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A RESULT ON THE MAPPING H OF S. S. DRAGOMIR WITH APPLICATIONS

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Abstract. In this paper, we establish a new result containing some inequalities for a convex and differentiable function f related to its associated mapping H introduced by S. S. Dragomir in the papers [4] and [5]. We give applications of this result to some special means.

1. Introduction

In the paper [5], S. S. Dragomir introduced two mappings associated to Hermite-Hadamard's inequalities connected to convex functions. Precisely, if $f : [a, b] \to \mathbb{R}$ is a convex function on the closed and finite interval $[a, b]$ of the real line then one can define the two following mappings on $[0, 1]$ by setting:

$$F(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)\, dx\, dy$$

and

$$H(t) := \frac{1}{b-a} \int_a^b f(tx + (1-t)\frac{a+b}{2})\, dx.$$

Concerning the mapping H, the following properties are known (see [4] and [5]):

1° H is convex;

2° H is monotonous nondecreasing.

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Therefore, we have:

$$\inf_{t\in[0,1]} H(t) = H(0) = f\Big(\frac{a+b}{2}\Big)$$

and

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The main aim of this paper is to prove for a convex differentiable mapping f some new inequalities involving the associated mapping H (see the result below). We give applications of these inequalities to some special means.

For other results in connection to Hermite–Hadamard's inequality see [1]–[16].

2. The Result

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be convex and differentiable. Then for all $t \in [0,1]$, we have the following inequalities:*

$$0 \leq (1-t)\left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)\right] \tag{2.1}$$

$$\leq \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{b-a}\int_a^b f\Big(tx+(1-t)\frac{a+b}{2}\Big)\,dx \tag{2.2}$$

$$\leq (1-t)\left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)\right] \tag{2.3}$$
$$+t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right].$$

Proof. f being convex, for all $t \in [0,1]$ and $x \in [a,b]$, we have

$$f\Big(tx+(1-t)\frac{a+b}{2}\Big) \leq tf(x) + (1-t)f\Big(\frac{a+b}{2}\Big).$$

Integrating after x, we get $H(t) \leq tH(1)+(1-t)H(0)$, from which we obtain

$$H(t) - H(1) \leq (1-t)(H(0)-H(1)).$$

By using Hermite–Hadamard's inequality,

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2},$$

we deduce

$$H(1) - H(t) \geq (1-t)(H(1) - H(0)) \geq 0,$$

and the inequalities (2.1)and (2.2) are proved.

In order to prove inequality (2.3), we observe by using the convexity and differentiability of f that for all $x \in [a,b]$ and $t \in (0,1]$ we have

$$f\left(tx + (1-t)\frac{a+b}{2}\right) - f(x) \geq (1-t)\left(\frac{a+b}{2} - x\right) f'(x)$$

and

$$f\left(tx + (1-t)\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right) \geq t\left(x - \frac{a+b}{2}\right) f'\left(\frac{a+b}{2}\right).$$

Consequently,

$$tf\left(tx + (1-t)\frac{a+b}{2}\right) - tf(x) \geq t(1-t)\left(\frac{a+b}{2} - x\right) f'(x)$$

and

$$\begin{aligned}(1-t)f\left(tx + (1-t)\frac{a+b}{2}\right) &- (1-t)f\left(\frac{a+b}{2}\right) \\ &\geq t(1-t)\left(x - \frac{a+b}{2}\right) f'\left(\frac{a+b}{2}\right).\end{aligned}$$

By summing the above inequalities, we get

$$\begin{aligned}&f\left(tx + (1-t)\frac{a+b}{2}\right) - tf(x) - (1-t)f\left(\frac{a+b}{2}\right) \\ &\quad\geq t(1-t)\left(\frac{a+b}{2} - x\right) f'(x) + t(1-t)\left(x - \frac{a+b}{2}\right) f'\left(\frac{a+b}{2}\right).\end{aligned}$$

By integrating after x, we get

$$\begin{aligned}H(t) &- tH(1) - (1-t)H(0) \\ &\geq \frac{t(1-t)}{b-a}\left[\int_a^b \left(\frac{a+b}{2} - x\right) f'(x)\,dx + f'\left(\frac{a+b}{2}\right)\int_a^b \left(x - \frac{a+b}{2}\right) dx\right].\end{aligned}$$

Now, integrating by parts, we get

$$\frac{1}{b-a}\int_a^b \left(\frac{a+b}{2} - x\right) f'(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}.$$

Also,

$$\int_a^b \Big(x - \frac{a+b}{2}\Big)\, dx = 0.$$

Therefore,

$$H(t) - tH(1) - (1-t)H(0) \geq t(1-t)\left[H(1) - \frac{f(a)+f(b)}{2}\right].$$

From the above inequality, we deduce that

$$H(1) - H(t) + (1-t)[H(0) - H(1)] \leq t(1-t)\left[H(1) - \frac{f(a)+f(b)}{2}\right],$$

which is equivalent to say that

$$H(1) - H(t) \leq (1-t)[H(1) - H(0)] + t(1-t)\left[\frac{f(a)+f(b)}{2} - H(1)\right],$$

and this is the inequality (2.3). $\square$

Remarks. Let f be as above. In [7], S. S. Dragomir proved the following inequalities:

$$0 \leq H(1) - H(t) \leq (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx\right] \tag{2.4}$$

for all $t \in [0,1]$. Thus, our result provides a better left bound for $H(1) - H(t)$. It is also possible to compare the right bounds of this expression given in (2.4) and in our theorem. Indeed, let us denote

$$C := \frac{1}{b-a}\int_a^b f(x)\, dx - f\Big(\frac{a+b}{2}\Big) \quad \text{and} \quad D := \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx.$$

By using Theorem 2.1 and the inequalities (2.4), we recapture the well known inequalities $0 \leq C \leq D$. We remark that if $0 < C < D$, then our right bound is less than the right bound of (5) for all $t \in [0, 1 - C/D]$.

Corollary 2.1. *Let $f : [a,b] \to \mathbb{R}$ be convex and differentiable. Then we have*

$$\begin{aligned} 0 \ &\leq \ \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\, dx - f(\frac{a+b}{2})\right] \leq \frac{1}{b-a}\int_a^b f(x)\, dx - \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\, dx \\ &\leq \ \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\, dx - f(\frac{a+b}{2})\right] + \frac{1}{4}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx\right] \end{aligned}$$

3. Applications to Some Special Means

The means considered here are recalled in the next subsection.

3.1. Recalls. 1° The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \quad a,b \geq 0;$$

2° The geometric mean:

$$G := G(a,b) := \sqrt{ab}, \qquad a,b \geq 0;$$

3° The harmonic mean:

$$H = H(a,b) := \frac{2}{\dfrac{1}{a} + \dfrac{1}{b}}, \qquad a,b \geq 0;$$

4° The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \dfrac{b-a}{\log b - \log a} & \text{if} \quad a \neq b,\ a,b > 0; \end{cases}$$

5° The identric mean:

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/b-a)} & \text{if} \quad a \neq b,\ a,b > 0; \end{cases}$$

6° The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if} \ a \neq b, \\ a & \text{if} \quad a = b, \end{cases}$$

where, $p \in \mathbb{R} \setminus \{-1, 0\},\ a,b > 0$.

The following inequalities involving these means are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

We recall also that the mean L_p is increasing in p with $L_0 = I$ and $L_{-1} = L$.

3.2. Applications. 1° Consider the function $f : x \mapsto x^p$ with $p > 1$ on any subinterval $[a, b]$ of $[0, +\infty)$, with $a < b$. We have $H(1) = L_p^p(a, b)$, $H(0) = [A(a, b)]^p$, and by easy computations, we have

$$H(t) = L_p^p\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right).$$

By application of Theorem 2.1, we get the following inequalities:

$$\begin{aligned}
0 &\leq (1-t)\left[L_p^p(a,b) - A^p(a,b)\right] \\
&\leq L_p^p(a,b) - L_p^p\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right) \\
&\leq (1-t)\left[L_p^p(a,b) - A^p(a,b)\right] + t(1-t)\left[A(a^p, b^p) - L_p^p(a,b)\right],
\end{aligned}$$

for all t in $[0, 1]$.

2° Consider the convex and differentiable function $f : x \mapsto 1/x$ on any subinterval $[a, b]$ of $]0, \infty)$, with $a < b$. We have $H(1) = L^{-1}(a, b)$, $H(0) = A^{-1}(a, b)$, and by easy computations, we have

$$H(t) = L^{-1}\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right).$$

By application of Theorem 2.1, we get the following inequalities:

$$\begin{aligned}
0 &\leq (1-t)\left[L^{-1}(a,b) - A^{-1}(a,b)\right] \\
&\leq L^{-1}(a,b) - L^{-1}\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right) \\
&\leq (1-t)\left[L^{-1}(a,b) - A^{-1}(a,b)\right] + t(1-t)\left[H^{-1}(a,b) - L^{-1}(a,b)\right],
\end{aligned}$$

for all t in $[0, 1]$.

3° Finally, let us consider the convex and differentiable function $f : x \mapsto -\log(x)$ on any subinterval $[a, b]$ of the interval $(0, +\infty)$ with $a < b$. We have $H(1) = -\log I(a, b)$, $H(0) = -\log A(a, b)$. By easy computations, for all $t \in [0, 1]$, we get

$$H(t) = -\log\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right).$$

By application of Theorem 2.1, we get the following inequalities:

$$1 \leq \left[\frac{A(a,b)}{I(a,b)}\right]^{1-t} \leq \frac{I\left(\frac{a+b}{2} - t\frac{b-a}{2}, \frac{a+b}{2} + t\frac{b-a}{2}\right)}{I(a,b)} \leq \left[\frac{A(a,b)}{I^{1-t}(a,b)\cdot G^t(a,b)}\right]^{1-t}$$

for all t in $[0,1]$.

In particular, if we choose $t=1/2$, then we get

$$I(a,b) \le A(a,b) \le \frac{I^2\left(\frac{3a+b}{4}, \frac{a+3b}{4}\right)}{I(a,b)} \le A(a,b)\sqrt{\frac{I(a,b)}{G(a,b)}}\,.$$

REFERENCES

1. S.S. Dragomir: *Two refinements of Hadamard's inequalities.* Coll. of Sci. Pap. of the Fac. of Sci., Kragujevac (Yougoslavia) **11** (1990), 23–26.
2. S.S. Dragomir, J.E. Pečarić and J. Sándor: *A note on the Jensen-Hadamard's inequality.* Anal. Numer. Theor. Approx. **19** (1990), 21–28.
3. S.S. Dragomir: *Some refinements of Hadamard's inequality.* Gaz. Mat. Metod. (Romania) **11** (1990), 189–191.
4. S.S. Dragomir: *A mapping connected with Hadamard's inequalities.* An. Öster. Akad. Wiss. Math.-Natur. (Wien) **123** (1991), 17–20.
5. S.S. Dragomir: *two mappings in connection to Hadamard's inequalities.* J. Math. Anal. Appl. **167** (1992), 49–56.
6. S.S. Dragomir and N.M. Ionescu: *Some intgral inequalities for differentiable convex functions.* Coll. of Sci. Pap. of the Fac. of Sci., Kragujevac (Yougoslavia) **13** (1992), 11–16.
7. S.S. Dragomir: *Some intgral inequalities for differentiable convex functions.* Contributions, Macedonian Acad. of Sci. and Arts **13** (1) (1992), 13–17.
8. S.S. Dragomir: *On Hadamard's inequalities for convex functions.* Math. Balkanica **6 (4)** (1992), 215–222.
9. S.S. Dragomir: *A refinement of On Hadamard's inequality for isotonic linear functionals.* Tamkang J. Math. **24** (1993), 101–106.
10. S.S. Dragomir: *A note On Hadamard's inequalities.* Mathematica (Romania) **35 (1)** (1993), 21–24.
11. S.S. Dragomir, D.M. Milošević and J. Sándor: *On some refinements of Hadamard's inequalities and applications.* Univ. Beograd Publ. Elek. Fak. Ser. Math. **4** (1993), 21–24.
12. S.S. Dragomir, D. Barbu and C. Buse: *A probabilistic argument for the convergence of some sequences associated to Hadamard's inequality.* Studia Univ. "Babeş–Bolyai" Math. **38 (1)** (1993), 29–33.
13. S.S. Dragomir: *Some remarks on Hadamard's inequalities for convex functions.* Extracta Math. **9(2)** (1994), 88–94.

14. S.S. Dragomir, Y.J. Cho and S.S. Kim: *Inequalities of Hadamard's type for Lipschitzian mappings and their applications.* J. Math. Anal. Appl. **245** (2000), 489–501.

15. J.E. Pečarić and S.S. Dragomir: *On some integral inequalities for convex functions.* Bull. Inst. Pol. Iasi. (Romania) **36** (1990), 19–23.

16. J.E. Pečarić and S.S. Dragomir:, *A generalization of Hadamard's inequality for isotonic linear functionals.* Radovi Mat. (Sarajevo) **7** (1991), 299–303.

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