A RESULT ON THE MAPPING $H$ OF S. S. DRAGOMIR WITH APPLICATIONS

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Abstract. In this paper, we establish a new result containing some inequalities for a convex and differentiable function $f$ related to its associated mapping $H$ introduced by S. S. Dragomir in the papers [4] and [5]. We give applications of this result to some special means.

1. Introduction

In the paper [5], S. S. Dragomir introduced two mappings associated to Hermite-Hadamard’s inequalities connected to convex functions. Precisely, if $f : [a, b] \to \mathbb{R}$ is a convex function on the closed and finite interval $[a, b]$ of the real line then one can define the two following mappings on $[0, 1]$ by setting:

$$F(t) := \frac{1}{(b - a)^2} \int_a^b \int_a^b f(tx + (1 - t)y) \, dx \, dy$$

and

$$H(t) := \frac{1}{b - a} \int_a^b f(tx + (1 - t)\frac{a + b}{2}) \, dx.$$ 

Concerning the mapping $H$, the following properties are known (see [4] and [5]):

1° $H$ is convex;

2° $H$ is monotonous nondecreasing.

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Therefore, we have:

\[
\inf_{t \in [0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right)
\]

and

\[
\sup_{t \in [0,1]} H(t) = H(1) = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

The main aim of this paper is to prove for a convex differentiable mapping \(f\) some new inequalities involving the associated mapping \(H\) (see the result below). We give applications of these inequalities to some special means.

For other results in connection to Hermite–Hadamard’s inequality see [1]–[16].

2. The Result

**Theorem 2.1.** Let \(f : [a, b] \to \mathbb{R}\) be convex and differentiable. Then for all \(t \in [0,1]\), we have the following inequalities:

\[
(2.1) \quad 0 \leq (1-t) \left[ \frac{1}{b-a} \int_a^b f(x) \, dx - f\left(\frac{a+b}{2}\right) \right]
\]

\[
(2.2) \quad \leq \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) \, dx
\]

\[
(2.3) \quad \leq (1-t) \left[ \frac{1}{b-a} \int_a^b f(x) \, dx - f\left(\frac{a+b}{2}\right) \right]
+ t(1-t) \left[ \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right].
\]

**Proof.** \(f\) being convex, for all \(t \in [0,1]\) and \(x \in [a, b]\), we have

\[f\left(tx + (1-t)\frac{a+b}{2}\right) \leq tf(x) + (1-t)f\left(\frac{a+b}{2}\right).\]

Integrating after \(x\), we get \(H(t) \leq tH(1) + (1-t)H(0)\), from which we obtain

\[H(t) - H(1) \leq (1-t)(H(0) - H(1)).\]

By using Hermite–Hadamard’s inequality,

\[f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2},\]
we deduce
\[ H(1) - H(t) \geq (1 - t)(H(1) - H(0)) \geq 0, \]
and the inequalities (2.1) and (2.2) are proved.

In order to prove inequality (2.3), we observe by using the convexity and differentiability of \( f \) that for all \( x \in [a, b] \) and \( t \in (0, 1] \) we have
\[
f \left( tx + (1 - t) \frac{a + b}{2} \right) - f(x) \geq (1 - t) \left( \frac{a + b}{2} - x \right) f'(x)
\]
and
\[
f \left( tx + (1 - t) \frac{a + b}{2} \right) - f \left( \frac{a + b}{2} \right) \geq t \left( x - \frac{a + b}{2} \right) f' \left( \frac{a + b}{2} \right).
\]
Consequently,
\[
t f \left( tx + (1 - t) \frac{a + b}{2} \right) - t f(x) \geq t(1 - t) \left( \frac{a + b}{2} - x \right) f'(x)
\]
and
\[
(1 - t) f \left( tx + (1 - t) \frac{a + b}{2} \right) - (1 - t) f \left( \frac{a + b}{2} \right) \geq t(1 - t) \left( x - \frac{a + b}{2} \right) f' \left( \frac{a + b}{2} \right).
\]
By summing the above inequalities, we get
\[
f \left( tx + (1 - t) \frac{a + b}{2} \right) - t f(x) - (1 - t) f \left( \frac{a + b}{2} \right) \geq t(1 - t) \left( \frac{a + b}{2} - x \right) f'(x) + t(1 - t) \left( x - \frac{a + b}{2} \right) f' \left( \frac{a + b}{2} \right).
\]
By integrating after \( x \), we get
\[
H(t) - tH(1) - (1 - t)H(0) \geq \frac{t(1 - t)}{b - a} \left[ \int_a^b (a + b) - x \right] f'(x) dx + f' \left( \frac{a + b}{2} \right) \int_a^b \left( x - \frac{a + b}{2} \right) dx.
\]
Now, integrating by parts, we get
\[
\frac{1}{b - a} \int_a^b \left( \frac{a + b}{2} - x \right) f'(x) dx = \frac{1}{b - a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2}.
\]
Also,
\[ \int_a^b \left( x - \frac{a + b}{2} \right) \, dx = 0. \]

Therefore,
\[ H(t) - tH(1) - (1-t)H(0) \geq t(1-t) \left( H(1) - \frac{f(a) + f(b)}{2} \right). \]

From the above inequality, we deduce that
\[ H(1) - H(t) + (1-t)[H(0) - H(1)] \leq t(1-t) \left( H(1) - \frac{f(a) + f(b)}{2} \right), \]
which is equivalent to say that
\[ H(1) - H(t) \leq (1-t)[H(1) - H(0)] + t(1-t) \left( \frac{f(a) + f(b)}{2} - H(1) \right), \]
and this is the inequality (2.3). \( \square \)

**Remarks.** Let \( f \) be as above. In [7], S. S. Dragomir proved the following inequalities:
\[ 0 \leq H(1) - H(t) \leq (1-t) \left( \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right) \]
for all \( t \in [0, 1] \). Thus, our result provides a better left bound for \( H(1) - H(t) \). It is also possible to compare the right bounds of this expression given in (2.4) and in our theorem. Indeed, let us denote
\[ C := \frac{1}{b-a} \int_a^b f(x) \, dx - f\left( \frac{a+b}{2} \right) \quad \text{and} \quad D := \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx. \]

By using Theorem 2.1 and the inequalities (2.4), we recapture the well known inequalities \( 0 \leq C \leq D \). We remark that if \( 0 < C < D \), then our right bound is less than the right bound of (5) for all \( t \in [0, 1 - C/D] \).

**Corollary 2.1.** Let \( f : [a, b] \to \mathbb{R} \) be convex and differentiable. Then we have
\[
0 \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f(x) \, dx - f\left( \frac{a+b}{2} \right) \right] \leq \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{2}{b-a} \int_{\frac{a+b}{4}}^{\frac{a+b}{2}} f(x) \, dx \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f(x) \, dx - f\left( \frac{a+b}{2} \right) \right] + \frac{1}{4} \left[ \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right]
\]
3. Applications to Some Special Means

The means considered here are recalled in the next subsection.

3.1. Recalls. 1° The arithmetic mean

\[ A = A(a, b) := \frac{a + b}{2}, \quad a, b \geq 0; \]

2° The geometric mean:

\[ G := G(a, b) := \sqrt{ab}, \quad a, b \geq 0; \]

3° The harmonic mean:

\[ H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b \geq 0; \]

4° The logarithmic mean:

\[ L = L(a, b) := \begin{cases} a & \text{if } a = b, \\ \frac{b - a}{\log b - \log a} & \text{if } a \neq b, a, b > 0; \end{cases} \]

5° The identric mean:

\[ I = I(a, b) := \begin{cases} a & \text{if } a = b, \\ \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{1/b-a} & \text{if } a \neq b, a, b > 0; \end{cases} \]

6° The \( p \)-logarithmic mean:

\[ L_p = L_p(a, b) := \begin{cases} \left[ \frac{b^{p+1} - a^{p+1}}{(p + 1)(b - a)} \right]^{1/p} & \text{if } a \neq b, \\ a & \text{if } a = b, \end{cases} \]

where, \( p \in \mathbb{R} \setminus \{-1, 0\}, \ a, b > 0. \)

The following inequalities involving these means are known in the literature

\[ H \leq G \leq L \leq I \leq A. \]

We recall also that the mean \( L_p \) is increasing in \( p \) with \( L_0 = I \) and \( L_{-1} = L \).
3.2. Applications. 1° Consider the function \( f : x \mapsto x^p \) with \( p > 1 \) on any subinterval \([a, b]\) of \([0, +\infty)\), with \( a < b \). We have \( H(1) = L_p^p(a, b) \), \( H(0) = [A(a, b)]^p \), and by easy computations, we have
\[
H(t) = L_p^p \left( \frac{a + b}{2} - \frac{b - a}{2}, \frac{a + b}{2} + \frac{b - a}{2} \right).
\]

By application of Theorem 2.1, we get the following inequalities:
\[
0 \leq (1 - t) \left[ L_p^p(a, b) - A^p(a, b) \right] \\
\leq L_p^p(a, b) - L_p^p \left( \frac{a + b}{2} - \frac{b - a}{2}, \frac{a + b}{2} + \frac{b - a}{2} \right) \\
\leq (1 - t) \left[ L_p^p(a, b) - A^p(a, b) \right] + t(1 - t) \left[ A(a, b) - L_p^p(a, b) \right],
\]
for all \( t \) in \([0, 1]\).

2° Consider the convex and differentiable function \( f : x \mapsto 1/x \) on any subinterval \([a, b]\) of \([0, \infty)\), with \( a < b \). We have \( H(1) = L^{-1}(a, b) \), \( H(0) = A^{-1}(a, b) \), and by easy computations, we have
\[
H(t) = L^{-1} \left( \frac{a + b}{2} - \frac{1}{2}, \frac{a + b}{2} + \frac{1}{2} \right).
\]

By application of Theorem 2.1, we get the following inequalities:
\[
0 \leq (1 - t) \left[ L^{-1}(a, b) - A^{-1}(a, b) \right] \\
\leq L^{-1}(a, b) - L^{-1} \left( \frac{a + b}{2} - \frac{1}{2}, \frac{a + b}{2} + \frac{1}{2} \right) \\
\leq (1 - t) \left[ L^{-1}(a, b) - A^{-1}(a, b) \right] + t(1 - t) \left[ H^{-1}(a, b) - L^{-1}(a, b) \right],
\]
for all \( t \) in \([0, 1]\).

3° Finally, let us consider the convex and differentiable function \( f : x \mapsto -\log(x) \) on any subinterval \([a, b]\) of the interval \((0, +\infty)\) with \( a < b \). We have \( H(1) = -\log I(a, b) \), \( H(0) = -\log A(a, b) \). By easy computations, for all \( t \in [0, 1] \), we get
\[
H(t) = -\log \left( \frac{a + b}{2} - \frac{b - a}{2}, \frac{a + b}{2} + \frac{b - a}{2} \right).
\]

By application of Theorem 2.1, we get the following inequalities:
\[
1 \leq \left[ \frac{A(a, b)}{I(a, b)} \right]^{-t} \leq \frac{I \left( \frac{a + b}{2} - \frac{b - a}{2}, \frac{a + b}{2} + \frac{b - a}{2} \right)}{I(a, b)} \leq \left[ \frac{A(a, b)}{I^{1-t}(a, b) \cdot G^t(a, b)} \right]^{1-t}
\]
for all $t$ in $[0, 1]$.

In particular, if we choose $t = 1/2$, then we get

$$I(a, b) \leq A(a, b) \leq \frac{I^2 \left( \frac{3a+b}{4}, \frac{a+3b}{4} \right)}{I(a, b)} \leq A(a, b) \sqrt{\frac{I(a, b)}{G(a, b)}}.$$ 

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