FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 17 (2002), 1–4

ALL GENERAL SOLUTIONS OF PREŠIĆ'S EQUATION

Dragić Banković

Abstract. S. Prešić ([6]) determined the all reproductive general solutions of the equation $r(x) = 1$, where r was a unary relation of a given set T. In this paper we determine the all general solutions of this equation.

S. Prešić ([6]) introduced the most general concept of equations. Namely, he considered the equation of the form $r(x) = 1$, where r was a unary relation of a non–empty set S. He gave the formula of the all reproductive general solutions, supposing that a general solution of this equation was known. This equation was also considered by Božić ([3]), Rudeanu ([8]), Banković ([1]) and Chvalinna ([4]).

S. Prešić ([5]) researched the finite equations i.e. the equations over a finite set. In the paper [7] he introduced the finite equation, the coefficients of which are from the set $\{0, 1\}$ (one can prove that every finite equation is equivalent to an equation of such type) and he described the all reproductive general solutions of this equation, if particular solutions were given. The descriptions of the all general solutions and all reproductive general solutions of this equations were obtained by Banković ([2]). Rudeanu ([9]) provided simpler characterizations of the general and reproductive general solutions.

Following Rudeanu's idea from [9], we generalize, in this paper, Prešić's result on reproductive solutions ([6]), i.e. we describe the all general solutions of the equation $r(x) = 1$, when a general solution of $r(x) = 1$ is given.

Let r be a unary relation of a non–empty set T, i.e. a mapping $r : T \to \{0, 1\}$. Denote by S the set of solutions of the equation $r(x) = 1$.

Received November 5, 2001.
2000 *Mathematics Subject Classification*. Primary 04A05.

Definition 1. Suppose equation $r(x) = 1$ is consistent (has a solution). The formula $x = f(t)$, where $f : T \to T$, represents a general solution of the equation $r(x) = 1$ if and only if

$$(\forall t \in T) r(f(t)) = 1 \wedge (\forall x \in T)(r(x) = 1 \Rightarrow (\exists t \in T)(x = f(t)))\,.$$

In the other words, $x = f(t)$ represents a general solution if and only if $f(T) = S$.

Definition 2. Suppose equation $r(x) = 1$ is consistent. The formula $x = f(t)$, where $f : T \to T$, represents a general solution of the equation $r(x) = 1$ if and only if

$$(\forall t \in T) r(f(t)) = 1 \wedge (\forall t \in T)(r(t) = 1 \Rightarrow t = f(t))\,.$$

In the other words, $x = f(t)$ represents a general solution if and only if $f(T) = S$ and $f(t) = t$ for all t from S.

Theorem 1. *Let r be a unary relation of a non-empty set T and let $g : T \to T$ be a mapping such that the formula $x = g(t)$ represents a general solution of the equation $r(x) = 1$. The formula*

$$x = f(t) \tag{1}$$

represents the general solution of $r(x) = 1$ if and only if there exist mappings $a : T \overset{\text{onto}}{\to} T$ and $b : T \to T$, such that

$$f(t) = r(a(t)) \cdot a(t) + r'(a(t)) \cdot g(b(t))\,. \tag{2}$$

Proof. Suppose there exist mappings $a : T \overset{\text{onto}}{\to} T$ and $b : T \to T$, such that (2) holds. Let $t \in T$. If $a(t) \in S$, then the formula (2) gives $f(t) = a(t) \in S$. If $a(t) \notin S$, then the formula (2) gives $f(t) = g(b(t)) \in S$. Let $x \in S$. Since a is "onto" there exists t such that $a(t) = x$. Then

$$f(t) = r(a(t)) \cdot a(t) + r'(a(t)) \cdot g(b(t)) = r(x) \cdot x + r'(x) \cdot g(b(x)) = 1 \cdot x + 0 = x\,.$$

Conversely, let $x = f(t)$ represents the general solution of $r(x) = 1$. Then $f(T) = S$. Since, for every $y \in S$, we can choose an element $z \in f^{-1}(\{y\})$, then there exists, by axiom of choice, an injective mapping $\phi : S \to T$ such that $f(\phi(y)) = y$ for all $y \in S$. Obviously, there exists a bijective mapping $\psi : T \setminus S \to T \setminus \phi(S)$. The mapping $h : T \to T$ determined by

$$h = \begin{cases} \phi(t), & t \in S \\ \psi(t), & t \notin S \end{cases}$$

is a bijection. Let $a = h^{-1}$. We determine the mapping $b : T \setminus \phi(S) \to T$ in the following way: let u be an arbitrary element of $T \setminus \phi(S)$ and

$$v = f(u), \qquad w \in g^{-1}(\{v\}), \qquad b(u) = w\,,$$

i.e.

$$g(b(u)) = g(w) = v = f(u)\,. \tag{3}$$

Extend the mapping b such that $b : T \to T$.

Let $t \in T$ and $a(t) = y$. Then $a(t) \in S \to t = \phi(y) \to f(t) = f(\phi(y)) = y = a(t)$, hence

$$r(a(t)) \cdot a(t) + r'(a(t)) \cdot g(b(t)) = 1 \cdot a(t) + 0 = f(t)$$

and $a(t) \notin S \to t = \psi(y) \to t \in T \setminus \phi(S) \to g(b(t)) = f(t)$, hence

$$r(a(t)) \cdot a(t) + r'(a(t)) \cdot g(b(t)) = 0 + 1 \cdot g(b(t)) = f(t)\,. \quad \square$$

Comment. If the mapping a, in Theorem 1, is not "onto," then formula (1), where g is given by (2), has not to represent the general solution. Namely, if for some $p \in S$ there doesn't exist q such that $a(q) = p$, then $r(a(t)) \cdot a(t) \neq p$ for all $t \in T$. Then $r(a(t)) \cdot a(t) + r'(a(t)) \cdot g(b(t))$ could be equal to p if there exists t such that $g(b(t)) = p$. However, if $a(t) \in S$ and $a(t) \neq p$, then we get

$$r(a(t)) \cdot a(t) + r'(a(t)) \cdot g(b(t)) = a(t) \neq p\,.$$

Therefore, in that case, (1) is not "onto".

Now it is easy to prove Prešić's theorem.

Theorem 2. ([5]) *Let r be a unary relation of a non–empty set T and let $g : T \to T$ be a mapping such that the formula $x = g(t)$ represents the general solution of the equation $r(x) = 1$. The formula $x = f(t)$ represents the reproductive general solution of $r(x) = 1$ if and only if there exists the mapping $b : S \to S$ such that*

$$f(t) = r(t) \cdot t + r'(t) \cdot g(b(t))\,.$$

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University of Kragujevac
Faculty of Science
Radoja Domanovića 12
34 000 Kragujevac
Yugoslavia