FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 16 (2001), 87–96

ON A TYPE OF SEMI–SYMMETRIC METRIC CONNECTION ON AN ALMOST CONTACT METRIC MANIFOLD

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Abstract. We find the expression for the curvature tensor of an almost contact metric manifold that admits a type of semi–symmetric metric connection. Also we study the properties of curvature tensor, conformal curvature tensor and projective curvature tensor.

1. Introduction

Let (M, g) be an n–dimensional Riemannian manifold of class C^∞ with metric tensor g and let ∇ be the Levi–Civita connection on M. A linear connection $\overline{\nabla}$ on (M, g) is said to be semi–symmetric ([1]) if the torsion tensor T of the connection $\overline{\nabla}$ satisfies

$$T(X,Y) = \pi(Y)X - \pi(X)Y\,, \tag{1.1}$$

where π is a 1–form on M with ρ as associated vector–field, i.e.

$$\pi(X) = g(X, \rho) \tag{1.2}$$

for any differentiable vector field X on M.

A semi–symmetric connection $\overline{\nabla}$ is called semi–symmetric metric connection ([2]) if it further satisfies

$$\overline{\nabla} g = 0\,. \tag{1.3}$$

Let M be an n–dimensional C^∞ manifold and let there exist in M a vector valued linear function Φ, a vector field ξ and a 1–form η such that

$$\Phi^2 X = -X + \eta(X)\xi\,, \tag{1.4}$$

$$\overline{X} \overset{\text{def}}{=} \Phi X\,, \tag{1.5}$$

Received September 13, 2000.
2000 *Mathematics Subject Classification*. Primary 53B15.

for arbitrary vector field X.

Then M is called an almost contact manifold. From (1.4) the following relations hold ([3]):

$$\Phi\xi = 0\,, \tag{1.6}$$

$$\eta(\Phi X) = 0\,, \tag{1.7}$$

$$\eta(\xi) = 1\,. \tag{1.8}$$

In addition, if in M, there exists a metric tensor g satisfying

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{1.9}$$

and

$$g(X,\xi) = \eta(X)\,, \tag{1.10}$$

then M is called an almost contact metric manifold.

In [4], Sharfuddin and Hussian defined a semi–symmetric metric connection in an almost contact manifold by identifying the 1–form π of (1.1) with the contact form η, i.e., by setting

$$T(X,Y) = \eta(Y)X - \eta(X)Y\,. \tag{1.11}$$

In 1995, Mileva Prvanović ([5]) studied a semi–symmetric metric connection in a locally decomposable Riemannian space whose torsion tensor T satisfies the condition

$$(\overline{\nabla}_X T)(Y,Z) = A(X)T(Y,Z) + A(FX)F\big(T(Y,Z)\big)\,, \tag{1.12}$$

where A is a 1–form and F is a tensor field of type (1,1).

In this paper we study a semi–symmetric metric connection on an almost contact metric manifold satisfying the condition (1.11) and

$$(\overline{\nabla}_X T)(Y,Z) = A(X)T(Y,Z) + A(\Phi X)\Phi\big(T(Y,Z)\big)\,, \tag{1.13}$$

where Φ is the tensor field of type (1,1) of the almost contact metric manifold. In Section 3, we find the expression for curvature tensor of $\overline{\nabla}$ and deduce some properties of the curvature tensor. It is proved that if the curvature tensor of $\overline{\nabla}$ vanishes then the manifold is of quasi–constant curvature ([6]). Next we prove that if the Ricci tensor of $\overline{\nabla}$ vanishes, then the manifold becomes an η–Einstein manifold. In Section 4, we prove that the Weyl conformal curvature tensor of $\overline{\nabla}$ is equal to the Weyl conformal curvature tensor of the manifold. In the last section, we obtain a necessary condition under which the projective curvature tensor of $\overline{\nabla}$ becomes equal to the projective curvature tensor of the manifold.

2. Preliminaries

The relation between the semi–symmetric metric connection $\overline{\nabla}$ and the Levi–Civita connection ∇ of (M^n, g) has been obtained by K. Yano ([7]), which is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)\rho\,. \tag{2.1}$$

Further, a relation between the curvature tensors R and $\overline{R}$ of type (1.3) of ∇ and $\overline{\nabla}$, respectively, are given by [7],

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z \\ &\quad + \alpha(X,Z)Y - \alpha(Y,Z)X - g(Y,X)LX + g(X,Z)LY\,,\end{aligned} \tag{2.2}$$

where

$$\alpha(Y,Z) = g(LY,Z) = (\nabla_Y \pi)(Z) - \pi(Y)\pi(Z) + \frac{1}{2}\pi(\rho)g(Y,Z)\,. \tag{2.3}$$

The Weyl conformal curvature tensor of type (1,3) of the manifold is defined by

$$\begin{aligned}C(X,Y)Z &= R(X,Y)Z \\ &\quad + \lambda(Y,Z)X - \lambda(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\,,\end{aligned} \tag{2.4}$$

where

$$\lambda(Y,Z) = g(QY,Z) = -\frac{1}{n-2}S(Y,Z) + \frac{r}{2(n-1)(n-2)}g(Y,Z)\,, \tag{2.5}$$

S and r denote respectively the (0,2) Ricci tensor and scalar curvature of the manifold.

The projective curvature tensor of the manifold is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\big[S(Y,Z)X - S(X,Z)Y\big]\,. \tag{2.6}$$

3. Curvature Tensor of the Semi–symmetric Metric Connection

We have

$$T(Y,Z) = \eta(Z)Y - \eta(Y)Z\,, \tag{3.1}$$

where

$$\eta(Z) = g(Z, \xi)\,. \tag{3.2}$$

From (3.1) we get by contracting Y,

$$(C', T)(Z) = (n-1)\eta(Z)\,. \tag{3.3}$$

Now,

$$(\overline{\nabla}_X C', T)(Z) = (n-1)(\overline{\nabla}_X \eta)(Z)\,. \tag{3.4}$$

Let

$$(\overline{\nabla}_X T)(Y, Z) = A(X)T(Y, Z) + A(\Phi X)\Phi\big(T(Y, Z)\big)\,, \tag{3.5}$$

where A is a 1–form and Φ is a tensor field of type (1,1). From (3.5) we get by contracting Y,

$$(\overline{\nabla}_X C', T)(Z) = (n-1)A(X)\eta(Z) + aA(\Phi X)\eta(Z)\,, \tag{3.6}$$

where

$$a = (C', \Phi)(Y)\,. \tag{3.7}$$

Combining (3.4) and (3.6) we get

$$(\overline{\nabla}_X \eta)(Z) = A(X)\eta(Z) + bA(\Phi X)\eta(Z)\,, \tag{3.8}$$

where

$$b = \frac{a}{n-1}\,. \tag{3.9}$$

Using (1.8) we get,

$$(\overline{\nabla}_X \eta)(Z) = (\nabla_X \eta)(Z) - \eta(X)\eta(Z) + g(X, Z)\,. \tag{3.10}$$

Combining (3.8) and (3.10) we get,

$$(\nabla_X \eta)(Z) = A(X)\eta(Z) + bA(\Phi X)\eta(Z) + \eta(X)\eta(Z) - g(X, Z)\,. \tag{3.11}$$

Then, from (2.3) and (3.11), it follows

$$\alpha(X,Z) = A(X)\eta(Z) + bA(\Phi X)\eta(Z) - \frac{1}{2}g(X,Z)\,. \tag{3.12}$$

From (2.3) and (3.12) we can say,

$$LX = A(X)\xi + bA(\Phi X)\xi - \frac{1}{2}X\,. \tag{3.13}$$

Therefore, the curvature tensor $\overline{R}$ of the manifold with respect to semi–symmetric metric connection $\overline{\nabla}$ is given by

$$\begin{aligned}\overline{R}(X,Y)Z =& R(X,Y)Z + \Big\{g(Y,Z)X - g(X,Z)Y\Big\}\\ &+ \Big\{A(X) + bA(\Phi X)\Big\}\Big\{\eta(Z)Y - g(Y,Z)\xi\Big\}\\ &- \Big\{A(Y) + bA(\Phi Y)\Big\}\Big\{\eta(Z)X - g(X,Z)\xi\Big\},\end{aligned} \tag{3.14}$$

where R denotes the curvature tensor of the manifold.

In view of the above, we can state the following:

Theorem 3.1. *The curvature tensor with respect to* $\overline{\nabla}$ *of an almost contact metric manifold admitting a semi–symmetric metric connection is of the form* (3.14).

From (3.14) it is obvious that

$$\overline{R}(Y,X)Z = -\overline{R}(X,Y)Z\,. \tag{3.15}$$

We now define a tensor ${}^\ulcorner R$ of type (0,4) by

$${}^\ulcorner R(X,Y,Z,V) = g\big(\overline{R}(X,Y)Z,V\big)\,. \tag{3.16}$$

From (3.14) and (3.16) it follows that

$${}^\ulcorner R(X,Y,Z,V) = -{}^\ulcorner R(X,Y,V,Z)\,. \tag{3.17}$$

Combining (3.17) and (3.15) one finds that

$${}^\ulcorner R(X,Y,Z,V) = {}^\ulcorner R(Y,X,V,Z)\,. \tag{3.18}$$

Again from (3.14) we get,

(3.19)
$$\begin{aligned}\overline{R}(X,Y)Z+&\overline{R}(Y,Z)X+\overline{R}(Z,X)Y\\ &=\big(A(X)+bA(\Phi X)\big)\big(\eta(Z)Y-\eta(Y)Z\big)\\ &\quad+\big(A(Y)+bA(\Phi Y)\big)\big(\eta(X)Z-\eta(Z)X\big)\\ &\quad+\big(A(Z)+bA(\Phi Z)\big)\big(\eta(Y)X-\eta(X)Y\big)\,.\end{aligned}$$

This is the first Bianchi identity with respect to $\overline{\nabla}$.

Let $\overline{S}$ and S denote respectively the Ricci tensor of the manifold with respect to $\overline{\nabla}$ and ∇. From (3.14) we get by contracting X

$$\begin{aligned}\overline{S}(Y,Z)&=S(Y,Z)+(n-1)g(Y,Z)\\ &\quad-(n-2)\big(A(Y)+bA(\Phi Y)\big)\eta(Z)-A(\xi)g(Y,Z)\,,\end{aligned}\tag{3.20}$$

since $\Phi\xi=0$.

In (3.20) we put $Y=e_i+Z$, $1\le i\le n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold.

Then summing over i we get

$$\overline{r}=r+n(n-1)-2(n-1)A(\xi)\,,\tag{3.21}$$

where $\overline{r}$ and r denote the scalar curvatures of the manifold with respect to $\overline{\nabla}$ and ∇ respectively.

From (3.20) it follows that $\overline{S}$ is symmetric if and only if

$$\eta(Y)\big(A(Z)+bA(\Phi Z)\big)=\eta(Z)\big(A(Y)+bA(\Phi Y)\big)\,.\tag{3.22}$$

In particular, if $\overline{S}=0$, then from (3.20) we have

$$\begin{aligned}S(Y,Z)&=(n-2)\big(A(Y)+bA(\Phi Y)\big)\eta(Z)\\ &\quad+A(\xi)g(Y,Z)-(n-1)g(Y,Z).\end{aligned}\tag{3.23}$$

Since S is symmetric, we get from (3.23),

$$\Big[A(Y)+bA(\Phi Y)\Big]\eta(Z)=\Big[A(Z)+bA(\Phi Z)\Big]\eta(Y)\,.\tag{3.24}$$

Putting $Z = \xi$, we get from the above relation

$$A(Y) + bA(\Phi Y) = A(\xi)\eta(Y)\,. \tag{3.25}$$

Now, if $\overline{R} = 0$, then $\overline{S} = 0$ and then from (3.14) and (3.25) we obtain

$$\begin{aligned}
{}'R(X,Y,Z,V) = &-\eta(\xi)\Big[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)\Big] \\
&+ A(\xi)\Big[\eta(Y)\eta(Z)g(X,V) - \eta(X)\eta(Z)g(Y,V) \\
&- \eta(Y)\eta(V)g(X,Z) + \eta(X)\eta(V)g(Y,Z)\Big],
\end{aligned} \tag{3.26}$$

since $\eta(\xi) = 1$ and where

$${}'R(X,Y,Z,V) = g\big(R(X,Y)Z,V\big)\,. \tag{3.27}$$

Hence we can state the following theorem.

Theorem 3.2. *If the curvature tensor of an almost contact metric manifold with respect to the semi–symmetric metric connection vanishes, then the manifold is of quasi–constant curvature.*

Next, let us assume that $\overline{R}$ is symmetric. Then (3.22) holds. Putting $Z = \xi$ in (3.22) we get

$$A(Y) + bA(\Phi Y) = A(\xi)\eta(Y)\,.$$

Using this result from (3.19) we get

$$\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0\,. \tag{3.28}$$

Conversely we assume that (3.28) holds, then in virtue of (3.19) we have

$$\begin{aligned}
&\big(A(X) + bA(\Phi X)\big)\big(\eta(Z)Y - \eta(Y)Z\big) \\
&+ \big(A(Y) + bA(\Phi Y)\big)\big(\eta(X)Z - \eta(Z)X\big) \\
&+ \big(A(Z) + bA(\Phi Z)\big)\big(\eta(Y)X - \eta(X)Y\big) = 0\,.
\end{aligned} \tag{3.29}$$

Contracting X, we get from (3.29)

$$\eta(Y)\big(A(Z) + bA(\Phi Z)\big) = \eta(Z)\big(A(Y) + bA(\Phi Y)\big)\,.$$

Hence by (3.22) $\overline{S}$ is symmetric.

Thus we can state:

Theorem 3.3. *A necessary and sufficient condition for the Ricci tensor of an almost contract metric manifold with respect to semi–symmetric metric connection to be symmetric is*

$$\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0\,.$$

Next, if $\overline{S} = 0$, then $\overline{r} = 0$ and so from (3.21) we get,

$$A(\xi) = \frac{1}{2}\Big\{\frac{r}{n-1} + n\Big\}\,. \tag{3.30}$$

Putting this value of $A(\xi)$ we get from (3.23),

$$S(Y,Z) = \mu g(Y,Z) + \nu\eta(Y)\eta(Z)\,, \tag{3.31}$$

where

$$\mu = \frac{1}{2}\Big\{\frac{r}{n-1} - n + 2\Big\} \tag{3.32}$$

and

$$\nu = \frac{1}{2}\cdot\frac{n-2}{n-1}(r + n^2 - n)\,. \tag{3.33}$$

So we can state:

Theorem 3.4. *If the Ricci tensor of an almost contact metric manifold with respect to the semi–symmetric metric connection vanishes, then the manifold becomes an η–Einstein manifold.*

4. Weyl Conformal Curvature Tensor

The Weyl conformal curvature tensor of type (1,3) of the almost contact metric manifold with respect to semi–symmetric metric connection is defined by

$$\begin{aligned}\overline{C}(X,Y)Z &= \overline{R}(X,Y)Z \\ &\quad + \overline{\lambda}(Y,Z)X - \overline{\lambda}(X,Z)Y + g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y\,,\end{aligned} \tag{4.1}$$

where

$$\overline{\lambda}(Y,Z) = \overline{g}(QY,Z) = -\frac{1}{n-1}S(Y,Z) + \frac{\overline{r}}{2(n-1)(n-2)}g(Y,Z)\,. \tag{4.2}$$

Putting the values of $\overline{S}$ and $\overline{r}$ from (3.20) and (3.21) respectively in (4.2) we get,

$$\overline{\lambda}(Y,Z) = \lambda(Y,Z) - \frac{1}{2}g(Y,Z) + \eta(Z) + \big(A(Y) + bA(\Phi Y)\big)\,. \tag{4.3}$$

Combining the results (4.1), (3.14) and (4.3) we get,

$$\overline{C}(X,Y)Z = C(X,Y)Z\,. \tag{4.4}$$

So, we can state:

Theorem 4.1. *The Weyl conformal curvature tensors of an almost contact metric manifold with respect to the Levi–Civita connection and semi–symmetric metric connection are equal.*

Next, if in particular $\overline{S} = 0$, then $\overline{r} = 0$. So from (4.2) we get

$$\overline{\lambda}(Y,Z) = 0\,. \tag{4.5}$$

Putting this result in (4.2) and using (4.4) we get,

$$C(X,Y)Z = \overline{R}(X,Y)Z\,. \tag{4.6}$$

Hence we can state:

Theorem 4.2. *If the Ricci tensor of an almost contact metric manifold with respect to semi–symmetric metric connection vanishes, then the Weyl conformal curvature tensor of the manifold is equal to the curvature tensor of the manifold with respect to semi–symmetric metric connection.*

5. Projective Curvature Tensor

The projective curvature tensor of type (1,3) of an almost contact metric manifold with respect to semi–symmetric metric connection is defined by

$$\overline{P}(X,Y)Z = \overline{R}(X,Y)Z - \frac{1}{n-1}\Big\{\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\Big\}\,. \tag{5.1}$$

Using (3.14) and (3.20) we get from (5.1),

$$\begin{aligned}\overline{P}(X,Y)Z = P(X,Y)Z &+ \frac{1}{n-1}A(\xi)\Big\{g(Y,Z)X - g(X,Z)Y\Big\}\\ &+ \Big\{A(X) + bA(\Phi X)\Big\}\Big\{\frac{1}{n-1}\eta(Z)Y - g(Y,Z)\xi\Big\}\\ &- \Big\{A(Y) + bA(\Phi Y)\Big\}\Big\{\frac{1}{n-1}\eta(Z)X - g(X,Z)\xi\Big\}\,.\end{aligned} \tag{5.2}$$

If, in particular, $\overline{S}$ is symmetric, then we already have,

$$A(Y) + bA(\Phi Y) = A(\xi)\eta(Y)\,.$$

Using the above result we get from (5.2),

$$\overline{P}(X,Y)Z = P(X,Y)Z + \frac{1}{n-1}A(\xi)\Big\{g(Y,Z)X - g(X,Z)Y\Big\} + A(\xi)\eta(X)\Big\{\frac{1}{n-1}\eta(Z)Y - g(Y,Z)\xi\Big\} - A(\xi)\eta(Y)\Big\{\frac{1}{n-1}\eta(Z)X - g(X,Z)\xi\Big\}. \tag{5.3}$$

From (5.3) it follows that $P = \overline{P}$ if $A(\xi) = 0$.

So, we have:

Theorem 5.1. *If the Ricci tensor of an almost contact metric manifold is symmetric, then a necessary condition for the projective curvature tensors of the manifold with respect to the Levi–Civita connection and the semi–symmetric metric connection to be equal is that* $A(\xi) = 0$.

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