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MAXIMAL ELEMENTS AND EQUILIBRIA FOR $\mathcal{U}$-MAJORISED PREFERENCES IN NOT NECESSARILY LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

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Abstract. The purpose of this paper is to give an existence theorem for maximal elements for a $\mathcal{U}$-majorised type of preference correspondences. As an application, an existence theorem of an equilibria for a qualitative game is obtained in Hausdorff not necessarily locally convex spaces.

1. Introduction and Preliminaries

The existence of equilibria is an abstract economy with compact strategy sets in $\mathbb{R}^n$ was proved in seminal paper of G. Debren. This results generalized the earlier work of Nash in game theory. Since then, there have been many generalizations of Debreu's theorem by considering preference correspondences that are not necessarily transitive or total, by allowing externalities in consumption and by assuming that the commodity space is not necessarily finite-dimensional. In these papers, the domain of preference and constraint correspondences is assumed to be compact or paracompact. Following the work of H. Sonnenschein existence theorems of maximal elements deal with preference correspondences which have open lower sections or are majorised by correspondences with open lower sections. The objective of this note is to give some existence theorems of maximal elements an equilibria in qualitative games without the compactness or paracompactness assumption on the domain of the preferences which are majorised by upper semicontinuous correspondences in not necessarily locally convex spaces.

Now we give some notations. Let A be a set. We shall denote by 2^A the family all subsets (including the empty subset $\varnothing$) of A. If A is a subset of a topological space X, we shall denote by $\mathrm{cl}_X(A)$ the closure of A in X. If A

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is a subset of a vector space, we shall denote by $\mathrm{co\,A}$ the convex hull of A. If A is a non-empty subset of a topological vector space E and $S, T : A \to 2^E$ are correspondences, then $\mathrm{co\,T}$, $T \cap S : A \to 2^E$ are correspondences defined by $(\mathrm{co\,T})(\mathrm{x}) = \mathrm{co\,T(x)}$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively.

If X and Y are topological spaces and $T : X \to 2^Y$ is a correspondence, then:

1. T is said to be upper semicontinuous (u.s.c.) at $x \in X$ if for any open subset U of Y containing $T(x)$, the set $\{z \in X : \ T(z) \subset U\}$ is an open neighborhood of x in X;

2. T is upper semicontinuous (on X) if T is upper semicontinuous at x for each $x \in X$;

3. the graph of T, denoted by Graph(T), is the set $\{(x, y) \in X \times Y : y \in T(x)\}$;

4. the correspondence $\bar{T} : \ X \to 2^Y$ is defined by $\bar{T}(x) = \{y \in Y : (x, y) \in \mathrm{cl}_{\mathrm{X}\times\mathrm{Y}}\mathrm{Graph\,(T)}\}$, and

5. the correspondence $\mathrm{cl\,T : X \to 2^Y}$ is defined by $\mathrm{(cl\,)T(x) = cl_Y(T(x))}$ for each $x \in X$.

It is easy to see that $\mathrm{(cl\,)T(x)} \subset \bar{\mathrm{T}}(\mathrm{x})$, for each $x \in X$. We remark there that in defining upper semicontinuity of T at $x \in X$, we do not require that $T(x)$ be non-empty.

Let X be a topological space, Y be a non-empty subset of a vector space E, $\theta : T \to E$ be a map and $\phi : X \to 2^Y$ be a correspondence. Then:

1. ϕ is said to be of class $\mathcal{U}_\theta$ if a) for each $x \in X$, $\theta(x) \notin \phi(x)$ and b) ϕ is upper semicontinuous with closed and convex values in Y;

2. ϕ_x is a $\mathcal{U}_\theta$-majorant of ϕ at x if there is an open neighborhood $\mathcal{N}(x)$ of x in X and $\phi_x : \mathcal{N}(x) \to 2^Y$ such that (a) for each $z \in \mathcal{N}(x)$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \phi_x(z)$ and (b) ϕ_x is upper semicontinuous with closed and convex values;

3. ϕ is said to be $\mathcal{U}_\theta$-majorised if for each $x \in X$ with $\phi(x) \neq \varnothing$, there exists a $\mathcal{U}_\theta$-majorant ϕ_x of ϕ at x. We remark that when $X = Y$ and $\theta = \mathbf{I}_X$, the identity map on X, our notions of a $\mathcal{U}_\theta$-majorant of ϕ at x and a $\mathcal{U}_\theta$-majorised correspondence are generalization of upper

semicontinuous correspondences which are irreflexive (that is, $x \notin \phi(x)$ for all $x \in X$) and have closed convex values.

In this paper, we shall deal mainly with either the case (I) $X = Y$ and X is a non-empty convex subset of the topological vector space E and $\theta = \mathbf{I_X}$, the identity map on X, or the case (II) $X = \prod_{i \in I} X_i$ and $\theta = \pi_j : X \to X_j$ is the projection of X onto X_j and $Y = X_j$ is a non-empty convex subset of a topological vector space. In both cases (I) and (II), we shall write $\mathcal{U}$ in place of $\mathcal{U}_\theta$.

Let I be a (possibly infinite) set of players. For each $i \in I$, let its choice or strategy set X_i be a non-empty subset of a topological vector space. Let $X = \prod_{i \in I} X_i$. For each $i \in I$, let $P_i : X \to 2^{X_i}$ be a preference correspondence. The collection $\Gamma = (X_i, P_i)_{i \in I}$ will be called a qualitative game (the notion of Gale and Mas-Colell [4]). A point $\hat{x} \in X$ is said to be an equilibrium of the game Γ if $P_i(\hat{x}) = \varnothing$ for all $i \in I$.

2. Existence of Maximal Elements

At first we recall some well known results.

Lemma 2.1 ([4]). *Let X and Y be two topological spaces, A be an open subset of X. Suppose $F_1 : X \to 2^Y$, $F_2 : A \to 2^Y$ are upper semicontinuous such that $F_2(x) \subset F_1(x)$, for all $x \in A$. Then the map $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x), & \text{if } x \notin A, \\ F_2(x), & \text{if } x \in A, \end{cases}$$

is also upper semicontinuous.

Lemma 2.2 ([4]). *Let X be a topological space and Y be a normal space. If $F, G : X \to 2^Y$ have closed values and are upper semicontinuous at $x \in X$, then $F \cap G$ is also upper semicontinuous at x.*

Lemma 2.3 ([1]). *Let D be a non-empty compact subset of a topological vector space E. Then* $\operatorname{co} \mathrm{D}$ *is σ-compact and hence is paracompact.*

Theorem 2.1 ([4]). *Let X be a paracompact space and Y be a non-empty normal subset of a topological vector space E. Let $\theta : X \to E$ and $P : X \to 2^Y \backslash \{\varnothing\}$ be $\mathcal{U}_\theta$-majorized. Then there exists a correspondence $\Psi : X \to 2^Y \backslash \{\varnothing\}$ of class $\mathcal{U}_\theta$ such that $P(x) \subset \Psi(x)$ for each $x \in X$.*

Definition 2.1 ([3]). Let X be a Hausdorff topological vector space, $K \subset X$ and $\mathcal{V}$ the fundamental system of neighborhoods of zero in X. The set K is said to be of Z- type if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{V}$ such that

$$\mathrm{co}\,(\mathrm{U} \cap (\mathrm{K} - \mathrm{K})) \subset \mathrm{V}.$$

Remark 2.1. Every subset $K \subset X$, where X is a locally convex topological vector space, is of Z-type. In [3] examples of subset $K \subset X$ of Z-type, where X is not locally convex topological vector spaces, are given.

Proposition 2.1 ([2]). *Let $\{K_i\}_{i \in I}$ be a family of nonempty convex subsets of Hausdorff topological vector spaces $\{X_i\}_{i \in I}$ $(K_i \subset X_i,\ for\ every\ i \in I)$, $X = \prod\limits_{i \in I} X_i$ and $K = \prod\limits_{i \in I} K$. If for every $i \in I$ the set K_i is of Z- type in X_i, then K is of Z- type in X.*

The next fixed point theorem will be an essential tool for proving the existence of solution in our optimization problems.

Theorem 2.2 ([3]). *Let K be a convex subset of a Hausdorff topological vector space X and D a non empty compact subset of K. Let $S : K \to 2^D$ be an u.s.c. mapping such that for each $x \in K, S(x)$ is a nonempty closed convex subset of D and $S(K)$ is of Z-type. Then there exists a point $\breve{x} \in D$ such that $\breve{x} \in S(\breve{x})$.*

We now prove the following theorem on existence of maximal element.

Theorem 2.3. *Let X be a non-empty convex subset of Hausdorff topological vector space and D be a non-empty compact subset of X. Let $P : X \to 2^D$ be $\mathcal{U}$-majorised and let $P(X)$ be of Z-type.Then there exists a point $x^* \in \mathrm{co}\,\mathrm{D}$ such that $P(x^*) = \varnothing$.*

Proof. Let us suppose, the contrary, that for all $x \in \mathrm{co}\,\mathrm{D}$, $P(x) \neq \varnothing$. Since $\mathrm{co}\,\mathrm{D}$ by Lemma 2.3 is paracompact, one can apply Theorem 2.1. So there exists a correspondence $\psi : \mathrm{co}\,\mathrm{D} \to 2^{\mathrm{D}}$, of class $\mathcal{U}$, such that for each $x \in \mathrm{co}\,\mathrm{D}$, $P(x) \subset \psi(x)$. The correspondence ψ is upper semicontinuous with non-empty closed convex values so by a fixed point theorem of O. Hadžić [3] there exists $x^* \in \mathrm{co}\,\mathrm{D}$ such that $x^* \in \psi(x^*)$. This contradicts that ψ is of a class $\mathfrak{U}$. The proof is complete. $\square$

3. Existence of Equilibria

In this section, we shall give some applications of Theorem 2.3. At first we prove the following result:

Theorem 3.1. *Let X be a non-empty convex subset of Hausdorff topological vector space and D be a non-empty compact subset of X. Let $P : X \to 2^D$ be $\mathcal{U}$-majorised and $A : X \to 2^D$ be upper semicontinuous with closed and convex values. If $P(X)$ is of Z-type then there exists a point $\hat{x} \in \operatorname{co} D$ such that either $\hat{x} \in A(\hat{x})$ and $P(\hat{x}) = \varnothing$ or $\hat{x} \notin A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \varnothing$.*

Proof. Let $F = \{x \in X : x \in A(x)\}$. Since A is upper semicontinuous with closed values note that F is closed in X. Define correspondence $\phi : X \to 2^D$ by

$$\phi(x) = \begin{cases} P(x), & x \in F, \\ A(x) \cap P(x), & x \notin F. \end{cases}$$

By assumption since P is on $\mathcal{U}$-majorised there exists an open neighborhood $\mathcal{N}(x)$ of x X and $\psi_x : \mathcal{N}(x) \to 2^D$ such that

1° for each $z \in \mathcal{N}(x)$, $P(z) \subset \psi_x(z)$ and $z \notin \psi_x(z)$;

2° ψ_x is upper semicontinuous with closed and convex values.

If $x \notin F$ (it means that $x \notin A(x)$) and $A(x) \cap P(x) \neq \phi$, then $X \backslash F$ is an open neighborhood of x in X, and without loss of generality, we may assume that $\mathcal{N}(x) \subset X \backslash F$. Now, let us define a new correspondence $\Psi :\ \mathcal{N}(x) \to 2^D$ by

$$\Psi_x(z) = A(z) \cap \phi_x(z)\ ,\ z \in \mathcal{N}(x)$$

One can prove using Lemma 2.2 that this mapping is a $\mathcal{U}$-majorant of ϕ at x.

If $x \in F$ (it means that $x \in A(x)$) and $P(x) \neq \varnothing$ the correspondence $\Psi'_x :\ \mathcal{N}(x) \to 2^D$ defined by

$$\Psi'_x(z) = \begin{cases} \psi_x(z), & z \in \mathcal{N}(x) \cap F \\ A(x) \cap \psi_x(z), & z \in \mathcal{N}(x) \backslash F \end{cases}$$

is $\mathcal{U}$-majorant of ϕ at x (use Lemma 2.1). For more details see [4].

Therefore ϕ is $\mathcal{U}$-majorised. By Theorem 2.3 there exists a point $\hat{x} \in \operatorname{co} D$ such that $\phi(\hat{x}) = \varnothing$. But, by the definition of ϕ, either $P(\hat{x}) = \varnothing$ and $\hat{x} \in A(\hat{x})$ or $A(\hat{x}) \cap P(\hat{x}) = \varnothing$ and $\hat{x} \notin A(\hat{x})$. $\square$

The following result is an equilibrium existence theorem of a qualitative game:

Theorem 3.2. *Let $\Gamma = (X_i, P_i)_{i\in I}$ be a qualitative game such that for each $i \in I$,*

a) *X_i is a non-empty convex Z-type subset of Hausdorff topological vector space E_i and D_i is a non-empty compact subset of X_i;*

b) *the set $E^i = \{x \in X : P_i(x) \neq \varnothing\}$ is open in X;*

c) *$P_i : E^i \to 2^{D_i}$ is $\mathcal{U}$-majorised;*

d) *there exists a non-empty compact and convex subset F_i of D_i, such that $F_i \cap P_i(x) \neq \varnothing$ for each $x \in E^i$.*

Then there exists a point $x \in X$ such that $P_i(x) = \varnothing$ for all $i \in I$.

Proof. Since D_i is non-empty compact for each $i \in I$ the set $D = \prod_{i\in I} D_i$ is also a non-empty compact in X. Let $I(x) = \{i \in I; P_i(x) \neq \varnothing\}$ for $x \in X$. As in [4] one can construct an $\mathcal{U}$-majorant of P in any point $x \in X$, $P(x) \neq \varnothing$. Now from Theorem 2.3 there exists a point $x^* \in \operatorname{co} \mathrm{D}$ such that $P(x^*) = \varnothing$ which implies that $P_i(x^*) = \varnothing$ for all $i \in I$. $\square$

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