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ON THE NUMBER OF SOLUTIONS OF NONLINEAR EQUATIONS

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Abstract. In this paper we study the existence and the finitness of the number of solutions to nonlinear equations as well as semilinear equations involving nonlinear perturbations of Fredholm maps of index zero.

1. Introduction

Let X and Y be Banach spaces and $T : X \to Y$ be a nonlinear map of A–proper type. Under various conditions on T, we study in Section 2 the surjectivity and the finitness of the solution set of the equation $Tx = f$. In particular, we also look at nonresonant semilinear equations of the form $Ax + Nx = f$ where A is a Fredholm map of index zero and the nonlinear map N is such that $A + N$ is (pseudo) A–proper and is either nondifferentiable or differentiable. We say that this equation is not at resonance if A and N are are such that it is solvable for each $f \in Y$. Our results extend the corresponding ones of Seda [9] who assumed that N is a compact (differentiable) map.

2. On the number of solutions of operator equations

In this section, we shall study the number of solutions of the equation $Tx = f$. The unique (approximation) solvability of it has been studied in detail in [8] using the A–proper mapping approach.

Recall the definition of (pseudo) A–proper maps.

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Definition 2.1. A map $T : D \subset X \to Y$ is (pseudo) A–proper w.r.t. a scheme $\Gamma = \{X_n, Y_n, Q_n\}$ with $\dim X_n = \dim Y_n$ on D if whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and such that $Q_{n_k}Tx_{n_k} - Q_{n_k}f \to 0$ for some $f \in Y$, then $\{x_n\}$ has a subsequence converging to $x \in D$ (there is $x \in D$) with $Tx = f$.

The classes of A–proper and pseudo A–proper maps are very general and we refer to [3]–[7] for many examples of such maps.

2.1. Nonlinear equations. We say that a map $T : X \to Y$ satisfies condition $(+)$ if $\{x_n\}$ is bounded whenever $Tx_n \to f$ in Y. We need the following result.

Proposition 2.1. *Let $T : X \to Y$ be a continuous A–proper map that satisfies condition $(+)$. Then T is a proper map.*

Proof. We have shown in [8] that the range $R(T)$ is a closed set. We also know that T is proper when restricted to bounded closed subsets D of X, i.e. $T^{-1}(K) \cap D$ is compact in X for each compact set K in Y. Next, let K be a compact subset in Y. Then $T^{-1}(K)$ is a bounded set in X. Indeed, if we had $\{x_k\} \in T^{-1}(K)$ with $||x_k|| \to +\infty$, then $Tx_k = y_k$ for some $y_k \in K$. We may assume that $y_k \to y \in K$ and so $Tx_k \to y \in K$ with $\{x_k\}$ unbounded, in contradiction to condition $(+)$. Hence, $T^{-1}(K)$ is bounded.

Next, we shall show that T^{-1} is closed. Let $x_k \in T^{-1}(K)$ and $x_k \to x$ in X. Then $Tx_k = y_k \in K$ and we may assume that $y_k \to y \in K$. Since T is continuous, $Tx_k \to Tx = y \in K$ and so $x \in T^{-1}(K)$. Hence, $T^{-1}(K)$ is closed. Thus, $T^{-1}(K)$ is bounded and closed and therefore compact since T is proper when restricted to bounded and closed subsets of X. □

Let Σ be the set of all points $x \in X$ where T is not locally invertible and $\operatorname{card} T^{-1}(\{f\})$ be the cardinal number of the set $T^{-1}(\{f\})$.

Theorem 2.1. *Let $T : X \to Y$ be continuous, A–proper and satisfy condition $(+)$. Then*

(a) *The set $T^{-1}(\{f\})$ is compact (possibly empty) for each $f \in Y$.*

(b) *The range $R(T)$ of T is closed and connected.*

(c) *Σ and $T(\Sigma)$ are closed subsets of X and Y, respectively, and $T(X \setminus \Sigma)$ is open in Y.*

(d) *$\operatorname{card} T^{-1}(\{f\})$ is constant and finite (it may be 0) on each connected component of the open set $Y \setminus T(\Sigma)$.*

(e) *If $\Sigma = \emptyset$, then T is a homeomorphism from X to Y.*

(f) *If* $\Sigma \neq \emptyset$*, then the boundary* $\partial T(X\setminus\Sigma)$ *of* $T(X\setminus\Sigma)$ *satisfies* $\partial T(X\setminus\Sigma) \subset T(\Sigma)$.

Proof. Since T is proper by Proposition 2.1, it is a closed map. Since $X \setminus \Sigma$ is an open set, Σ is a closed set. Hence (a)–(c) hold, where $T(X \setminus \Sigma)$ is open since T is locally invertible on $X \setminus \Sigma$. (d) follows from the Ambrosetti theorem [1] and (e) follows from the global inversion theorem. Next, (b) and (c) imply that

$$T(X) = T(\Sigma) \cup T(X \setminus \Sigma) = T(\Sigma) \cup \overline{F(X \setminus \Sigma)} = \overline{T(X)}\,. \tag{2.1}$$

Moreover, $\partial T(X\setminus\Sigma) = \overline{T(X \setminus \Sigma)}\setminus T(X\setminus\Sigma)$, which together with (2.1) imply (f). □

As a consequence of Theorem 2.1, we have the following surjectivity result.

Corollary 2.1. *Let* T *be continuous,* A*–proper and satisfy condition* (+)*. Then* T *is surjective if either one of the following conditions hold:*

(a) $T(\Sigma) \subset T(X \setminus \Sigma)$.

(b) $Y \setminus T(\Sigma)$ *is connected and* $T(X \setminus \Sigma) \setminus T(\Sigma) \neq \emptyset$.

Proof. Let (a) hold. Then (2.1) implies that $T(X) = T(X \setminus \Sigma)$ and so $R(R)$ is open and closed subset of a connected space Y. Hence, $T(X) = Y$.

Next, let (b) hold. Then $\operatorname{card} T^{-1}(\{f\}) = k \geq 0$ on $Y \setminus T(\Sigma)$ by (d) in Theorem 2.1. If $k = 0$, then $T(X) = T(\Sigma)$ and $T(X \setminus \Sigma) \subset T(\Sigma)$ a contradiction to our assumption. Hence, $k > 0$ and T is surjective. □

Regarding the local invertibility property in Theorem 2.1, we have

Proposition 2.2 ([8])**.** *If* $T : X \to Y$ *is an open* A*–proper map* (*in particular, if it has the invariance of domain property*)*, then* T *is locally invertible at* $x \in X$ *if and only if* T *is locally injective at* x*.*

Next, we shall look at another surjectivity result. Let $J : X \to 2^{X^*}$ be the normalized duality map and $G : X \to Y$ be a bounded map such that $Gx \neq 0$ for all x with $||x|| \geq r_0$ for some $r_0 > 0$ and

$$\deg\,(Q_n G, B(0,r) \cap X_n, 0) \neq 0 \tag{2.2}$$

for each large $r > 0$ and for all large n.

Theorem 2.2. *Let $T : X \to Y$ satisfy condition (+), (2.2) hold and*

(i) *for each $f \in Y$ there is an $r_f > 0$ such that*

$$Tx \neq \lambda Gx \quad \text{for} \quad x \in \partial B(0, r_f)\,, \quad \lambda < 0\,, \tag{2.3}$$

(ii) *$H(t, x) = tTx + (1 - t)Gx$ is an A–proper w.r.t. Γ homotopy on $[0, 1] \times X$.*

Then T is surjective. If T is also continuous, then $T{-}1(\{f\})$ is compact for each $f \in Y$ and the cardinal number $\operatorname{card} T^{-1}(\{f\})$ is constant, finite and different from zero on each connected component of the set $Y \setminus T(\Sigma)$.

Proof. The surjectivity of T has been established in [7]. Moreover, T is continuous and proper by Proposition 2.1. Hence, the other assertions of the theorem follow from Theorem 2.1. □

Corollary 2.2. *Let $F, K : X \to X$ be continuous ball–condensing maps and $T = I - F$ and $G = I - K$ satisfy (2.2)–(2.3). Then the conclusions of Theorem 2.2 hold for T.*

Corollary 2.2 is also valid for general condensing maps (see [9]).

Our next result is based on the following continuation result ([5]–[6]).

Theorem 2.3. *Let V be dense subspace of a Banach space X, $D \subset X$ be open and bounded subset, Y be a Banach space and a homotopy $H : [0, 1] \times (\bar{D} \cap V) \to Y$ be such that:*

(i) *H is an A–proper homotopy w.r.t. $\Gamma = \{X_n, Y_n, Q_n\}$ on $[0, \varepsilon] \times (\partial D \cap V)$ for each $\varepsilon \in (0, 1)$ and H_1 is pseudo A–proper w.r.t. Γ,*

(ii) *$H(t, x)$ is continuous at 1 uniformly for $x \in \partial D \cap V$,*

(iii) *$H(t, x) \neq f$ and $H(0, x) \neq tf$ for $t \in [0, 1]$, $x \in \partial D \cap V$,*

(iv) *$\deg(Q_n H_0, D \cap X_n, 0) \neq 0$ for all large n.*

Then the equation $H(1, x) = f$ is solvable in $\bar{D} \cap V$.

For a map M, define its quasinorm by $|M| = \limsup_{||x|| \to +\infty} ||Mx||/||x||$.

Theorem 2.4. *Let $A : D(A) \subset X \to Y$ be a linear densely defined map and $N : X \to Y$ be bounded and of the form $Nx = B(x)x + Mx$ for some linear maps $B(x) : X \to X$. Assume that there is a $c > |M|$ and a positively homogeneous map $C : X \to Y$ such that:*

$$||Ax - (1 - t)Cx - tB(x)x|| \geq c||x||\,, \quad x \in D(A) \setminus B(0, R)\,, \tag{2.4}$$

(i) $H_t = A - (1-t)C - tN$ *is* A*–proper w.r.t.* $\Gamma = \{X_n, Y_n, Q_n\}$ *for* $t \in [0,1)$ *and* $A - N$ *is pseudo* A*–proper,*

(ii) *for all* $r > R$, $\deg(Q_n(A-C), B(0,r) \cap X_n, 0) \neq 0$ *for each large* n.

Then the equation $Ax - Nx = f$ *is solvable for each* $f \in Y$*. If, in addition,* $A - N$ *is continuous and* A*–proper, then* $(A-N)^{-1}(\{f\})$ *is compact for each* $f \in Y$ *and* $\operatorname{card}(A-N)^{-1}(\{f\})$ *is constant, finite and positive on each connected component of the set* $Y \setminus (A-N)(\Sigma)$.

Proof. Regarding the surjectivity of $A-N$, it suffices to solve $Ax - Nx = 0$. Define $H(t,x) = Ax - (1-t)Cx - tNx$ on $[0,1] \times D(A)$. Then there is an $r > 0$ such that

$$H(t,x) \neq 0 \quad \text{for} \quad x \in \partial B(0,r) \cap D(A)\,, \quad t \in [0,1]\,. \tag{2.5}$$

If not, then there are $x_n \in H$ and $t_n \in [0,1]$ such that $||x_n|| \to +\infty$ and $H(t_n, x_n) = 0$. Let $\varepsilon > 0$ be small such that $|M| \leq (|M| + \varepsilon)||x||$ for $||x|| \geq R_1$ and $|M| + \varepsilon < c$. For each x_n with $||x_n|| \geq R_1$ we have that

$$c||x_n|| \leq ||Ax_n - (1-t)Cx_n - tB(x_n)x_n|| \leq (|M| + \varepsilon)||x_n||\,.$$

Dividing by $||x_n||$, this leads to a contradiction and (2.5) holds. Hence, $A-N$ is surjective by Theorem 2.3. Next, it is easy to see that $||(A-N)x|| \to +\infty$ as $||x|| \to +\infty$ by (2.4). Hence, the other assertions follow Theorem 2.1. □

Let U be a convex subset of the set of positively homogeneous maps from X to Y.

Corollary 2.3. *Let* $A : D(A) \subset X \to Y$ *be a linear densely defined map and* $N : X \to Y$ *be bounded and of the form* $Nx = B(x)x + Mx$ *with* $B(x) \in U$ *for all* $x \in X$ *and* $|M|$ *sufficiently small. Let there exist* $\delta > 0$ *such that*

$$||Ax - Sx|| \geq \delta ||x||\,, \quad x \in D(A)\,, \quad S \in U\,. \tag{2.6}$$

Suppose that

(i) $H_t = A - (1-t)C - tN$ *is* A*–proper w.r.t.* $\Gamma = \{X_n, Y_n, Q_n\}$ *for* (X,Y) *for* $t \in [0,1)$ *and some* $C \in U$ *and* $H_1 = A - N$ *is pseudo* A*–proper,*

(ii) *for each large* $r > 0$, $\deg(Q_n(A-C), B(0,r) \cap X_n, 0) \neq 0$ *for* n *large.*

Then the conclusions of Theorem 2.4 *are valid.*

Proof. It suffices to show that (2.4) of Theorem 2.4 holds. Since the maps $S = (1-t)C + tB(x) \in U$ for each $x \in X$, (2.6) implies (2.4) of Theorem 2.4. □

2.2. On nonlinear perturbations of Fredholm maps of index zero. In this subsection we shall look at a class of nonlinear perturbations of Fredholm maps of index zero of the form $A+N$ where A is a linear Fredholm map of index zero and N is either nondifferentiable or a differentiable nonlinear map.

When N is not differentiable, we have the following special case of Theorem 2.2.

Corollary 2.4. *Let $A : X \to Y$ be a linear Fredholm map of index zero and $N : X \to Y$ be nonlinear such that the map $T = A + N$ satisfy conditions* (+), (2.2) *and* (i)–(ii) *of Theorem* 2.2 *for some G. Then T is surjective. If T is also continuous, then $T{-}1(\{f\})$ is compact for each $f \in Y$ and* $\operatorname{card} T^{-1}(\{y\})$ *is constant, finite and different from zero on each connected component of the set $Y \setminus T(\Sigma)$.*

Next, we shall look at the case of differentiable N. Recall that if D is a nonempty open subset of X and $T : D \to Y$ is differentiable, then $x_0 \in D$ is a regular point of T if $T'(x_0)$ is a linear homeomorphism of X onto Y. If x_0 is not a regular point of T, then it is a singular point of T. A singular value of T is the image by T of a singular point. If $T : X \to Y$ and S is the set of all singular points of T, then $T(S)$ is the set of all singular values of T and $R_T = Y \setminus T(S)$ is called the set of all regular values of T. Hence, for each $y \in R_T$ either $T^{-1}(\{y\})$ is empty or it consists solely of regular points of T. Next, a set S in a topological space Z is residual if it is a countable intersection of dense open subsets of Z. By the Baire theorem, if Z is a complete metric space or if Z is a locally compact Hausdorff topological space, then a residual set S is dense in Z.

A map $T \in C^1(D, Y)$ is a Fredholm map if its Frechet derivative $T'(x)$ is a linear Fredholm map at each $x \in D$. If D is connected, hence a region in X, then the difference $\dim N(T'(x)) - \operatorname{codim} R(T'(x))$ has a constant value in D. Thus, the index ind T of T is well defined by the equality

$$\operatorname{ind} T = \dim N(T'(x)) - \operatorname{codim} R(T'(x)) \ , \quad x \in D \, .$$

In case of a Fredholm map T of index zero, x_0 is a singular point of T if and only if $T'(x_0)h = 0$ has a solution $h \neq 0$.

The following result provides a large class of nonlinear Fredholm maps of index zero.

Proposition 2.3. *Let $T = A + N : X \to Y$ be such that A is a linear Fredholm map of index zero and N is continuously Frechet differentiable* *such that* $A + tN'(x) : X \to Y$ *is* A*–proper w.r.t.* Γ *for each* $x \in X$ *and* $t \in [0, 1]$*. Then*

(a) $T \in C^1(X, Y)$ *is a Fredholm map of index zero.*

(b) *If* $S = \{x \in X \mid T'(x)h = 0$ *has a solution* $h \neq 0\}$*, then* S *is the set of all singular points of* T *and the set* $R_T = Y \setminus T(S)$ *of all regular points of* T *is dense in* Y*.*

Proof. (a) Since $A+tN'(x)$ is a continuous linear A–proper map for each $t \in [0, 1]$, it is proper when restricted to closed bounded subsets of X. Hence, $A + tN'(x) \in \Phi_+(X)$, the set of semifredholm maps (cf. [2]) for $t \in [0, 1]$ and each $x \in X$. Thus, $\mathrm{ind}(A + N'(x)) = \mathrm{ind}A$ for each $x \in X$ and therefore T is a Fredholm map of index zero.

(b) Note that $T'(x) : X \to Y$ is a linear homeomorphism if and only if it is bijective. Since $T'(x)$ is a Fredholm map of index zero, it is bijective if and only if it is injective. Hence, x is a singular point of T if and only if $x \in S$. Since T is C^1 and proper when restricted to closed bounded subsets of X, hence σ–proper, the set R_T of all regular values of T is residual in Y by the Smale–Quinn theorem. Thus, R_T is dense in Y. □

Now, Propositions 2.1 and 2.3 and the main theorem on nonlinear proper Fredholm maps (Theorem 29.E in [10], p. 665) imply the following result.

Theorem 2.5. *Let* $T = A + N : X \to Y$ *satisfy condition* $(+)$ *and be* A*–proper and such that* A *is a linear Fredholm map of index zero, and* N *is continuously Frechet differentiable such that* $A+tN'(x) : X \to Y$ *is* A*–proper w.r.t.* Γ *for each* $x \in X$ *and each* $t \in [0, 1]$*. Then*

(a) T *is a proper* C^1*–Fredholm map of index zero.*

(b) $\mathrm{card}\, T^{-1}(\{y\})$ *is constant and finite (it may be zero) on each connected component of the open and dense subset* R_T *of* Y*.*

(c) T *is a local* C^1*–diffeomorphism at* x *for each* x *in the open set* $X \setminus S$*.*

(d) *If* $S = \emptyset$*, then* $T : X \to Y$ *is a* C^1*–diffeomorphism.*

(e) *The set* $T(S)$ *of all singular values of* T *is closed and nowhere dense in* Y*.*

Corollary 2.5. *Suppose that* T *is as in Theorem* 2.5*. Then*

(a) *If* $S \neq \emptyset$*, then* $\partial T(X \setminus S) \subset T(S)$*.*

(b) *If* $T(S) \subset T(X \setminus S)$*, then* $R(T) = Y$*.*

(c) *If* $Y \setminus T(S)$ *is connected and* $X \setminus S \neq \emptyset$*, then again* $R(R) = Y$*.*

Proof. Note that $T(X)$ and $T(S)$ are closed and $T(X \setminus S)$ is open by (e) and (c) of Theorem 2.5. Hence,

$$T(X) = T(S) \cup T(X \setminus S) = T(S) \cup \overline{T(X \setminus S)} = \overline{T(X)}\,.$$

This and (c) and (e) of Theorem 2.5 imply the corollary. $\square$

Remark 2.1. If $T \in C^1(X,Y)$, the fact that T is a local C^1–diffeomorphism at x if and only if x is a regular point of T implies that $\Sigma \subset S$. Moreover, if $\dim Y \geq 3$ and $T : X \to Y$ is a Fredholm map of index zero, it is known that if x is an isolated singular point of T, then T is locally invertible at x. Hence, in general, $\Sigma \neq S$.

Corollary 2.6. *Let $T = A + N : X \to Y$ be as in Theorem 2.5, $\dim Y \geq 3$ and assume that each point $x \in X$ is either a regular point or an isolated critical point of T. Then T is a homeomorphism of X onto Y.*

Proof. Since T is a C^1–Fredholm map of index zero by Proposition 2.3, the observations in Remark 2.1 imply that T is a locally homeomorphic map from X into Y. Then T is a homeomorphism from X onto Y by the global inversion theorem. $\square$

Next, we shall extend Theorem 2.5 to closed linear maps $A : D(A) \subset X \to Y$. Define the graph norm $||x||_A = ||x|| + ||Ax||$ on $D(A)$. Then $X_A = (D(A), ||.||_A)$ is a Banach space.

Theorem 2.6. *Let $A : D(A) \subset X \to Y$ be a linear closed Fredholm map of index zero and $N : D(N) \subset X \to Y$ be a nonlinear map with $D(A) \subset D(N)$ that is continuously Frechet differentiable from $D(A)$ into Y and $A + tN'(x) : X_A \to Y$ is A–proper w.r.t Γ for each $x \in X_A$ and $t \in [0,1]$. Then $T = A + N$ satisfies assertions (a)–(b) in Proposition 2.3 on X_A. Moreover, if T also satisfies condition $(+)$ as a map from $D(A)$ into Y, $T : X_A \to Y$ is A–proper and N maps bounded set in $D(A) \subset X$ into bounded sets of Y, then all the assertions of Theorem 2.5 hold for $T : X_A \to Y$.*

Proof. It is clear that $T : X_A \to Y$ is a continuous Fredholm map of index zero. Since $||x|| \leq ||x||_A$ on X_A, then each continuous linear map $L : D(A) \subset X \to Y$ is also continuous from X_A into Y. Hence, $N \in C^1(X_A, Y)$ as does $A + N$ and therefore assertions (a)–(b) in Proposition 2.3 are true.

Next, in view of Theorem 2.5, it remains to show that $T : X_A \to Y$ satisfies condition $(+)$. Let $Tx_n = Ax_n + Nx_n \to f$ in Y. Then $\{x_n\}$ is bounded in X and $\{Nx_n\}$ is bounded in Y by hypothesis. Thus, $\{Ax_n\}$ is bounded in Y. Since $x_n = x_{0n} + x_{1n}$ with $x_{0n} \in N(A)$, then $\{Ax_{1n}\}$

is bounded in Y. Moreover, by the continuity of the partial inverse $A^{-1}: R(A) \subset Y \to X_A$, we get that $||x_{1n}||_A \leq ||A^{-1}||\,||Ax_{1n}|| \leq$constant. Hence, $\{x_{0n}\}$ is also bounded in X_A since $\dim N(A)$ is finite and therefore $\{x_n\}$ is bounded in X_A. $\square$

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