and $i(\lambda - A - J) = 0$. Since $J \in \mathcal{N}_A(X)$ we infer that $I + J(\lambda - A - J)^{-1}$ is a Fredholm operator and $i(I + J(\lambda - A - J)^{-1}) = 0$. Using the equality $\lambda - A = (I + J(\lambda - A - J)^{-1})(\lambda - A - J)$ together with Atkinson's theorem ([32, Proposition 2.c.7.(ii), p. 77]) one gets $\lambda \in \Phi_A$ and $i(\lambda - A) = 0$. Finally, the use of Proposition 1.1, shows that $\lambda \notin \sigma_{\text{ess}}(A)$ which completes the proof of theorem. $\square$

Corollary 3.1. *Let X be a Banach space, $A \in \mathcal{C}(X)$ and let $\mathcal{M}(X)$ be any subset of $\mathcal{J}(X)$ (not necessarily an ideal) satisfying the condition*

$$\mathcal{K}(X) \subset \mathcal{M}(X) \subset \mathcal{J}(X). \tag{3.1}$$

Then

$$\sigma_{\text{ess}}(A) = \bigcap_{K \in \mathcal{H}_A(X)} \sigma(A+K),$$

where $\mathcal{H}_A(X) = \{J \in \mathcal{C}(X)$ *such that* J *is* A*-bounded and* $J(\lambda - A - J)^{-1} \in \mathcal{M}(X)$ *for all* $\lambda \in \rho(A+J)\}$.

Remark 3.4. a) Note that any subset $\mathcal{M}(X)$ of $\mathcal{L}(X)$ (not necessarily an ideal) satisfying the condition (3.1) may characterize the Schechter essential spectrum.

b) For all $K \in \mathcal{H}_A(X)$, $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$.

Proof of Corollary 3.1. Set $\mathcal{Z} := \bigcap_{K \in \mathcal{H}_A(X)} \sigma(A+K)$. We already know from (3.1) that $\mathcal{K}(X) \subset \mathcal{H}_A(X) \subset \mathcal{N}_A(X)$. From this we infer that $\sigma_a(A) \subset \mathcal{Z} \subset \sigma_{\text{ess}}(A)$. Now, the result follows from Theorem 3.1. □

By Lemma 4.1 of [23] and Theorem 3.1 we have:

Corollary 3.2. *Let X be a Banach space and let $A \in \mathcal{C}(X)$. Then*

$$\sigma C(A) \subset \sigma_a(A) \quad \text{and} \quad \sigma R(A) \subset \sigma_a(A).$$

The following result provides a characterization of the set $\sigma_a(.)$ on a Banach space X.

Corollary 3.3. *Let X be a Banach space and let $A \in \mathcal{C}(X)$. Then*

$$\lambda \notin \sigma_a(A) \ \text{ if and only if } \ \lambda \in \Phi_A \ \text{ and } \ i(\lambda - A) = 0.$$

Proof. This corollary immediately follows from Theorem 2.1 and Proposition 1.1. □

Corollary 3.4. *Let X be a Banach space and let $A \in \mathcal{C}(X)$ such that $\sigma_{\text{ess}}(A) = \varnothing$ (i.e., $\sigma(A) = \sigma P(A)$). If $K \in \mathcal{N}_A(X)$ then $\sigma(A+K) = \sigma P(A+K)$.*

Proof. This corollary immediately follows from Corollary 3.2 and Remark 3.3 (b) . □

Let J be an arbitrary A-bounded operator. Hence we can regard A and J as operators from X_A into X. They will be denoted by $\hat{A}$ and $\hat{J}$ respectively. These belong to $\mathcal{L}(X_A, X)$. Furthermore, we have the obvious relations

$$(3.2) \qquad \begin{cases} \alpha(\hat{A}) = \alpha(A), \quad \beta(\hat{A}) = \beta(A), \quad R(\hat{A}) = R(A), \\ \alpha(\hat{A}+\hat{J}) = \alpha(A+J), \\ \beta(\hat{A}+\hat{J}) = \beta(A+J) \ \text{ and } \ R(\hat{A}+\hat{J}) = R(A+J). \end{cases}$$

Definition 3.2. Let X be a Banach space, $A \in \mathcal{C}(X)$ and let F be an A-defined linear operator on X. We say that F is an A-Fredholm perturbation if $\hat{F} \in \mathcal{F}^b(X_A, X)$. F is called a upper (resp. lower) A-semi-Fredholm perturbation if $\hat{F} \in \mathcal{F}_+^b(X_A, X)$ (resp. $\hat{F} \in \mathcal{F}_-^b(X_A, X)$).

The sets of A-Fredholm, upper A-semi-Fredholm and lower A-semi-Fredholm perturbations are denoted by $A\mathcal{F}(X)$, $A\mathcal{F}_+(X)$ and $A\mathcal{F}_-(X)$, respectively.

Definition 3.3. Let X be a Banach space, $A \in \mathcal{C}(X)$ and let J be an A-defined linear operator on X. We say that J is A-compact (resp. A-weakly compact, A-strictly singular, A-strictly cosingular) if $\hat{J} \in \mathcal{K}(X_A, X)$ (resp. $\hat{J} \in \mathcal{W}(X_A, X)$, $\hat{J} \in \mathcal{S}(X_A, X)$, $\hat{J} \in C\mathcal{S}(X_A, X)$).

Let $A\mathcal{K}(X)$, $A\mathcal{W}(X)$, $A\mathcal{S}(X)$ and $AC\mathcal{S}(X)$ denote, respectively, the sets of A-compact, A-weakly compact, A-strictly singular and A-strictly cosingular on X.

Remark 3.5. If J is A-defined and compact (resp. weakly compact, strictly singular, strictly cosingular) then J is A-compact (resp. A-weakly compact, A-strictly singular, A-strictly cosingular).

Definition 3.4. Let X be a Banach space, $A \in \mathcal{C}(X)$, $\rho(A) \neq \varnothing$ and let F be an A-defined linear operator on X. We say that F is an A-resolvent Fredholm perturbation if $(\lambda - \hat{A})^{-1}\hat{F} \in \mathcal{F}^b(X_A)$ for some $\lambda \in \rho(A)$.

Let $A\mathcal{RF}(X)$, designate the set of A-resolvent Fredholm perturbation.

Remark 3.6. Observe that, in the definition of the set $A\mathcal{RF}(X)$, if an operator satisfies the required condition for a fixed $\lambda \in \rho(A)$, then it satisfies it for every $\lambda \in \rho(A)$.

Definition 3.5. Let X be a Banach space, $A \in \mathcal{C}(X)$, F be an A-defined linear operator on X and $\rho(A+F) \neq \varnothing$. We say that F is an A-resolvent Fredholm perturbation which zero index if $(\lambda - \hat{A} - \hat{F})^{-1}\hat{F} \in \mathcal{J}(X_A, X_{A+F})$ for all $\lambda \in \rho(A+F)$.

Let $A\mathcal{J}(X)$ designate the set of A-resolvent Fredholm perturbation which zero index.

Remark 3.7. (i) For all $\lambda \in \rho(A+F)$, the operator $(\lambda - \hat{A} + \hat{F})^{-1} \in \mathcal{L}(X, X_{A+F})$. In fact, let $x \in X$ and put $y = (\lambda - A + F)^{-1}x$. It follows from the estimate

$$\|y\|_{A+F} = \|y\| + \|(\hat{A} + \hat{F})y\|$$

$$
\begin{aligned}
&= \|y\| + \|\lambda y - x\| \\
&= \|(\lambda - \hat{A} - \hat{F})^{-1}x\| + \|\lambda(\lambda - \hat{A} - \hat{F})^{-1}x - x\| \\
&\leq \Big(1 + (1 + |\lambda|)\|(\lambda - \hat{A} - \hat{F})^{-1}\|\Big)\,\|x\|
\end{aligned}
$$

that

$$(\lambda - \hat{A} - \hat{F})^{-1} \in \mathcal{L}(X, X_{A+F}). \tag{3.3}$$

(ii) If F is an A-Fredholm perturbation then F is an A-resolvent Fredholm perturbation (see [17, Remark 1.6 (ii)]).

(iii) If F is an A-resolvent Fredholm perturbation then F is an A-resolvent Fredholm perturbation which zero index. In fact, let $\mathcal{I}$ denote the imbedding operator which maps every $x \in X_A$ onto the same element $x \in X_{A+F}$. Clearly we have $N(\mathcal{I}) = \{0\}$ and $R(\mathcal{I}) = X_{A+F}$. So, the estimate

$$
\begin{aligned}
\|\mathcal{I}(x)\|_{X_{A+F}} = \|x\|_{A+F} &\leq \|x\| + \|Ax\|_X + \|Fx\|_X \\
&\leq \big(1 + \|F\|_{\mathcal{L}(X_A, X)}\big)\,\|x\|_{X_A}, \quad \forall x \in X_A,
\end{aligned}
$$

leads to

$$\mathcal{I} \in \Phi^b(X_A, X_{A+F}) \ \text{ and } \ i(\mathcal{I}) = 0.$$

Let $\mu \in \rho(A)$, we have

$$(\lambda - \hat{A} - \hat{F})^{-1}\hat{F} = [\mathcal{I} + (\lambda - \hat{A} - \hat{F})^{-1}(\mu - \lambda + \hat{F})](\mu - \hat{A})^{-1}\hat{F} \tag{3.4}$$

Since $(\mu - \lambda + \hat{F}) \in \mathcal{L}(X_A, X)$, applying (3.3) we infer that

$$(\lambda - \hat{A} - \hat{F})^{-1}(\mu - \lambda + \hat{F}) \in \mathcal{L}(X_A, X_{A+F}),$$

and therefore

$$\mathcal{I} + (\lambda - \hat{A} - \hat{F})^{-1}(\mu - \lambda + \hat{F}) \in \mathcal{L}(X_A, X_{A+F}).$$

Using (3.4), F is an A-resolvent Fredholm perturbation and Proposition 1.2 (i) we infer that $(\lambda - \hat{A} - \hat{F})^{-1}\hat{F} \in \mathcal{F}^b(X_A, X_{A+F})$. This proves that $F \in A\mathcal{J}(X_A, X_{A+F})$.

(iv) A consequence of Definition 3.5, Remark 3.7 (ii) (iii) and the inclusions in [5, p. 69] that

$$A\mathcal{K}(X) \subset A\mathcal{S}(X) \subset A\mathcal{F}_+(X) \subset A\mathcal{F}(X) \subset A\mathcal{RF}(X) \subset A\mathcal{J}(X),$$

$$A\mathcal{K}(X) \subset AC\mathcal{S}(X) \subset A\mathcal{F}_-(X) \subset A\mathcal{F}(X) \subset A\mathcal{RF}(X) \subset A\mathcal{J}(X).$$

The inclusion $A\mathcal{S}(X) \subset A\mathcal{F}_+(X)$ (resp. $AC\mathcal{S}(X) \subset A\mathcal{F}_-(X)$) was established in [21] (resp. [41]).

Theorem 3.2. *Let X be a Banach space, $A \in \mathcal{C}(X)$. Then*

$$\sigma_{\text{ess}}(A) = \bigcap_{J \in A\mathcal{J}(X)} \sigma(A+J).$$

Remark 3.8. (i) This theorem may be viewed as an extension of [24, Theorem 4.2], [25, Theorem 2.2], [31, Theorem 2.2] and [17, Theorem 2.3].

(ii) For all $K \in A\mathcal{J}(X)$, $\sigma_{\text{ess}}(A+K) = \sigma_{\text{ess}}(A)$.

Proof of Theorem 3.2. Set $\mathcal{O} := \bigcap_{J \in A\mathcal{J}(X)} \sigma(A+J)$. We first claim that $\sigma_{\text{ess}}(A) \subset \mathcal{O}$. Indeed, if $\lambda \notin \mathcal{O}$ then there exists $J \in A\mathcal{J}(X)$ such that $\lambda \in \rho(A+J)$.

Since $\lambda \in \rho(A+J)$, it follows from (3.2) and Remark 3.7 (iii) that

$$(\lambda - \hat{A} - \hat{J}) \in \Phi^b(X_{A+J}, X) \quad \text{and} \quad i(\lambda - \hat{A} - \hat{J}) = 0. \tag{3.5}$$

Writing $\lambda - \hat{A}$ in the form $\lambda - \hat{A} = (\lambda - \hat{A} - \hat{J})(\mathcal{I} + (\lambda - \hat{A} - \hat{J})^{-1}\hat{J})$ and using (3.4), (3.5) and the Atkinson's theorem ([32, Proposition 2.c.7.(ii), p. 77]) one gets $\lambda - \hat{A} \in \Phi^b(X_A, X)$ and $i(\lambda - \hat{A}) = 0$. Now using (3.2) we infer that $\lambda \in \Phi_A$ and $i(\lambda - A) = 0$. Finally, the use of Proposition 1.1, shows that $\lambda \notin \sigma_{\text{ess}}(A)$ which proves the claim.

On the other hand, since $\mathcal{K}(X) \subset A\mathcal{J}(X)$ we infer that $\mathcal{O} \subset \sigma_{\text{ess}}(A)$ which completes the proof of theorem. □

We close this section by the following result:

Corollary 3.5. *Let X be a Banach space, $A \in \mathcal{C}(X)$ and let $\mathcal{M}(X)$ be any subset of $A\mathcal{J}(X)$ (not necessarily an ideal) satisfying the condition*

$$\mathcal{K}(X) \subset \mathcal{M}(X) \subset A\mathcal{J}(X).$$

Then

$$\sigma_{\text{ess}}(A) = \bigcap_{K \in \mathcal{M}(X)} \sigma(A+K).$$

4. Application to Transport Equations

In this section we are concerned with the Schechter essential spectrum of singular transport operators

$$A\psi(x,v) = -v.\nabla_x\psi(x,v) - \sigma(v)\psi(x,v) + \int_{\mathbb{R}^n} \kappa(v,v')\psi(x,v')d\mu(v')$$

with vacuum boundary conditions, i.e., $\psi_{\Gamma_-} = 0$ with

$$\Gamma_- = \{(x,v) \in \partial D \times \mathbb{R}^n \text{ such that } v.\nu_x < 0\},$$

where ν_x stands for the outer unit normal vector at $x \in \partial D$.

Here $x \in D$ and $v \in \mathbb{R}^n$ where D is an open bounded subset of $\mathbb{R}^n$, $d\mu(.)$ is a bounded positive Radon measure on $\mathbb{R}^n$. The functions $\sigma(.)$ and $\kappa(.,.)$ represent, respectively, the collision frequency and the scattering kernel and will be assumed to be unbounded and satisfies the conditions (1.1)–(1.2).

We introduce the different notions and notations which we shall need in the sequel. Let us first make precise the functional setting of the problem:

Let $X_p := L_p(D \times \mathbb{R}^n, dxd\mu(v)),\ \ 1 \leq p < +\infty$,

$X_p^\sigma := L_p(D \times \mathbb{R}^n, \sigma(v)dxd\mu(v))$ and $L_p^\sigma(\mathbb{R}^n) := L_p(\mathbb{R}^n, \sigma(v)d\mu(v))$.

We define the partial Sobolev space W_p by

$$W_p = \{\psi \in X_p \ \text{ such that } \ v.\nabla_x \in X_p\}.$$

Next we introduce the following subspace of W_p by

$$W_p^o = \left\{\psi \in W_p \ \text{ such that } \ \psi_{|\Gamma_-} = 0\right\}.$$

Now, we define the streaming operator T by

$$\begin{cases} T\psi(x,v) = -v.\nabla_x\psi(x,v) - \sigma(v)\psi(x,v), & \psi \in D(T), \\ D(T) = W_p^o \cap X_p^\sigma. \end{cases}$$

The transport operator A can be formulated as follows $A = T + K$, where K is the following collision operator

$$K : \psi \longrightarrow K\psi(v) := \int_{\mathbb{R}^n} \kappa(v,v')\psi(x,v')d\mu(v') \in L_p(\mathbb{R}^n),$$

with $L_p(\mathbb{R}^n) := L_p(\mathbb{R}^n, d\mu(v))$. It follows from the assumption (1.3) that $K \in \mathcal{L}(L_p^\sigma(\mathbb{R}^n), L_p(\mathbb{R}^n))$ and

$$\|K\|_{\mathcal{L}(L_p^\sigma(\mathbb{R}^n),L_p(\mathbb{R}^n))} \leq \left\| \left[\int_{\mathbb{R}^n} \left(\frac{\kappa(.,v')}{\sigma(v')^{1/p}}\right)^q d\mu(v')\right]^{1/q} \right\|_{L_p(\mathbb{R}^n)}.$$

Moreover, using the boundedness of D we find that $K \in \mathcal{L}(X_p^\sigma, X_p)$, with

$$\|K\|_{\mathcal{L}(X_p^\sigma,X_p)} \leq \left\| \left[\int_{\mathbb{R}^n} \left(\frac{\kappa(.,v')}{\sigma(v')^{1/p}} \right)^q d\mu(v') \right]^{1/q} \right\|_{L_p(\mathbb{R}^n)}.$$

Note that a simple calculation using the assumption (1.1) shows that X_p^σ is a subset of X_p and the embedding $X_p^\sigma \hookrightarrow X_p$ is continuous.

Let $\varphi \in X_p$ and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\sigma_0$. We seek ψ in $D(T)$ satisfying

$$(\lambda - T)\psi = \varphi \tag{4.1}$$

The solution of (4.1) reads as follows

$$\psi(x,v) = \int_0^{t^-(x,v)} e^{-(\lambda+\sigma(v))s}\varphi(x - sv, v)ds,$$

where $t^-(x,v) = \sup\{t > 0,\ x - sv \in D,\ 0 < s < t\}$. So, we have $\{\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\sigma_0\} \subset \rho(T)$, where $\rho(T)$ denotes the resolvent set of T.

Since $\sigma(.)$ is bounded below by σ_0, a similar reasoning as in [20, Corollary 12.11, p. 272] shows that

$$\sigma(T) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leq -\sigma_0\}.$$

In fact, by [20, Chapter 12] we can easily check that $\sigma(T)$ is reduced to $\sigma C(T)$ (the continuous spectrum of T), i.e., $\sigma(T) = \sigma C(T)$. Consequently, it follows from Corollary 3.2 and Theorem 3.1 that

$$\sigma_{\text{ess}}(T) = \sigma C(T) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leq -\sigma_0\}.$$

For the details we refer to [31].

Lemma 4.1 **(**[31, Proposition 4.1]**).** *Let D be a bounded subset of $\mathbb{R}^n$ and $1 < p < +\infty$. If, the hypotheses* (1.1) *and* (1.2) *are satisfied, the measure $d\mu$ satisfies*

$$\begin{cases} \textit{the hyperplanes have zero } d\mu \textit{ measure, i.e.,} \\ \textit{for each } e \in S^{n-1},\ d\mu\{v \in \mathbb{R}^n,\ v.e = 0\} = 0 \end{cases}$$

where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ and the collision operator $K : L_p^\sigma(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n)$ is compact, then for any λ satisfying $\operatorname{Re}\lambda > -\sigma_0$, the operator $K(\lambda - T)^{-1}$ is compact on X_p.

Theorem 4.1. *Assume that the hypotheses of Lemma* 4.1 *are satisfied. Then*

(i) *For all* $\lambda \in \rho(T+K)$ *we have* $K(\lambda - T - K)^{-1}$ *is compact on* X_p*;*

(ii) $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(T) = \{\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}\lambda \leq -\sigma_0\}$*.*

Proof. (i) Let $\lambda \in \rho(T+K)$ and $\mu \in \rho(T)$, we have

$$K(\lambda - T - K)^{-1} = K(\mu - T)^{-1}[I + (\mu - \lambda + K)(\lambda - T - K)^{-1}]. \tag{4.2}$$

Using Lemma 4.1 and Eq. (4.2) we have $K(\lambda - T - K)^{-1}$ is compact on X_p.

(ii) The hypothesis on K together with Theorem 4.1 (i) implies that $K \in \mathcal{N}_T(X_p)$. Now the result follows from Theorem 3.1 and Remark 3.3 (b). $\square$

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