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ON SEMICLASSICAL ORTHOGONAL POLYNOMIALS: A QUASI-DEFINITE FUNCTIONAL OF CLASS 1

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Abstract. In this paper the integral representation of any solution of the distributional equation $D\big((x^3-x)u\big)+\big((\mu-2\alpha-s-3)x^2-sx-\mu+1\big)u=0$ is obtained in an alternative and more natural way than the one derived from the method given in [4]. A particular quasi–definite case is studied and some properties for the corresponding sequence of orthogonal polynomials are obtained. Explicit expressions for the moments and for the recurrence coefficients are given using the Laguerre–Freud equations as the basic tool.

1. Introduction

Semiclassical linear functionals are characterized by the distributional equation $D(\phi L)+\psi L=0$ where ϕ and ψ are arbitrary polynomials with $\deg(\psi)\geq 1$. In order to give integral representations for such functionals, F. Marcellán and I. A. Rocha consider in [4], [5] two cases

$$\begin{aligned}(A)\ &:\quad \deg(\phi)>\deg(\psi)\,,\\ (B)\ &:\quad \deg(\phi)\leq\deg(\psi)\,.\end{aligned}$$

The equation

$$D\big((x^3-x)u\big)+\big((\mu-2\alpha-s-3)x^2-sx-\mu+1\big)u=0\,, \tag{1}$$

which is going to be studied, belongs to the case (A) and any solution, as we will see, can be written as

$$\langle u(\mu,s,\alpha),f(x)\rangle=\int_{-1}^{1}w(x)\big(a\cdot\chi_{[-1,0]}(x)+b\cdot\chi_{[0,1]}(x)\big)f(x)\,dx\,, \tag{2}$$

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where $w(x) = |x|^{-\mu}(1-x^2)^\alpha(1-x)^s$, $\mu < 1$, $\alpha > -1$, $s > -1$, and $\chi_{[-1,0]}(x)$ and $\chi_{[0,1]}$ denote the characteristic functions of intervals $[-1, 0]$ and $[0, 1]$ respectively, i.e.

$$\chi_C(x) = 1 \quad \text{if} \quad x \in C\,, \qquad 0 \quad \text{if} \quad x \notin C\,.$$

The notation for the linear functionals as well as the main results of [1] will be used in this paper.

When $\mu = -(2p + 1)$ for nonnegative integer values of p, $a = -k$, and $b = k$, (2) becomes

$$\langle v(p,s), f(x)\rangle = k \int_{-1}^{1} x^{2p+1}(1-x^2)^\alpha(1-x)^s f(x)\,dx\,. \tag{3}$$

In our contribution, in section 3, the moments of this functional are explicitly given for positive integer values of s. Furthermore, in section 4, the coefficients γ_n and β_n of the three-term recurrence relation $xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$ are obtained in cases $s = 1, 2, 3$. Because of coefficients γ_n are negative, $v(p, s)$ are quasi–definite functionals and thus there exists the sequence of orthogonal polynomials.

2. Integral Representation

If u is a solution of (1), its moments, $(u)_n = \langle u, x^n\rangle$, $n = 0, 1, 2\ldots$ satisfy

$$\big[n - (\mu - 2\alpha - s - 3)\big](u)_{n+2} = -s(u)_{n+1} + (1 - \mu + n)(u)_n, \quad n = 0, 1, \ldots\,,$$

and the set of solutions is a 2–dimensional linear space.

Because of $w(x) = |x|^{-\mu}(1-x^2)^\alpha(1-x)^s$ is a solution of the differential equation

$$\big((x^3 - x)w(x)\big)' + \big((\mu - 2\alpha - s - 3)x^2 - sx - \mu + 1\big)w(x) = 0$$

in $(-1, 0)$ and in $(0, 1)$, integration by parts gives that the functionals u_1 and u_2 such that

$$\langle u_1, f(x)\rangle = \int_{-1}^{0} w(x)f(x)\,dx \quad \text{and} \quad \langle u_2, f(x)\rangle = \int_{0}^{1} w(x)f(x)\,dx\,,$$

are solutions of (2). The same is true for

$$\langle au_1 + bu_2, f(x)\rangle = \int_{-1}^{1} w(x)\big(a\cdot\chi_{[-1,0]}(x) + b\cdot\chi_{[0,1]}(x)\big)f(x)\,dx$$

for any constants a and b and for any polynomial $f(x)$.

If the functional $au_1 + bu_2$ is zero for some constants a and b, the Cauchy integral

$$\begin{aligned} F(z) &= \int_{-1}^{1} \frac{w(x)\big(a \cdot \chi_{[-1,0]}(x) + b \cdot \chi_{[0,1]}(x)\big)}{z-x}\, dx \\ &= \sum_{n=0}^{+\infty} \frac{\langle au_1 + bu_2, x^n \rangle}{z^{n+1}} = 0 \end{aligned}$$

and thus the *one-sided limits* of $F(z)$, $F^+(x)$ and $F^-(x)$ are equal to zero. As a consequence, $w(x)\big(a\cdot\chi_{[-1,0]}(x)+b\cdot\chi_{[0,1]}(x)\big) = 0$ for x in $(-1,1)$ which means $a = b = 0$. Then we have:

Theorem 2.1. *A linear functional u is a solution of*

$$D\big((x^3 - x)u\big) + \big((\mu - 2\alpha - s - 3)x^2 - sx - \mu + 1\big)u = 0$$

if and only if

$$\langle u, f(x) \rangle = \int_{-1}^{1} w(x)\big(a \cdot \chi_{[-1,0]}(x) + b \cdot \chi_{[0,1]}(x)\big) f(x)\, dx$$

for any polynomial $f(x)$ and for any constants a and b, where $w(x) = |x|^{-\mu}(1-x^2)^{\alpha}(1-x)^{s}$, $\mu < 1$, $\alpha > -1$, $s > -1$.

Remark. Notice that the functional u is semiclassical of class 1 (see [2]). In another paper [3], the Laguerre–Freud equations for exponential weights are obtained. Later on, in [6], the asymptotics for the recurrence coefficients in case of the simplest generalized Jacobi measure is explored from a numerical point of view. Our contribution is focussed in the use of Laguerre–Freud equations for a signed measure with bounded support related to the Jacobi measure. Thus a quasi–definite functional (which is not positive–definite as in the previous examples) is considered.

3. Moments and Recurrence Coefficients

Now we consider the functional defined in (3) which will be denoted by $u(s)$. More precisely,

$$\langle u(s), f(x) \rangle = c \int_{-1}^{1} x^{2p+1}(1-x^2)^{\alpha}(1-x)^{s} f(x)\, dx\,,$$

with $\alpha > -1$, and p and s nonnegative integer values. The normalization factor c is given by

$$c = -\frac{1}{\displaystyle\sum_{k=0}^{E(\frac{s-1}{2})} \binom{s}{2k+1} B\Big(\frac{2k+2p+3}{2}, \alpha+1\Big)}, \tag{4}$$

where $E(x)$ is the integer part of x and $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$.

For $s = 2m$ we have

$$\begin{aligned} \langle u(2m), 1\rangle &= c\int_{-1}^{1} x^{2p+1}(1-x^2)^{\alpha}(1-x)^{2m}\,dx \\ &= c\int_{-1}^{1} x^{2p+1}(1-x^2)^{\alpha}\sum_{k=0}^{2m}\binom{2m}{k}(-x)^k\,dx \\ &= -2c\int_{0}^{1} x^{2p+1}(1-x^2)^{\alpha}\sum_{k=0}^{m-1}\binom{2m}{2k+1}x^{2k+1}\,dx \\ &= -2c\sum_{k=0}^{m-1}\binom{2m}{2k+1}\int_{0}^{1} x^{2(k+p+1)}(1-x^2)^{\alpha}\,dx \\ &= -c\sum_{k=0}^{m-1}\binom{2m}{2k+1}B\Big(\frac{2k+2p+3}{2}, \alpha+1\Big). \end{aligned} \tag{5}$$

In a similar way, for $s = 2m+1$,

$$\begin{aligned} \langle u(2m+1), 1\rangle &= c\int_{-1}^{1} x^{2p+1}(1-x^2)^{\alpha}(1-x)^{2m+1}\,dx \\ &= -c\sum_{k=0}^{m}\binom{2m+1}{2k+1}B\Big(\frac{2k+2p+3}{2}, \alpha+1\Big) \end{aligned} \tag{6}$$

and from (5) and (6), for nonnegative integer values of s, $(u(s))_0 = 1$ if and only if c is given by (4).

With the same calculations we obtain for the even moments

$$\begin{aligned} \frac{1}{c}\langle u(s), x^{2n}\rangle &= \int_{-1}^{1} x^{2p+1}(1-x^2)^{\alpha}(1-x)^{s}x^{2n}\,dx \\ &= -\sum_{k=0}^{E(\frac{s-1}{2})}\binom{s}{2k+1}B\Big(\frac{2p+2n+2k+3}{2}, \alpha+1\Big), \end{aligned}$$

as well as

$$\begin{aligned}\frac{1}{c}\langle u(s), x^{2n+1}\rangle &= \int_{-1}^{1} x^{2p+1}(1-x^2)^\alpha (1-x)^s x^{2n+1}\, dx \\ &= \sum_{k=0}^{E(\frac{s-1}{2})} \binom{s}{2k} B\Big(\frac{2p+2n+2k+3}{2}, \alpha+1\Big) .\end{aligned}$$

Denoting $\langle u(s), x^k\rangle = ((u(s))_k$, we get:

Proposition 3.1. *The moments of $u(s)$ are given by*

$$(u(s))_{2n} = \frac{\sum_{k=0}^{E(\frac{s-1}{2})} \binom{s}{2k+1} B(\frac{2p+2n+2k+3}{2}, \alpha+1)}{\sum_{k=0}^{E(\frac{s-1}{2})} \binom{s}{2k+1} B(\frac{2p+2k+3}{2}, \alpha+1)} \ , \quad n \geq 0 \, ,$$

$$(u(s))_{2n+1} = -\frac{\sum_{k=0}^{E(\frac{s-1}{2})} \binom{s}{2k} B(\frac{2p+2n+2k+3}{2}, \alpha+1)}{\sum_{k=0}^{E(\frac{s-1}{2})} \binom{s}{2k+1} B(\frac{2p+2k+3}{2}, \alpha+1)} \ , \quad n \geq 0 \, .$$

The following relations also hold:

Proposition 3.2. *The moments of the linear functional $u(s)$ satisfy the recurrence relation*

$$\sum_{k=0}^{s} \binom{s}{k} (u(s))_{2n+k} = 0 \ , \quad n = 0, 1, \dots \, .$$

Proof. Because of

$$\langle u(s), x^{2n}(1+x)^s\rangle = c\int_{-1}^{1} x^{2p+2n+1}(1-x^2)^\alpha (1-x)^s (1+x)^s\, dx = 0 \, ,$$

the formula holds. □

3.1. Laguerre–Freud equations. In this paragraph, we use Laguerre–Freud equations which, with the moments of order one $u(s)_1 = \beta_0$ given in the previous section, allows us to obtain the recurrence coefficients when $s = 1, 2, 3$.

If $D(\phi u(s)) + \psi u(s) = 0$, with $\phi(x) = x^3 - x$ and $\psi(x) = (-2p - 2\alpha - s - 4)x^2 - sx + 2p + 2$, denotes the distributional equation of $u(s)$ then the Laguerre–Freud equations (see [3] formulas (39) and (40) but notice that in (40) there is a misprint in the right hand side, $\beta_k + \beta_{k+1}$ must be read as $\beta_k + \beta_{k-1}$) become

$$
\begin{aligned}
(7) \quad & (-2n - 2\alpha - 2p - s - 4)(\gamma_n + \gamma_{n+1}) \\
& \qquad = 4\sum_{k=1}^{n-1}\gamma_k + 2\sum_{k=0}^{n-1}(\theta_{\beta_n}\phi)(\beta_k) - \psi(\beta_n), \quad n \geq 2,
\end{aligned}
$$

with the initial conditions

$$
\begin{aligned}
(-2p - 2\alpha - s - 6)(\gamma_1 + \gamma_2) &= 2(\theta_{\beta_1}\phi)(\beta_0) - \psi(\beta_1)\,, \\
(-2p - 2\alpha - s - 4)\gamma_1 &= -\psi(\beta_0)
\end{aligned}
$$

as well as

$$
\begin{aligned}
(8) \quad & (-2p-2\alpha - s - 4 - (2n+1))\gamma_{n+1}\beta_{n+1} \\
& = \sum_{k=0}^{n}\phi(\beta_k) + \big(2\gamma_{n+1}(n\beta_n + \sum_{k=0}^{n}\beta_k) + 3\sum_{k=1}^{n}\gamma_k(\beta_k + \beta_{k-1})\big) \\
& \quad - (-2p - 2\alpha - s - 4)\beta_n\gamma_{n+1} - s\gamma_{n+1}\,, \quad n \geq 1\,,
\end{aligned}
$$

with the initial condition

$$
(-2p - 2\alpha - s - 4 - 1)\gamma_1\beta_1 = \phi(\beta_0) + \gamma_1\big(2\beta_0 - (-2p - 2\alpha - s - 4)\beta_0\big)\,.
$$

Here $\theta_{\beta_n}\phi$ means

$$
(\theta_{\beta_n}\phi)(x) = \frac{\phi(x) - \phi(\beta_n)}{x - \beta_n}\,.
$$

Substituting $(\theta_{\beta_n}\phi)(\beta_k)$ in (7) we get

$$\begin{aligned}
&(-2p-1-2n-2\alpha-s-3)\gamma_{n+1}\\
=&-(-2p-1-2n-2\alpha-s-3)\gamma_n+4\sum_{k=1}^{n-1}\gamma_k+2\sum_{k=0}^{n-1}(\beta_n^2+\beta_k^2+\beta_n\beta_k-1)\\
&-(-2p-1-2\alpha-s-3)\beta_n^2+s\beta_n-(2p+2)\\
=&-(-2p-1-2n-2\alpha-s-3)\gamma_n+4\sum_{k=1}^{n-1}\gamma_k+2\sum_{k=0}^{n-1}\beta_k^2+2\beta_n\sum_{k=0}^{n-1}\beta_k\\
&+(2n+2\alpha+s+3+2p+1)\beta_n^2+s\beta_n+(-2p-1-2n-1)\ , \quad n\geq 2\,.
\end{aligned}$$

Shifting n in $n+1$ and replacing again, using notation $E_n=\sum_{k=0}^{n}\beta_k$ we obtain

$$\begin{aligned}
&(-2p-1-2n-2\alpha-s-5)\gamma_{n+2}\\
&=2\gamma_{n+1}+(-2p-1-2n-2\alpha-s+1)\gamma_n+2\beta_{n+1}E_n-2\beta_nE_{n-1}\\
&+(2n+2\alpha+s+5+2p+1)\beta_{n+1}^2\\
&-(2n+2\alpha+s+1+2p+1)\beta_n^2+s(\beta_{n+1}-\beta_n)-2\ , \quad n\geq 1\,.
\end{aligned}\tag{9}$$

Substituting in (8) $\sum_{k=0}^{n}\phi(\beta_k)$ we get, for $n\geq 1$,

$$\begin{aligned}
&(-2p-1-2n-2\alpha-s-4)\gamma_{n+1}\beta_{n+1}\\
&=\sum_{k=0}^{n}\phi(\beta_k)+\big(2\gamma_{n+1}(n\beta_n+\sum_{k=0}^{n}\beta_k)+3\sum_{k=1}^{n}\gamma_k(\beta_k+\beta_{k-1})\big)\\
&-\big((-2p-1-2\alpha-s-3)\beta_n-s\big)\gamma_{n+1}\\
&=\sum_{k=0}^{n}(\beta_k^3-\beta_k)+\big(2\gamma_{n+1}(n\beta_n+\sum_{k=0}^{n}\beta_k)+3\sum_{k=1}^{n}\gamma_k(\beta_k+\beta_{k-1})\big)\\
&-\big((-2p-1-2\alpha-s-3)\beta_n-s\big)\gamma_{n+1}\,.
\end{aligned}\tag{10}$$

From (10), the same technique gives

$$\begin{aligned}
&(-2p-1-2n-2\alpha-s-6)\gamma_{n+2}\beta_{n+2}\\
&=(-2p-1-2n-2\alpha-s-4)\gamma_{n+1}\beta_{n+1}+\beta_{n+1}^3-\beta_{n+1}\\
&+3\gamma_{n+1}(\beta_{n+1}+\beta_n)+2\gamma_{n+2}\big((n+1)\beta_{n+1}+E_{n+1})\big)\\
&-\big(2\gamma_{n+1}(n\beta_n+E_n)\big)-\big((-2p-1-2\alpha-s-3)\beta_{n+1}-s\big)\gamma_{n+2}\\
&+\big((-2p-1-2\alpha-s-3)\beta_n-s\big)\gamma_{n+1}
\end{aligned}\tag{11}$$

for $n \geq 0$. Using (9) and (11), the system becomes

$$\begin{aligned}(12)\qquad &(-2p-1-2n-2\alpha-s-5)\gamma_{n+2}\\ &= 2\gamma_{n+1} + (-2p-1-2n-2\alpha-s+1)\gamma_n\\ &+ 2\beta_{n+1}E_n - 2\beta_n E_{n-1} + (2n+2\alpha+s+5+2p+1)\beta_{n+1}^2\\ &- (2n+2\alpha+s+1+2p+1)\beta_n^2 + s(\beta_{n+1}-\beta_n) - 2\ , \quad n \geq 1\,,\end{aligned}$$

with initial conditions

$$\begin{aligned}&(-2p-1-2\alpha-s-5)(\gamma_1+\gamma_2)\\ &= 2(\beta_1^2+\beta_0\beta_1+\beta_0^2-1) - (-2p-1-2\alpha-s-3)\beta_1^2 + s\beta_1 - 2p - 2\,,\\ &(-2p-1-2\alpha-s-3)\gamma_1\\ &= -(-2p-1-2\alpha-s-3)\beta_0^2 + s\beta_0 - 2p - 2\,,\end{aligned}$$

and

$$\begin{aligned}(13)\qquad &(-2p-1-2n-2\alpha-s-6)\gamma_{n+2}\beta_{n+2}\\ &= \beta_{n+1}^3 - \beta_{n+1} + (2n+2\alpha+s+7+2p+1)\gamma_{n+2}\beta_{n+1}\\ &+ (-2p-1-2n-2\alpha-s-1)\gamma_{n+1}\beta_{n+1}\\ &+ (-2p-1-2n-2\alpha-s)\gamma_{n+1}\beta_n + 2E_{n+1}(\gamma_{n+2}-\gamma_{n+1})\ , \quad n \geq 0\,,\end{aligned}$$

with the initial condition

$$(-2p-1-2\alpha-s-4)\gamma_1\beta_1 = \beta_0^3 - \beta_0 + \gamma_1\big(2\beta_0 - (-2p-1-2\alpha-s-3)\beta_0 + s\big)\,.$$

This nonlinear system of equations allows us to obtain the coefficients of the three–term recurrence relation.

Proposition 3.3. *For $s = 1$, the recurrence coefficients are given by*

$$\begin{aligned}\beta_n &= (-1)^{n+1}\ , \quad n \geq 0\,,\\ \gamma_{2n+1} &= -2\frac{(n+\alpha+1)(2n+2\alpha+2p+3)}{(4n+2\alpha+2p+3)(4n+2\alpha+2p+5)}\ , \quad n \geq 0\,,\\ \gamma_{2n+2} &= -\frac{(2n+2)(2n+2p+3)}{(4n+2\alpha+2p+5)(4n+2\alpha+2p+7)}\ , \quad n \geq 0\,.\end{aligned}$$

Proof. From (12) one has

$$\gamma_1 = \frac{-(-2p-2\alpha-5)\beta_0^2 + \beta_0 - 2p - 2}{-2p-2\alpha-5} = -\frac{2\alpha+2}{2p+2\alpha+5}$$

because $\beta_0 = -1$. We also have

$$\beta_1 = \frac{\beta_0^3 - \beta_0 + \gamma_1\bigl(-(-2p-2\alpha-7)\beta_0 + 1\bigr)}{(-2p-2\alpha-6)\gamma_1} = 1\,.$$

The same formula gives

$$\begin{aligned}\gamma_2 &= -\gamma_1 + \frac{2(\beta_1^2 + \beta_0\beta_1 + \beta_0^2 - 1) - (-2p-2\alpha-5)\beta_1^2 + \beta_1 - (2p+2)}{-2p-2\alpha-7} \\ &= -\frac{2(2p+3)}{(2\alpha+2p+5)(2\alpha+2p+7)}\,.\end{aligned}$$

In order to calculate β_2, from (13) we have

$$\begin{aligned}&-(2p+2\alpha+8)\gamma_2\beta_2 \\ &= \beta_1^3 - \beta_1 + (2\alpha+2p+9)\gamma_2\beta_1 - (2p+2\alpha+3)\gamma_1\beta_1 \\ &- (2p+2\alpha+2)\gamma_1\beta_0 + (2\beta_0+1)(\gamma_2-\gamma_1) \\ &= (2\alpha+2p+9)\gamma_2 - (2p+2\alpha+3)\gamma_1 + (2p+2\alpha+2)\gamma_1 - (\gamma_2-\gamma_1) \\ &= (2\alpha+2p+8)\gamma_2\end{aligned}$$

and $\beta_2 = -1$ follows.

From (9),

$$\gamma_3 = -2(\alpha+2)\frac{2\alpha+2p+5}{(2\alpha+2p+7)(2\alpha+2p+9)}$$

and now with (12) we obtain $\beta_3 = 1$. Using (9) again

$$\gamma_4 = -\frac{4(2p+5)}{(2\alpha+2p+9)(2\alpha+2p+11)}\,.$$

Assume that, for $0 \le k \le n-1$,

$$\begin{aligned}\beta_k &= (-1)^{k+1}\,, \\ \gamma_{2k+1} &= -2\frac{(k+\alpha+1)(2k+2\alpha+2p+3)}{(4k+2\alpha+2p+3)(4k+2\alpha+2p+5)}\,, \\ \gamma_{2k+2} &= -\frac{(2k+2)(2k+2p+3)}{(4k+2\alpha+2p+5)(4k+2\alpha+2p+7)}\,.\end{aligned}$$

Replacing n by $2n$ and for $2n+1$ in (12) we get

$$\begin{aligned}&-(2p+2\alpha+4n+7)\gamma_{2n+2}\\&=2\gamma_{2n+1}-(2p+2\alpha+4n+1)\gamma_{2n}+2\beta_{2n+1}E_{2n}-2\beta_{2n}E_{2n-1}\\&+(4n+2p+2\alpha+7)\beta_{2n+1}^2-(4n+2p+2\alpha+3)\beta_{2n}^2+(\beta_{2n+1}-\beta_{2n})-2\end{aligned}$$

and

$$\begin{aligned}&-(2p+2\alpha+4n+9)\gamma_{2n+3}\\&=2\gamma_{2n+2}-(2p+2\alpha+4n+3)\gamma_{2n+1}+2\beta_{2n+2}E_{2n+1}-2\beta_{2n+1}E_{2n}\\&+(4n+2p+2\alpha+9)\beta_{2n+2}^2-(4n+2p+2\alpha+5)\beta_{2n+1}^2\\&+(\beta_{2n+2}-\beta_{2n+1})-2\,.\end{aligned}$$

By induction, $E_{2k}=-1$ and $E_{2k+1}=0$. Moreover, replacing n by $2n$ in (13) and using MAPLE (see Appendix 1) induction gives

$$\begin{aligned}&(-2p-2\alpha-4n-8)\gamma_{2n+2}\beta_{2n+2}\\&=\beta_{2n+1}^3-\beta_{2n+1}+(2p+2\alpha+4n+9)\beta_{2n+1}\gamma_{2n+2}\,,\\&-(2p+2\alpha+4n+3)\beta_{2n+1}\gamma_{2n+1}\\&-(2p+2\alpha+4n+2)\beta_{2n}\gamma_{2n+1}+(2E_{2n}+1)(\gamma_{2n+2}-\gamma_{2n+1})\\&=(2p+2\alpha+4n+8)\gamma_{2n+2}\,,\end{aligned}$$

and $\beta_{2n+2}=-1$ follows.

In the same way, replacing n by $2n+2$ in (13),

$$\begin{aligned}&(-2p-2\alpha-4n-10)\gamma_{2n+3}\beta_{2n+3}\\&=\beta_{2n+2}^3-\beta_{2n+2}+(2p+2\alpha+4n+11)\beta_{2n+2}\gamma_{2n+3}\,,\\&-(2p+2\alpha+4n+5)\beta_{2n+2}\gamma_{2n+2}\\&-(2p+2\alpha+4n+4)\beta_{2n+1}\gamma_{2n+2}+(2E_{2n+1}+1)(\gamma_{2n+3}-\gamma_{2n+2})\\&=-(2p+2\alpha+4n+10)\gamma_{2n+3}\,.\end{aligned}$$

Then $\beta_{2n+3}=1$.

Coefficients γ_n are obtained with MAPLE (see Appendix 1). □

For $s=2$ and $s=3$, also with MAPLE in Appendix 2 and 3, we have

Proposition 3.4. *For $s = 2$, the recurrence coefficients are given by*

$$\begin{aligned}
\beta_{2n} &= -\frac{4n+\alpha+2p+4}{4n+2\alpha+2p+5} - \frac{2n(2n+2p+1)}{(\alpha+1)(4n+2\alpha+2p+3)}\,, \quad n \geq 0\,,\\
\beta_{2n+1} &= \frac{4n+\alpha+2p+4}{4n+2\alpha+2p+5} + \frac{(2n+2)(2n+2p+3)}{(\alpha+1)(4n+2\alpha+2p+7)}\,, \quad n \geq 0\,,\\
\gamma_{2n} &= -\frac{4n(2n+2p+1)(n+\alpha+1)(2n+2\alpha+2p+3)}{(\alpha+1)^2(4n+2\alpha+2p+3)^2}\,, \quad n \geq 1\,,\\
\gamma_{2n+1} &= -\frac{(\alpha+1)^2}{(4n+2\alpha+2p+5)^2}\,, \quad n \geq 0\,.
\end{aligned}$$

Proposition 3.5. *For $s = 3$, the recurrence coefficients are given by*

$$\begin{aligned}
E_{2n} &= -\frac{4(n+\alpha+2)p+\alpha^2+(8n+9)\alpha+4n^2+18n+14}{4(n+\alpha+2)p+3\alpha^2+(8n+15)\alpha+4n^2+18n+18}\,, \quad n \geq 0\,,\\
E_{2n+1} &= -\frac{4(n+1)(2n+2p+3)}{4(n+1)p-\alpha^2-3\alpha+4n^2+10n+4}\,, \quad n \geq 0\,,\\
\gamma_{2n+1} &= -\frac{2(n+\alpha+2)(2n+2\alpha+2p+5)}{(4n+2\alpha+2p+3)(4n+2\alpha+2p+5)} \cdot X\,, \quad n \geq 0\,,\\
\gamma_{2n} &= -\frac{2n(2n+2p+1)}{(4n+2\alpha+2p+5)(4n+2\alpha+2p+7)} \cdot Y\,, \quad n \geq 1\,,
\end{aligned}$$

where X and Y denote

$$\begin{aligned}
X &= \frac{(4n+2p+1-\tau)(4n+2p+1+\tau)}{(4n+2p+2\alpha+9-\tau)^2(4n+2p+2\alpha+9+\tau)^2}\\
&\quad \times (4n+2p+5-\tau)(4n+2p+5-\tau)\,,\\
Y &= \frac{(4n+2p+2\alpha+5-\tau)(4n+2p+2\alpha+5+\tau)}{(4n+2p+1-\tau)^2(4n+2p+2\alpha+1+\tau)^2}\\
&\quad \times (4n+2p+9-\tau)(4n+2p+2\alpha+9-\tau)
\end{aligned}$$

and $\tau = \sqrt{4p^2+4\alpha^2+12\alpha+4p+9}$.

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REFERENCES

1. M. J. Atia: *An example of non-symmetric semi-classical form of class s=1. Generalization of a case of Jacobi sequence.* Int. J. Math. and Math. Sci. **24**(10) (2000), 673–689.
2. S. Belmehdi: *On semi–classical linear functionals of class s=1. Classification and integral representations.* Indag. Mathem. New Series **3**(3) (1992), 253–275.
3. S. Belmehdi and A. Ronveaux: *Laguerre–Freud's equations for the recurrence coefficients of semi–classical orthogonal polynomials.* J. of Approx. Theory **76** (1994), 351–368.
4. F. Marcellan and I. A. Rocha: *On semiclassical linear functionals: Integral representations.* J. Comp. Appl. Math. **57** (1995), 239–249.
5. F. Marcellan and I. A. Rocha: *Complex path integral representation for semiclassical linear functionals.* J. of Approx. Theory **94** (1998), 107–127.
6. A. P. Magnus: *Asymptotics for the simplest generalized Jacobi polynomials recurrence coefficients from Freud's equations: numerical explorations.* Annals of Num. Math. **2** (1995), 311–325.

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Appendix 1

```
> restart;                                    ANNEX1:
> mu:=-2*p-1;beta0:=-1;
```

$$\mu := -2\,p - 1$$

$$\beta 0 := -1$$

```
> gamma1:=factor(simplify(1/(mu-2*alpha-4)*((2*alpha+4-mu)*beta0^2+beta
> 0+mu-1)));
```

$$\gamma 1 := -2\,\frac{\alpha + 1}{2\,\alpha + 5 + 2\,p}$$

```
> beta1:=collect(factor(simplify(1/((mu-2*alpha-5)*gamma1)*(beta0^3-bet
> a0+gamma1*(-(mu-2*alpha-6)*beta0+1)))),p);E1:=collect(simplify(beta0+b
> eta1),p);
```

$$\beta 1 := 1$$

$$E1 := 0$$

```
> gamma2:=factor(simplify(-gamma1+1/(mu-2*alpha-6)*(2*beta1^2+2*beta0*b
> eta1+2*beta0^2-2-(mu-2*alpha-4)*beta1^2+beta1-1+mu)));
```

$$\gamma 2 := -2\,\frac{2\,p + 3}{(2\,\alpha + 5 + 2\,p)\,(2\,p + 7 + 2\,\alpha)}$$

```
> beta2:=collect(factor(simplify(1/((mu-2*alpha-4-3)*gamma2)*(beta1^3-b
> eta1+(4-mu+2*alpha+4)*beta1*gamma2+(2+mu-2*alpha-4)*beta1*gamma1+(3+mu
> -2*alpha-4)*beta0*gamma1+(2*beta0+1)*(gamma2-gamma1)))),p);E2:=collect
> (simplify(beta2+E1),p);
```

$$\beta 2 := -1$$

$$E2 := -1$$

```
> gamma3:=factor(simplify(1/(mu-2*alpha-8)*(2*gamma2+(mu-2*alpha-2)*gam
> ma1+2*beta2*E1-2*beta1*beta0+(
> 8-mu+2*alpha)*beta2^2-(2*alpha+4-mu)*beta1^2+beta2-beta1-2)));
```

$$\gamma 3 := -2\,\frac{(2 + \alpha)\,(2\,\alpha + 5 + 2\,p)}{(2\,p + 7 + 2\,\alpha)\,(9 + 2\,p + 2\,\alpha)}$$

```
> beta3:=collect(factor(simplify(1/((mu-2*alpha-4-5)*gamma3)*(beta2^3-b
> eta2+(6-mu+2*alpha+4)*beta2*gamma3+(mu-2*alpha-4)*beta2*gamma2+(1+mu-2
> *alpha-4)*beta1*gamma2+(2*E1+1)*(gamma3-gamma2)))),p);E3:=collect(simp
> lify(beta3+E2),p);
```

$$\beta 3 := 1$$

$$E3 := 0$$

```
> gamma4:=factor(simplify(1/(mu-2*alpha-10)*(2*gamma3+(mu-2*alpha-4)*ga
> mma2+2*beta3*E2-
> 2*beta2*E1+(10-mu+2*alpha)*beta3^2-(2*alpha+6-mu)*beta2^2
> +beta3-beta2-2)));
```

$$\gamma 4 := -4\,\frac{5 + 2\,p}{(9 + 2\,p + 2\,\alpha)\,(11 + 2\,p + 2\,\alpha)}$$

```
> beta4:=collect(factor(simplify(1/((mu-2*alpha-4-4*1-3)*gamma4)*(beta3
> ^3-beta3+(4*1+4-mu+2*alpha+4)*beta3*gamma4+(-4*1+2+mu-2*alpha-4)*beta3
> *gamma3+(-4*1+3+mu-2*alpha-4)*beta2*gamma3+(2*E2+1)*(gamma4-gamma3))))
> ,p);
```

$$\beta 4 := -1$$

We suppose that:

```
> gamma2n:=-2*n*(2*n+2*p+1)/((4*n+2*alpha+2*p+1)*(4*n+2*alpha+2*p+3));
```

$$gamma2n := -2\,\frac{n\,(2\,n+2\,p+1)}{(4\,n+2\,\alpha+2\,p+1)\,(4\,n+2\,\alpha+2\,p+3)}$$

```
> gamma2n :=
> -2*n*(2*n+2*p+1)/((4*n+2*alpha+2*p+1)*(4*n+2*alpha+2*p+3));
```

$$gamma2n := -2\,\frac{n\,(2\,n+2\,p+1)}{(4\,n+2\,\alpha+2\,p+1)\,(4\,n+2\,\alpha+2\,p+3)}$$

```
> gamma2np1:=-2*(n+alpha+1)*(2*n+2*alpha+2*p+3)/((4*n+2*alpha+2*p+3)*(4
> *n+2*alpha+2*p+5));
```

$$gamma2np1 := -2\,\frac{(n+\alpha+1)\,(2\,n+2\,\alpha+2\,p+3)}{(4\,n+2\,\alpha+2\,p+3)\,(4\,n+2\,\alpha+2\,p+5)}$$

```
> beta2n:=-1;
```

$$beta2n := -1$$

```
> beta2np1:=1;
```

$$beta2np1 := 1$$

```
> E2n:=-1;
```

$$E2n := -1$$

```
> E2np1:=0;E2nm1:=0;
```

$$E2np1 := 0$$

$$E2nm1 := 0$$

and we prove by reccurence:

```
> gamma2np2:=simplify(factor(simplify(1/(mu-2*alpha-4*n-6)*(2*gamma2np1
> +(mu-2*alpha-4*n)*gamma2n+2*beta2np1*E2n-2*beta2n*E2nm1+(4*n+6-mu+2*al
> pha)*beta2np1^2-(4*n+2*alpha+2-mu)*beta2n^2+beta2np1-beta2n-2))));
```

$$gamma2np2 := -2\,\frac{(n+1)\,(2\,n+2\,p+3)}{(4\,n+2\,\alpha+2\,p+5)\,(4\,n+7+2\,p+2\,\alpha)}$$

```
> beta2np2:=factor(simplify(1/((mu-2*alpha-4-4*n-3)*gamma2np2)*(beta2np
> 1^3-beta2np1+(4*n+4-mu+2*alpha+4)*beta2np1*gamma2np2+(-4*n+2+mu-2*alph
> a-4)*beta2np1*gamma2np1+(-4*n+3+mu-2*alpha-4)*beta2n*gamma2np1+(2*E2n+
> 1)*(gamma2np2-gamma2np1))));
```

$$beta2np2 := -1$$

```
> gamma2np3:=factor(simplify(1/(mu-2*alpha-4*n-8)*(2*gamma2np2+(mu-2*al
> pha-4*n-2)*gamma2np1+2*beta2np2*E2np1-2*beta2np1*E2n+(4*n+8-mu+2*alpha
> )*beta2np2^2-(4*n+2*alpha+4-mu)*beta2np1^2+beta2np2-beta2np1-2)));
```

$$gamma2np3 := -2\,\frac{(2+n+\alpha)\,(2\,\alpha+2\,p+2\,n+5)}{(4\,n+7+2\,p+2\,\alpha)\,(4\,n+9+2\,p+2\,\alpha)}$$

```
> beta2np3:=(factor(simplify(1/((mu-2*alpha-4-4*n-5)*gamma2np3)*(beta2n
> p2^3-beta2np2+(4*n+6-mu+2*alpha+4)*beta2np2*gamma2np3+(-4*n+mu-2*alpha
> -4)*beta2np2*gamma2np2+(-4*n+1+mu-2*alpha-4)*beta2np1*gamma2np2+(2*E2n
> p1+1)*(gamma2np3-gamma2np2)))));
```

$$beta2np3 := 1$$

Appendix 2

```
> restart;                                    ANNEX2:
> mu:=-2*p-1;beta0:=-(alpha+2*p+4)/(2*alpha+2*p+5);
```

$$\mu := -2\,p - 1$$

$$\beta 0 := -\frac{\alpha + 2\,p + 4}{2\,\alpha + 2\,p + 5}$$

```
> gamma1:=factor(simplify(1/(mu-2*alpha-5)*((2*alpha+5-mu)*beta0^2+2*be
> ta0+mu-1)));
```

$$\gamma 1 := -\frac{(\alpha + 1)^2}{(2\,\alpha + 2\,p + 5)^2}$$

```
> beta1:=collect(factor(simplify(1/((mu-2*alpha-6)*gamma1)*(beta0^3-bet
> a0+gamma1*(-(mu-2*alpha-7)*beta0+2)))),p);E1:=collect(simplify(beta0+b
> eta1),p);
```

$$\beta 1 := \frac{(4\,\alpha + 12)\,p^2 + (36\,\alpha + 6\,\alpha^2 + 54)\,p + 2\,\alpha^3 + 17\,\alpha^2 + 58 + 55\,\alpha}{(2\,\alpha + 2\,p + 5)\,(\alpha + 1)\,(2\,p + 7 + 2\,\alpha)}$$

$$E1 := 2\,\frac{2\,p + 3}{(\alpha + 1)\,(2\,p + 7 + 2\,\alpha)}$$

```
> gamma2:=factor(simplify(-gamma1+1/(mu-2*alpha-7)*(2*beta1^2+2*beta0*b
> eta1+2*beta0^2-2-(mu-2*alpha-5)*beta1^2+2*beta1-1+mu)));
```

$$\gamma 2 := -4\,\frac{(2 + \alpha)\,(2\,p + 3)\,(2\,\alpha + 2\,p + 5)}{(2\,p + 7 + 2\,\alpha)^2\,(\alpha + 1)^2}$$

```
> beta2:=collect(factor(simplify(1/((mu-2*alpha-5-3)*gamma2)*(beta1^3-b
> eta1+(4-mu+2*alpha+5)*beta1*gamma2+(2+mu-2*alpha-5)*beta1*gamma1+(3+mu
> -2*alpha-5)*beta0*gamma1+(2*beta0+2)*(gamma2-gamma1)))),p);E2:=collect
> (simplify(beta2+E1),p);
```

$$\beta 2 := -\frac{(4\,\alpha + 12)\,p^2 + (78 + 6\,\alpha^2 + 44\,\alpha)\,p + 110 + 25\,\alpha^2 + 91\,\alpha + 2\,\alpha^3}{(2\,p + 7 + 2\,\alpha)\,(\alpha + 1)\,(2\,p + 9 + 2\,\alpha)}$$

$$E2 := -\frac{\alpha + 2\,p + 8}{2\,p + 9 + 2\,\alpha}$$

```
> gamma3:=factor(simplify(1/(mu-2*alpha-9)*(2*gamma2+(mu-2*alpha-3)*gam
> ma1+2*beta2*E1-2*beta1*beta0+(
> 9-mu+2*alpha)*beta2^2-(2*alpha+5-mu)*beta1^2+2*(beta2-beta1)-2)));
```

$$\gamma 3 := -\frac{(\alpha + 1)^2}{(2\,p + 9 + 2\,\alpha)^2}$$

```
> beta3:=collect(factor(simplify(1/((mu-2*alpha-5-5)*gamma3)*(beta2^3-b
> eta2+(6-mu+2*alpha+5)*beta2*gamma3+(mu-2*alpha-5)*beta2*gamma2+(1+mu-2
> *alpha-5)*beta1*gamma2+(2*E1+2)*(gamma3-gamma2)))),p);E3:=collect(simp
> lify(beta3+E2),p);
```

$$\beta 3 := \frac{(20 + 4\,\alpha)\,p^2 + (6\,\alpha^2 + 60\,\alpha + 150)\,p + 2\,\alpha^3 + 268 + 29\,\alpha^2 + 155\,\alpha}{(2\,p + 9 + 2\,\alpha)\,(\alpha + 1)\,(2\,p + 11 + 2\,\alpha)}$$

$$E3 := 4\,\frac{2\,p + 5}{(\alpha + 1)\,(2\,p + 11 + 2\,\alpha)}$$

```
> gamma4:=factor(simplify(1/(mu-2*alpha-11)*(2*gamma3+(mu-2*alpha-5)*ga
> mma2+2*beta3*E2-
> 2*beta2*E1+(11-mu+2*alpha)*beta3^2-(2*alpha+7-mu)*beta2^2+2*(beta3-bet
> a2)-2)));
```

$$\gamma 4 := -8\,\frac{(\alpha+3)\,(2\,p+5)\,(2\,p+7+2\,\alpha)}{(2\,p+11+2\,\alpha)^2\,(\alpha+1)^2}$$

We suppose that:

```
> gamma2n:= -4*n*
> (2*n+2*p+1)*(n+alpha+1)*(2*n+2*alpha+2*p+3)/((alpha+1)^2*(4*n+2*alpha+
> 2*p+3)^2);
```

$$\mathit{gamma2n} := -4\,\frac{n\,(2\,n+2\,p+1)\,(n+\alpha+1)\,(2\,n+2\,\alpha+2\,p+3)}{(\alpha+1)^2\,(4\,n+2\,\alpha+2\,p+3)^2}$$

```
> gamma2np1:=-(alpha+1)^2/((4*n+2*alpha+2*p+5)^2);
```

$$\mathit{gamma2np1} := -\frac{(\alpha+1)^2}{(4\,n+2\,\alpha+2\,p+5)^2}$$

```
> beta2n:=-(4*n+alpha+2*p+4)/(4*n+2*alpha+2*p+5)-2*n*(2*n+2*p+1)/((alph
> a+1)*(4*n+2*alpha+2*p+3));
```

$$\mathit{beta2n} := -\frac{4\,n+\alpha+2\,p+4}{4\,n+2\,\alpha+2\,p+5} - 2\,\frac{n\,(2\,n+2\,p+1)}{(\alpha+1)\,(4\,n+2\,\alpha+2\,p+3)}$$

```
> beta2np1:=(4*n+alpha+2*p+4)/(4*n+2*alpha+2*p+5)+(2*n+2)*(2*n+2*p+3)/(
> (alpha+1)*(4*n+2*alpha+2*p+7));
```

$$\mathit{beta2np1} := \frac{4\,n+\alpha+2\,p+4}{4\,n+2\,\alpha+2\,p+5} + \frac{(2\,n+2)\,(2\,n+2\,p+3)}{(\alpha+1)\,(4\,n+2\,\alpha+2\,p+7)}$$

```
> E2np1:=(2*n+2)*(2*n+2*p+3)/((alpha+1)*(4*n+2*alpha+2*p+7));
```

$$\mathit{E2np1} := \frac{(2\,n+2)\,(2\,n+2\,p+3)}{(\alpha+1)\,(4\,n+2\,\alpha+2\,p+7)}$$

```
> E2nm1:=2*n*(2*n+2*p+1)/((alpha+1)*(4*n+2*alpha+2*p+3));
```

$$\mathit{E2nm1} := 2\,\frac{n\,(2\,n+2\,p+1)}{(\alpha+1)\,(4\,n+2\,\alpha+2\,p+3)}$$

```
> E2n:=factor(simplify(E2np1-beta2np1));
```

$$\mathit{E2n} := -\frac{4\,n+\alpha+2\,p+4}{4\,n+2\,\alpha+2\,p+5}$$

and we prove by reccurrence:

```
> gamma2np2:=factor(simplify(1/(mu-2*alpha-4*n-7)*(2*gamma2np1+(mu-2*al
> pha-4*n-1)*gamma2n+2*beta2np1*E2n-2*beta2n*E2nm1+(4*n+7-mu+2*alpha)*be
> ta2np1^2-(4*n+2*alpha+3-mu)*beta2n^2+2*beta2np1-2*beta2n-2)));
```

$$\mathit{gamma2np2} := -4\,\frac{(n+1)\,(n+2+\alpha)\,(2\,n+2\,p+3)\,(2\,p+2\,n+2\,\alpha+5)}{(4\,n+2\,\alpha+2\,p+7)^2\,(\alpha+1)^2}$$

```
> beta2np2:=(factor(simplify(1/((mu-2*alpha-5-4*n-3)*gamma2np2)*(beta2n
> p1^3-beta2np1+(4*n+4-mu+2*alpha+5)*beta2np1*gamma2np2+(-4*n+2+mu-2*alp
> ha-5)*beta2np1*gamma2np1+(-4*n+3+mu-2*alpha-5)*beta2n*gamma2np1+(2*E2n
> +2)*(gamma2np2-gamma2np1)))));
> E2np2:=factor(beta2np2+E2np1);
```

$$beta2np2 := -(12\,p^2 + 4\,\alpha\,p^2 + 8\,n\,p^2 + 24\,n^2\,p + 88\,n\,p + 44\,\alpha\,p + 24\,n\,\alpha\,p + 78\,p + 6\,p\,\alpha^2 + 92\,n^2 + 110 + 174\,n + 25\,\alpha^2 + 91\,\alpha + 2\,\alpha^3 + 24\,n^2\,\alpha + 12\,n\,\alpha^2 + 16\,n^3 + 92\,n\,\alpha)/((4\,n + 2\,\alpha + 2\,p + 7)\,(\alpha + 1)\,(2\,p + 9 + 2\,\alpha + 4\,n))$$

$$E2np2 := -\frac{\alpha + 4\,n + 2\,p + 8}{2\,p + 9 + 2\,\alpha + 4\,n}$$

```
> gamma2np3:=factor(simplify(1/(mu-2*alpha-4*n-9)*(2*gamma2np2+(mu-2*al
> pha-4*n-3)*gamma2np1+2*beta2np2*E2np1-2*beta2np1*E2n+(4*n+9-mu+2*alpha
> )*beta2np2^2-(4*n+2*alpha+5-mu)*beta2np1^2+2*beta2np2-2*beta2np1-2)));
```

$$gamma2np3 := -\frac{(\alpha + 1)^2}{(2\,p + 9 + 2\,\alpha + 4\,n)^2}$$

```
> beta2np3:=collect(factor(simplify(1/((mu-2*alpha-5-4*n-5)*gamma2np3)*
> (beta2np2^3-beta2np2+(4*n+6-mu+2*alpha+5)*beta2np2*gamma2np3+(-4*n+mu-
> 2*alpha-5)*beta2np2*gamma2np2+(-4*n+1+mu-2*alpha-5)*beta2np1*gamma2np2
> +(2*E2np1+2)*(gamma2np3-gamma2np2)))),p);
> E2np3:=factor(beta2np3+E2np2);
```

$$beta2np3 := ((20 + 4\,\alpha + 8\,n)\,p^2 + (6\,\alpha^2 + 24\,n\,\alpha + 120\,n + 150 + 60\,\alpha + 24\,n^2)\,p + 29\,\alpha^2 + 2\,\alpha^3 + 155\,\alpha + 124\,n^2 + 24\,n^2\,\alpha + 124\,n\,\alpha + 12\,n\,\alpha^2 + 268 + 318\,n + 16\,n^3)/((2\,p + 9 + 2\,\alpha + 4\,n)\,(\alpha + 1)\,(2\,p + 11 + 2\,\alpha + 4\,n))$$

$$E2np3 := 2\,\frac{(n + 2)\,(2\,n + 2\,p + 5)}{(\alpha + 1)\,(2\,p + 11 + 2\,\alpha + 4\,n)}$$

Appendix 3

```
> restart;            ANNEX3:
> beta:=(4*p^2+4*p+9+4*alpha^2+12*alpha)^(1/2);
> mu:=-2*p-1:;beta0:=factor(simplify(-(1+3*(2*p+3)/(2*p+2*alpha+5))/(3+(
> 2*p+3)/(2*p+2*alpha+5))));
```

$$\beta := \sqrt{4\,p^2 + 4\,p + 9 + 4\,\alpha^2 + 12\,\alpha}$$

$$\beta 0 := -\frac{4\,p + \alpha + 7}{4\,p + 3\,\alpha + 9}$$

```
> gamma1:=factor(simplify(1/(mu-2*alpha-6)*(-(mu-2*alpha-6)*beta0^2+3*b
> eta0-1+mu)));
```

$$\gamma 1 := 2\,\frac{(1+\alpha)\,(-\alpha^2 - 3\,\alpha + 4 + 4\,p)}{(2\,p + 7 + 2\,\alpha)\,(4\,p + 3\,\alpha + 9)^2}$$

```
> beta1:=collect(factor(simplify(1/((mu-2*alpha-7)*gamma1)*(beta0^3-bet
> a0+gamma1*(-(mu-2*alpha-8)*beta0+3)))),a):E1:=collect(simplify(beta0+b
> eta1),a);
```

$$E1 := -4\,\frac{2\,p + 3}{-\alpha^2 - 3\,\alpha + 4 + 4\,p}$$

```
> gamma2:=factor(simplify(-gamma1+1/(mu-2*alpha-8)*(2*beta1^2+2*beta0*b
> eta1+2*beta0^2-2-(mu-2*alpha-6)*beta1^2+3*beta1-1+mu)));
```

$$\gamma 2 := -2\,\frac{(\alpha+2)\,(2\,p+3)\,(4\,p\,\alpha + 12\,p + 23\,\alpha + 40 + 3\,\alpha^2)\,(4\,p + 3\,\alpha + 9)}{(-\alpha^2 - 3\,\alpha + 4 + 4\,p)^2\,(2\,p + 9 + 2\,\alpha)\,(2\,p + 7 + 2\,\alpha)}$$

```
> beta2:=collect(factor(simplify(1/((mu-2*alpha-6-3)*gamma2)*(beta1^3-b
> eta1+(4-mu+2*alpha+6)*beta1*gamma2+(2+mu-2*alpha-6)*beta1*gamma1+(3+mu
> -2*alpha-6)*beta0*gamma1+(2*beta0+3)*(gamma2-gamma1)))),a):E2:=collect
> (simplify(beta2+E1),a);
```

$$E2 := -\frac{4\,p\,\alpha + 12\,p + \alpha^2 + 17\,\alpha + 36}{4\,p\,\alpha + 12\,p + 23\,\alpha + 40 + 3\,\alpha^2}$$

```
> gamma3:=factor(simplify(1/(mu-2*alpha-10)*(2*gamma2+(mu-2*alpha-4)*ga
> mma1+2*beta2*E1-2*beta1*beta0+(10-mu+2*alpha)*beta2^2-(2*alpha+6-mu)*b
> eta1^2+3*beta2-3*beta1-2)));
```

$$\gamma 3 := -2\,\frac{(\alpha+3)\,(2\,p + 7 + 2\,\alpha)\,(8\,p + 18 - 3\,\alpha - \alpha^2)\,(-\alpha^2 - 3\,\alpha + 4 + 4\,p)}{(4\,p\,\alpha + 12\,p + 23\,\alpha + 40 + 3\,\alpha^2)^2\,(2\,p + 9 + 2\,\alpha)\,(11 + 2\,p + 2\,\alpha)}$$

```
> beta3:=collect(factor(simplify(1/((mu-2*alpha-6-5)*gamma3)*(beta2^3-b
> eta2+(6-mu+2*alpha+6)*beta2*gamma3+(mu-2*alpha-6)*beta2*gamma2+(1+mu-2
> *alpha-6)*beta1*gamma2+(2*E1+3)*(gamma3-gamma2)))),a):
> E3:=collect(simplify(beta3+E2),a);
```

$$E3 := -8\,\frac{2\,p + 5}{8\,p + 18 - 3\,\alpha - \alpha^2}$$

```
> gamma4:=factor(simplify(1/(mu-2*alpha-4-8)*(2*gamma3+(mu-2*alpha-4-2)
> *gamma2+2*beta3*E2-
> 2*beta2*E1+(4+8-mu+2*alpha)*beta3^2-(4+2*alpha+4-mu)*beta2^2
> +3*beta3-3*beta2-2)));
```

$$\gamma 4 := -4\,\frac{(2\,p+5)\,(3\,\alpha^2 + 4\,p\,\alpha + 31\,\alpha + 16\,p + 70)\,(4\,p\,\alpha + 12\,p + 23\,\alpha + 40 + 3\,\alpha^2)}{(8\,p + 18 - 3\,\alpha - \alpha^2)^2\,(11 + 2\,p + 2\,\alpha)\,(13 + 2\,p + 2\,\alpha)}$$

```
> beta4:=collect(factor(simplify(1/((mu-2*alpha-6-4-3)*gamma4)*(beta3^3
> -beta3+(4+4-mu+2*alpha+6)*beta3*gamma4+(-4+2+mu-2*alpha-6)*beta3*gamma
> 3+(-4+3+mu-2*alpha-6)*beta2*gamma3+(2*E2+3)*(gamma4-gamma3)))),a):
> E4:=collect(simplify(beta4+E3),a);
```

$$E4 := -\frac{4\,p\,\alpha + 16\,p + 25\,\alpha + 66 + \alpha^2}{3\,\alpha^2 + 4\,p\,\alpha + 31\,\alpha + 16\,p + 70}$$

```
> gamma5:=collect(factor(simplify(1/(mu-2*alpha-4-10)*(2*gamma4+(mu-2*a
> lpha-4-4)*gamma3+2*beta4*E3-2*beta3*E2+(4+10-mu+2*alpha)*beta4^2-(4+2*
> alpha+6-mu)*beta3^2+3*beta4-3*beta3-2))),a);
```

$$\gamma 5 := -2\,\frac{(2\,p + 9 + 2\,\alpha)\,(\alpha + 4)\,(12\,p - \alpha^2 - 3\,\alpha + 40)\,(8\,p + 18 - 3\,\alpha - \alpha^2)}{(3\,\alpha^2 + 4\,p\,\alpha + 31\,\alpha + 16\,p + 70)^2\,(13 + 2\,p + 2\,\alpha)\,(15 + 2\,p + 2\,\alpha)}$$

```
> beta5:=collect(factor(simplify(1/((mu-2*alpha-5-4-6)*gamma5)*(beta4^3
> -beta4+(4+6-mu+2*alpha+6)*beta4*gamma5+(-4+mu-2*alpha-6)*beta4*gamma4+
> (-4+1+mu-2*alpha-6)*beta3*gamma4+(2*E3+3)*(gamma5-gamma4)))),a):E5:=co
> llect(simplify(beta5+E4),a);
```

$$E5 := -12\,\frac{2\,p + 7}{12\,p - \alpha^2 - 3\,\alpha + 40}$$

```
> gamma6:=collect(factor(simplify(1/(mu-2*alpha-4-12)*(2*gamma5+(mu-2*a
> lpha-4-6)*gamma4+2*beta5*E4-2*beta4*E3+(4+12-mu+2*alpha)*beta5^2-(4+2*
> alpha+8-mu)*beta4^2+3*beta5-3*beta4-2))),p);
```

$$\gamma 6 := -6\,\frac{(2\,p + 7)\,((4\,\alpha + 20)\,p + 3\,\alpha^2 + 39\,\alpha + 108)\,((4\,\alpha + 16)\,p + 3\,\alpha^2 + 31\,\alpha + 70)}{(12\,p - \alpha^2 - 3\,\alpha + 40)^2\,(15 + 2\,p + 2\,\alpha)\,(17 + 2\,p + 2\,\alpha)}$$

```
> beta6:=collect(factor(simplify(1/((mu-2*alpha-5-4-8)*gamma6)*(beta5^3
> -beta5+(4+6-mu+2*alpha+8)*beta5*gamma6+(-4+mu-2*alpha-8)*beta5*gamma5+
> (-4+1+mu-2*alpha-8)*beta4*gamma5+(2*E4+3)*(gamma6-gamma5)))),p):E6:=co
> llect(simplify(beta6+E5),p);
```

$$E6 := -\frac{(4\,\alpha + 20)\,p + 33\,\alpha + 104 + \alpha^2}{(4\,\alpha + 20)\,p + 3\,\alpha^2 + 39\,\alpha + 108}$$

```
> beta0;E2;E4;
```

$$-\frac{4\,p + \alpha + 7}{4\,p + 3\,\alpha + 9}$$

$$-\frac{4\,p\,\alpha + 12\,p + \alpha^2 + 17\,\alpha + 36}{4\,p\,\alpha + 12\,p + 23\,\alpha + 40 + 3\,\alpha^2}$$

$$-\frac{4\,p\,\alpha + 16\,p + 25\,\alpha + 66 + \alpha^2}{3\,\alpha^2 + 4\,p\,\alpha + 31\,\alpha + 16\,p + 70}$$

```
> E1;E3;E5;
```

$$-4\,\frac{2\,p + 3}{-\alpha^2 - 3\,\alpha + 4 + 4\,p}$$

$$-8\,\frac{2\,p + 5}{8\,p + 18 - 3\,\alpha - \alpha^2}$$

$$-12\,\frac{2\,p + 7}{12\,p - \alpha^2 - 3\,\alpha + 40}$$

We suppose that

```
> E2n:=simplify(-(4*(alpha+(n+2))*p+alpha^2+(8*n+9)*alpha+4*n^2+18*n+14
> )/(4*(alpha+(n+2))*p+3*alpha^2+(8*n+15)*alpha+4*n^2+18*n+14+4));
```

$$E2n := -\frac{4\,p\,\alpha + 4\,p\,n + 8\,p + \alpha^2 + 8\,\alpha\,n + 9\,\alpha + 4\,n^2 + 18\,n + 14}{4\,p\,\alpha + 4\,p\,n + 8\,p + 3\,\alpha^2 + 8\,\alpha\,n + 15\,\alpha + 4\,n^2 + 18\,n + 18}$$

```
> E2np2test:=simplify(-(4*(alpha+(n+3))*p+alpha^2+(8*n+17)*alpha+4*(n+1
> )^2+18*(n+1)+14)/(4*(alpha+(n+3))*p+3*alpha^2+(8*(n+1)+15)*alpha+4*(n+
> 1)^2+18*(n+1)+14+4));
```

$$E2np2test := -\frac{4\,p\,\alpha + 4\,p\,n + 12\,p + \alpha^2 + 8\,\alpha\,n + 17\,\alpha + 4\,n^2 + 26\,n + 36}{4\,p\,\alpha + 4\,p\,n + 12\,p + 3\,\alpha^2 + 8\,\alpha\,n + 23\,\alpha + 4\,n^2 + 26\,n + 40}$$

```
> E2np1:=-4*(n+1)*(2*p+2*n+3)/(4*(n+1)*p-alpha^2-3*alpha+4*n^2+10*n+4);
```

$$E2np1 := -4\,\frac{(n+1)\,(2\,p+2\,n+3)}{4\,(n+1)\,p - \alpha^2 - 3\,\alpha + 4\,n^2 + 10\,n + 4}$$

```
> E2np3test:=-4*(n+2)*(2*p+2*n+5)/(4*(n+2)*p-alpha^2-3*alpha+4*(n+1)^2+
> 10*(n+1)+4);
```

$$E2np3test := -4\,\frac{(n+2)\,(2\,p+2\,n+5)}{4\,(n+2)\,p - \alpha^2 - 3\,\alpha + 4\,(n+1)^2 + 10\,n + 14}$$

```
> E2nm1:=-4*(n)*(2*p+2*n+1)/(4*(n)*p-alpha^2-3*alpha+4*(n-1)^2+10*(n-1)
> +4);
```

$$E2nm1 := -4\,\frac{n\,(2\,p+2\,n+1)}{4\,p\,n - \alpha^2 - 3\,\alpha + 4\,(n-1)^2 + 10\,n - 6}$$

```
> beta2np1:=simplify(E2np1-E2n):
> beta2n:=simplify(E2n-E2nm1):
> gamma2np1:=
> factor(simplify(-2*(n+alpha+2)*(2*n+2*alpha+2*p+5)*(4*n*p-alpha^2-3*al
> pha+4*(n-1)^2+10*(n-1)+4)*(4*(n+1)*p-alpha^2-3*alpha+4*(n-1)^2+18*(n-1
> )+18)/((4*n+2*alpha+2*p+5)*(4*n+2*alpha+2*p+7)*(4*alpha*p+3*alpha^2+(8
> *n+15)*alpha+(4*n+8)*p+4*(n+1)^2+10*(n+1)+4)^2)),beta);
```

$$gamma2np1 := -2((\alpha+n+2)\,(-2\,p-4\,n+\%1-5)\,(-2\,p-4\,n-1+\%1)$$
$$(4\,n+2\,p+1+\%1)\,(2\,n+2\,\alpha+2\,p+5)\,(4\,n+2\,p+5+\%1))/($$
$$(4\,n+2\,\alpha+2\,p+7)\,(4\,n+2\,\alpha+2\,p+5)\,(4\,n+2\,p+4\,\alpha+9+\%1)^2$$
$$(-2\,p-4\,n-9-4\,\alpha+\%1)^2)$$
$$\%1 := \sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha}$$

```
> gamma2np1:=
> simplify(-2*(((4*n+2*p+5)^2-(4*p^2+4*p+9+4*alpha^2+12*alpha))*(2*n+2*a
> lpha+2*p+5)*((2*p+4*n+1)^2-(4*p^2+4*p+9+4*alpha^2+12*alpha))*(n+alpha+
> 2)/((4*n+2*alpha+2*p+7)*(4*n+2*alpha+2*p+5)*((4*n+2*p+4*alpha+9)^2-(4*
> p^2+4*p+9+4*alpha^2+12*alpha))^2)));
```

$$gamma2np1 := -2((\alpha+n+2)\,(2\,n+2\,\alpha+2\,p+5)\,(4\,p\,n-\alpha^2-3\,\alpha+4\,n^2+2\,n-2)$$
$$(4\,p\,n+4\,p-\alpha^2-3\,\alpha+4\,n^2+10\,n+4))\Big/((4\,n+2\,\alpha+2\,p+5)$$
$$(4\,n+2\,\alpha+2\,p+7)$$
$$(4\,p\,\alpha+4\,p\,n+8\,p+3\,\alpha^2+8\,\alpha\,n+15\,\alpha+4\,n^2+18\,n+18)^2)$$

```
> solve(-4*p*n+alpha^2+3*alpha-4*n^2-2*n+2=0,n);
> solve(-4*p*n-4*p+alpha^2+3*alpha-4*n^2-10*n-4=0,n);
> solve(4*p*alpha+4*p*n+8*p+3*alpha^2+8*alpha*n+15*alpha+4*n^2+18*n+18=0
> ,n);
```

$$-\frac{1}{2}\,p-\frac{1}{4}+\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha},\ -\frac{1}{2}\,p-\frac{1}{4}-\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha}$$

$$-\frac{1}{2}\,p-\frac{5}{4}+\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha},\ -\frac{1}{2}\,p-\frac{5}{4}-\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha}$$

$$-\frac{1}{2}\,p-\frac{9}{4}-\alpha+\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha},$$
$$-\frac{1}{2}\,p-\frac{9}{4}-\alpha-\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha}$$

```
> gamma2n:=factor(-2*n*(2*p+2*n+1)*(4*alpha*p+3*alpha^2+(8*n+7)*alpha+4
> *(n+1)*p+4*(n)^2+10*(n)+4)*(4*alpha*p+(8*n+15)*alpha+4*(n+2)*p+3*alpha
> ^2+4*(n)^2+18*(n)+18)/((4*n+2*alpha+2*p+3)*(4*n+2*alpha+2*p+5)*(-alpha
> ^2-3*alpha+(4*n)*p+4*(n-1)^2+10*(n-1)+4)^2),beta);
```

$$\mathit{gamma2n} := -2n\,(4\,n+2\,p+4\,\alpha+9+\%1)\,(-2\,p-4\,n-9-4\,\alpha+\%1)$$
$$(-2\,p+\%1-5-4\,n-4\,\alpha)\,(2\,p+2\,n+1)\,(4\,n+4\,\alpha+2\,p+5+\%1)/($$
$$(4\,n+2\,\alpha+2\,p+5)\,(4\,n+2\,\alpha+2\,p+3)\,(-2\,p-4\,n-1+\%1)^2$$
$$(4\,n+2\,p+1+\%1)^2)$$
$$\%1 := \sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha}$$

```
> gamma2n :=simplify( -2*(2*p+2*n+1)*(
> (4*n+2*p+4*alpha+9)^2-(4*p^2+4*p+9+4*alpha^2+12*alpha))*((4*n+4*alpha+
> 2*p+5)^2-(4*p^2+4*p+9+4*alpha^2+12*alpha))*n/((4*n+2*alpha+2*p+3)*(4*n
> +2*alpha+2*p+5)*((2*p+4*n+1)^2-(4*p^2+4*p+9+4*alpha^2+12*alpha))^2));
```

$$\mathit{gamma2n} := -2((2\,p+2\,n+1)$$
$$(4\,p\,\alpha+4\,p\,n+8\,p+3\,\alpha^2+8\,\alpha\,n+15\,\alpha+4\,n^2+18\,n+18)$$
$$(4\,p\,\alpha+3\,\alpha^2+8\,\alpha\,n+7\,\alpha+4\,p\,n+4\,p+4\,n^2+10\,n+4)\,n)\Big/($$
$$(4\,n+2\,\alpha+2\,p+3)\,(4\,n+2\,\alpha+2\,p+5)\,(4\,p\,n-\alpha^2-3\,\alpha+4\,n^2+2\,n-2)^2)$$

```
> solve(4*p*alpha+3*alpha^2+8*alpha*n+7*alpha+4*p*n+4*p+4*n^2+10*n+4=0,
> n);
> solve(4*p*alpha+4*p*n+8*p+3*alpha^2+8*alpha*n+15*alpha+4*n^2+18*n+18=0
> ,n);solve(-4*p*n+alpha^2+3*alpha-4*n^2-2*n+2=0,n);
```

$$-\alpha-\frac{5}{4}-\frac{1}{2}\,p+\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha},$$
$$-\alpha-\frac{5}{4}-\frac{1}{2}\,p-\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha}$$

$$-\frac{1}{2}\,p-\frac{9}{4}-\alpha+\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha},$$
$$-\frac{1}{2}\,p-\frac{9}{4}-\alpha-\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha}$$

$$-\frac{1}{2}\,p-\frac{1}{4}+\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha},\ -\frac{1}{2}\,p-\frac{1}{4}-\frac{1}{4}\,\sqrt{4\,p^2+4\,p+9+4\,\alpha^2+12\,\alpha}$$

and we prove by reccurence:

```
> gamma2np2:=factor(simplify((1/(mu-2*alpha-4*n-8)*(2*gamma2np1+(mu-2*a
> lpha-4*n-2)*gamma2n+2*beta2np1*E2n-2*beta2n*E2nm1+(4*n+8-mu+2*alpha)*b
> eta2np1^2-(4*n+2*alpha+4-mu)*beta2n^2+3*beta2np1-3*beta2n-2))));
```

$$\begin{aligned}&\mathit{gamma2np2} := -2((n+1)\,(2\,p+2\,n+3)\\&(4\,p\,\alpha+4\,p\,n+12\,p+3\,\alpha^2+8\,\alpha\,n+23\,\alpha+4\,n^2+26\,n+40)\\&(4\,p\,\alpha+4\,p\,n+8\,p+3\,\alpha^2+8\,\alpha\,n+15\,\alpha+4\,n^2+18\,n+18))\Big/(\\&(4\,p\,n+4\,p-\alpha^2-3\,\alpha+4\,n^2+10\,n+4)^2\,(4\,n+2\,\alpha+2\,p+7)\\&(4\,n+9+2\,p+2\,\alpha))\end{aligned}$$

```
> beta2np2:=collect(factor(simplify(1/((mu-2*alpha-6-4*n-3)*gamma2np2)*
> (beta2np1^3-beta2np1+(4*n+4-mu+2*alpha+6)*beta2np1*gamma2np2+(-4*n+2+m
> u-2*alpha-6)*beta2np1*gamma2np1+(-4*n+3+mu-2*alpha-6)*beta2n*gamma2np1
> +(2*E2n+3)*(gamma2np2-gamma2np1)))),p):
> E2np2:=factor(simplify(E2np1+beta2np2));factor(E2np2-E2np2test);
```

$$\mathit{E2np2} := -\frac{4\,p\,\alpha+4\,p\,n+12\,p+\alpha^2+8\,\alpha\,n+17\,\alpha+4\,n^2+26\,n+36}{4\,p\,\alpha+4\,p\,n+12\,p+3\,\alpha^2+8\,\alpha\,n+23\,\alpha+4\,n^2+26\,n+40}$$

$$0$$

```
> gamma2np3:=factor(simplify(1/(mu-2*alpha-4*n-10)*(2*gamma2np2+(mu-2*a
> lpha-4*n-4)*gamma2np1+2*beta2np2*E2np1-2*beta2np1*E2n+(4*n+10-mu+2*alp
> ha)*beta2np2^2-(4*n+2*alpha+6-mu)*beta2np1^2+3*beta2np2-3*beta2np1-2))
> );
```

$$\begin{aligned}&\mathit{gamma2np3} := -2((\alpha+3+n)\,(4\,p\,n+4\,p-\alpha^2-3\,\alpha+4\,n^2+10\,n+4)\\&(4\,p\,n+8\,p+18+18\,n+4\,n^2-3\,\alpha-\alpha^2)\,(2\,p+7+2\,n+2\,\alpha))\Big/(\\&(4\,p\,\alpha+4\,p\,n+12\,p+3\,\alpha^2+8\,\alpha\,n+23\,\alpha+4\,n^2+26\,n+40)^2\\&(4\,n+9+2\,p+2\,\alpha)\,(4\,n+11+2\,p+2\,\alpha))\end{aligned}$$

```
> beta2np3:=collect(factor(simplify(1/((mu-2*alpha-6-4*n-5)*gamma2np3)*
> (beta2np2^3-beta2np2+(4*n+6-mu+2*alpha+6)*beta2np2*gamma2np3+(-4*n+mu-
> 2*alpha-6)*beta2np2*gamma2np2+(-4*n+1+mu-2*alpha-6)*beta2np1*gamma2np2
> +(2*E2np1+3)*(gamma2np3-gamma2np2)))),p):
> E2np3:=factor(simplify(E2np2+beta2np3));factor(E2np3-E2np3test);
```

$$\mathit{E2np3} := -4\,\frac{(n+2)\,(2\,p+2\,n+5)}{4\,p\,n+8\,p+18+18\,n+4\,n^2-3\,\alpha-\alpha^2}$$

$$0$$