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SOME PROBLEMS OF CONVERGENCE IN NORMED SPACES

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Abstract. The existence of the very know Banach limits has been shown in [1, pp. 28–29]. In [2] G.G. Lorenz defined, by Banach limits, a method of summability of real sequences (called almost convergence of sequences). In [3] were shown the existence of the family of functionals (of the kind of the Banach limits) defined on the real vector space $\mathbf{m}$ of all bounded sequences in a real normed space X. In [4] by these functionals the almost convergence of real sequences was extended to the vector sequences $(x_n) \in \mathbf{m}$.

This paper is devoted to the convergence of sequences $(x_n) \in \mathbf{m}$ and it is organized as follows. First, we will show the existence of an another family of functionals on the space $\mathbf{m}$ and define the convergence of a sequence $(x_n) \in \mathbf{m}$ by these functionals. We also show that our definition is equivalent to the corresponding usual definition. Further, we will show three theorems, which contain necessary and sufficient conditions for a sequence $(x_n) \in \mathbf{m}$ to be convergent. These theorems are new and fundamental. Next, we will give a new definition of a Cauchy sequence. Further, we also define the notion of improper Cauchy sequences and prove that every Cauchy sequence is an improper Cauchy and that there are improper Cauchy sequences which are not Cauchy.

1. A New Family of Functionals

Let us define the functional q on the space $\mathbf{m}$ by

$$(1.1)\quad q(x) \equiv q(x_n) = \sup_{p,\,(k_i)} \left\{ \overline{\lim_{n\to+\infty}}\ \frac{1}{p} \left\| \sum_{i=0}^{p-1} x_{n+k_i} \right\|_X \right\}, \quad (x_n) = x \in \mathbf{m},$$

where the supremum is taken over all possible choices of natural numbers $p\,(= 1, 2, \ldots)$ and all sequences (k_i), $k_i \in \{0, 1, 2, \ldots\}$ $(i = 0, 1, 2, \ldots)$.

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The functional q is seen to be real-valued and clearly it satisfies the conditions

$$q(x) \geq 0, \quad q(ax) = |a|q(x), \quad q(x+y) \leq q(x) + q(y) \quad (a \in \mathbb{R};\ x, y \in \mathbf{m}),$$

that is, q is symmetric convex functional on the space $\mathbf{m}$; so, by corollary of Hahn-Banach theorem (see, also, [5, Exercise 11.2, p. 187]) there exists a nontrivial linear functional L on the space $\mathbf{m}$ such that

$$|L(x_n)| \leq q(x_n), \quad (x_n) \in \mathbf{m}. \tag{1.2}$$

Let us show now that the functional L is not unique. To do this let $\mathbf{m}_0$ be the space of all sequences $(x_n) \in \mathbf{m}$ having $\lim\limits_{n\to+\infty} x_n = 0$. Then clearly we have

$$q(x_n) = L(x_n) = 0, \quad (x_n) \in \mathbf{m}_0. \tag{1.3}$$

Further, for some $\xi \in X$ $(\xi \neq 0)$, define the sequence $y = (y_n)$ by

$$y_n = \xi \quad (n = 1, 2, \ldots).$$

Then $y \in \mathbf{m} \setminus \mathbf{m}_0$ and $q(y_n) = \|\xi\| > 0$. To extend the functional L to the space spanned by $\mathbf{m}_0$ and $\{y\}$ (that is, the space $\mathbf{m}_0 \cup \{y\}$), we can choose the value $L(y_n)$ arbitrarily in the segment $[-q(y_n), q(y_n)]$. Thus, we can extend the functional $L : \mathbf{m}_0 \to \mathbb{R}$ in a such way that it has distinct values at the point $y \in \mathbf{m}$. In other words, the functional L satisfying the above conditions is not unique. Indeed, we can take the value $L(y_n)$ arbitrarily in the segment $[k, K]$, where

$$k = \sup_{x\in\mathbf{m}_0} \{-q(x+y)\}, \quad M = \inf_{x\in\mathbf{m}_0} \{q(x+y)\}$$

(see [6, p. 222]). Further, by (1.1), we have $q(x+y) = q(y) = q(y_n)$, since the sequence $x + y = x + (y_n)$ converges to s.

We show now the following auxiliary result:

Lemma 1.1. *Let X be a real linear space and $q : X \to \mathbb{R}$ a functional such that the following assertions are valid*

$$q(x) \geq 0,\ q(ax) = |a|\, q(x), \quad q(x+y) = q(x) + q(y) \quad (a \in \mathbb{R}; x, y \in \mathbf{m}).$$

Then, for each $x_0 \in X$, there exists a linear functional L on X such that

$$(\forall x \in X) \quad |L(x)| \leq q(x),\ L(x_0) = q(x_0).$$

Proof. The set $X_0 = \{\alpha x_0 : \alpha \in \mathbb{R}\}$ clearly is a subspace of the space X, and L_0, defined by

$$L_0(\alpha x_0) = \alpha q(x_0) \quad (\alpha \in \mathbb{R}),$$

is a linear functional on the subspace X_0 satisfying the condition

$$|L_0(\alpha x_0)| = |\alpha q(x_0)| = |\alpha| q(x_0) = q(\alpha x_0) \quad (\alpha \in \mathbb{R}).$$

By a corollary of the Hahn-Banach theorem (see [5, Theorem 11.2, p. 181]) there exists a linear functional L on the space X extending L_0 and satisfying the condition

$$(\forall x \in X) \quad |L(x)| \le q(x).$$

Also, we have

$$L(x_0) = L_0(x_0) = 1 \cdot q(x_0) = q(x_0),$$

which completes the proof. □

Now, denoting by Π the family of all functionals L on the space $\mathbf{m}$ satisfying the above conditions, then for each $x \in X$ we obtain

$$(1.4) \qquad (\forall L \in \Pi)\ L(x_n - s) = 0 \ \text{ iff } \ q(x_n - s) = 0 \quad ((x_n) \in \mathbf{m}).$$

Indeed, $q(x_n - s) = 0$ clearly implies $L(x_n - s) = 0$, for all $L \in \Pi$. Further, the implication

$$(\forall L \in \Pi) \quad L(x_n - s) = 0 \ \Rightarrow \ q(x_n - s) = 0$$

is equivalent to the implication

$$q(x_n - s) > 0 \ \Rightarrow \ (\exists L \in \Pi) \quad L(x_n - s) \ne 0$$

which, by the lemma proved before, is valid. So, (4) is true.

Thus, we have proved the following statement.

Theorem 1.1. *There exists the family Π of nontrivial functionals L defined on the space $\mathbf{m}$ such that for all $a, b \in \mathbb{R}$, each $s \in X$ and all $(x_n), (y_n) \in \mathbf{m}$ the following assertions are valid*

1° $L(ax_n + by_n) = aL(x_n) + bL(y_n)$,

2° $|L(x_n)| \le q(x_n)$,

3° $(\forall L \in \Pi)$ $L(x_n - s) = 0$ *if and only if* $q(x_n - s) = 0$.

2. Definition of Convergence in a Normed Space

By Banach shift-invariant functionals in [4] were defined the almost convergence of a sequence $(x_n) \in \mathbf{m}$. Analogously, we here define the convergence of a sequence $(x_n) \in \mathbf{m}$ by the functionals from Theorem 1.1.

Definition 2.1. A sequence $(x_n) \in \mathbf{m}$ converges to $s \in X$ and s is called its limit (written, as usual, $\lim\limits_{n\to+\infty} x_n = s$ or $x_n \to s$ as $n \to +\infty$) if

$$(\forall L \in \Pi) \quad L(x_n - s) = 0. \tag{2.1}$$

We show now that the limit of a sequence $(x_n) \in \mathbf{m}$, defined in a such way, is uniquely determined. To do this suppose s' and s'' are any two limits of a sequence $(x_n) \in \mathbf{m}$ and define the sequences (y_n) and (z_n) by

$$y_n = s', \quad z_n = s'' \qquad (n = 1, 2, \ldots)$$

Then, by (2.1),

$$(\forall L \in \Pi) \quad L(z_n - y_n) = L(x_n - y_n) - L(x_n - z_n) = L(x_n - s') - L(x_n - s'') = 0$$

which, by (1.4) and (1.1), implies

$$q(z_n - y_n) = \|s'' - s'\| = 0 \ \text{ or } \ s' = s''.$$

Now, the following theorem is important:

Theorem 2.1. *Definition* 2.1 *and the corresponding standard definition are equivalent.*

Proof. Let $x_n \to s$ as $n \to +\infty$ in the sense of Definition 2.1. Then, by (2.1) and (1.4), we have $q(x_n - s) = 0$ which, by (1.1), for all p and (k_i) implies

$$\overline{\lim_{n\to+\infty}} \frac{1}{p} \left\| \sum_{i=0}^{p-1} (x_{n+k_i} - s) \right\| = 0.$$

Whence, for $p = 1$ and $k_i = 0$ $(i = 0, 1, 2, \ldots)$, we have

$$\overline{\lim_{n\to+\infty}} \|x_n - s\| = 0 \ \text{ or } \ x_n \to s \ \text{ as } \ n \to +\infty$$

in the sense of the usual definition.

Conversely, let $x_n \to s$ as $n \to +\infty$ in the sense of the standard (usual) definition. Then for any $\varepsilon > 0$ there exists an integer $n_0 > 0$ such that for all (k_i) we have

$$\|x_{n+k_i} - s\| < \varepsilon, \quad n > n_0.$$

Hence, for all p and (k_i), we obtain

$$\frac{1}{p}\left\|\sum_{i=0}^{p-1}(x_{n+k_i} - s)\right\| \le \frac{1}{p}\sum_{i=0}^{p-1}\|x_{n+k_i} - s\| < \varepsilon, \quad n > n_0.$$

Then

$$q(x_n - s) = \sup_{p,(k_i)}\left\{\overline{\lim_{n\to+\infty}}\frac{1}{p}\left\|\sum_{i=0}^{p-1}(x_{n+k_i} - s)\right\|_X\right\} \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily, we have $q(x_n - s) = 0$. Hence, by (1.4), we have

$$(\forall L \in \Pi) \quad L(x_n - s) = 0$$

which, by (2.1), means that $x_n \to s$ as $n \to +\infty$ in the sense of Definition 2.1. □

Remark 2.1. It is clear, from expressions and results obtained before, as in S. Banach, G.G. Lorentz and other papers, that it is possible to obtain not only a generalization of usual convergence, but usual convergence itself, too.

3. Two Statements on Usual Convergence in Normed Space

The sense of the following two statements is to additionally characterize usual convergence in a normed space, by some expressions analogous to proceeding ones.

Theorem 3.1. *A sequence (x_n) of points in a normed space X converges to $s \in X$ as $n \to +\infty$ if and only if*

$$\left\|\frac{1}{p}\sum_{i=0}^{p-1}x_{n+k_i} - s\right\|_X \to 0 \quad as \quad n \to +\infty \tag{3.1}$$

uniformly in p and (k_i), $p = 1, 2, \ldots$; $k_i \in \{0, 1, 2, \ldots\}$, $i = 0, 1, 2, \ldots$.

Proof. Let $x_n \to s$ as $n \to +\infty$. Then, for any $\varepsilon > 0$, there exists an integer $n_0 > 0$ such that for all (k_i) we have

$$\|x_{n+k_i} - s\| < \varepsilon, \quad n > n_0$$

which, for all $p\,(=1,2,\ldots)$ and all $k_i \in \{0,1,2,\ldots\}$, implies

$$\left\| \frac{1}{p} \sum_{i=0}^{p-1} x_{n+k_i} - s \right\| < \frac{1}{p} \sum_{i=0}^{p-1} \|x_{n+k_i} - s\| < \varepsilon, \quad n > n_0.$$

Hence,

$$\left\| \frac{1}{p} \sum_{i=0}^{p-1} x_{n+k_i} - s \right\|_X \to 0 \text{ as } n \to +\infty$$

uniformly in p and (k_i); that is, the condition (6) is necessary.

Conversely, if the condition (3.1) is true, then for $p = 1$ and $k_i = 0$ $(i = 0,1,2,\ldots)$ we have $x_n \to s$ as $n \to +\infty$; that is, the condition (3.1) is sufficient. $\square$

The following result is an interesting modification of Theorem 3.1.

Theorem 3.2. *A sequence (x_n) of points in a normed space X converges to $s \in X$ as $n \to +\infty$ if and only if*

$$\left\| \frac{1}{p} \sum_{i=\nu p}^{(\nu+1)p-1} x_{n+k_i} - s \right\|_X \to 0 \quad \textit{as} \quad n \to +\infty \tag{3.2}$$

uniformly in (k_i), p and ν $(\nu = 0,1,2,\ldots)$.

Proof. Let $x_n \to s$ as $n \to +\infty$. Then, for any $\varepsilon > 0$, there exists an integer $n_0 > 0$ such that for all $p, \nu, (k_i)$ we have

$$\|x_{n+k_i+\nu p} - s\| < \varepsilon, \quad n > n_0;$$

so, for all $p, \nu, (k_i)$, we obtain

$$\left\| \frac{1}{p} \sum_{i=\nu p}^{(\nu+1)p-1} x_{n+k_i} - s \right\|_X \le \frac{1}{p} \sum_{i=0}^{p-1} \|x_{n+k_i+\nu p} - s\| < \varepsilon, \quad n > n_0$$

uniformly in p, ν and (k_i); that is, the condition (3.2) is necessary.

Conversely, suppose that the condition (3.2) is valid. Then for $\nu = 0$ from (3.2) follows (3.1). Hence, by Theorem 3.1, the condition (3.2) is sufficient. $\square$

4. Cauchy Sequences

Having obtained the above results we give the following definition.

Definition 4.1. A sequence (x_n) of points in a normed space X is Cauchy if for any $\varepsilon > 0$ there exists an integer $n_0 > 0$ such that for all $n, m\,(= 1, 2, \dots)$

$$n, m \geq n_0 \;\Rightarrow\; \sup_{p,\,(k_i)} \frac{1}{p} \left\| \sum_{i=0}^{p-1} (x_{n+k_i} - x_{m+k_i}) \right\| < \varepsilon. \tag{4.1}$$

Now, the following statement is important.

Theorem 4.1. *Definition* 4.1 *and a standard definition of the Cauchy sequence in a normed space X are equivalent.*

Proof. Let (x_n), $x_n \in X$ be a Cauchy sequence in the sense of Definition 2.1. Then for $p = 1$ and $k_i = 0$ $(i = 0, 1, 2, \dots)$, by (8), we obtain

$$n, m \geq n_0 \Rightarrow \|x_n - x_m\| < \varepsilon$$

which means that (x_n) is Cauchy in the sense of the corresponding usual definition.

Conversely, suppose (x_n) is Cauchy in the sense of the usual definition. Then, for any $\varepsilon > 0$, there exists a natural number n_0 such that

$$n, m \geq n_0 \Rightarrow \|x_n - x_m\| < \varepsilon$$

which, for all (k_i), implies

$$n, m \geq n_0 \Rightarrow \|x_{n+k_i} - x_{m+k_i}\| < \varepsilon.$$

Therefore

$$n, m \geq n_0 \Rightarrow \sup_{p,(k_i)} \frac{1}{p} \left\| \sum_{i=0}^{p-1} (x_{n+k_i} - x_{m+k_i}) \right\| < \varepsilon$$

which, by (4.1), means that the sequence (x_n) is Cauchy in the sense of Definition 4.1. $\square$

Since a sequence (x_n) of points in a complete normed space X is Cauchy sequence if and only if it is convergent, from Definition 2.1 and Theorem 4.1 directly follows the following statement.

Theorem 4.2. *A sequence (x_n) of points in a complete normed space X converges as $n \to +\infty$ if and only if*

$$\sup_{p,\,(k_i)} \frac{1}{p} \left\| \sum_{i=0}^{p-1} (x_{n+k_i} - x_{m+k_i}) \right\| \to 0 \quad \text{as} \quad n, m \to +\infty. \tag{4.2}$$

5. Improper Cauchy Sequence

Definition 5.1. A sequence (x_n) of points in a normed space X is a improper Cauchy sequence if, for any $\varepsilon > 0$, there exists a natural number n_0 such that for all $n, m\,(= 1, 2, \ldots)$

$$n, m \geq n_0 \Rightarrow \inf_{p,(k_i)} \frac{1}{p} \left\| \sum_{i=0}^{p-1} (x_{n+k_i} - x_{m+k_i}) \right\| < \varepsilon. \tag{5.1}$$

Now, a question on the connection between the class of Cauchy sequences and the class of improper Cauchy sequences can be stated. An answer gives the following statement:

Theorem 5.1. *Let X be any real normed space. Then*

1° *Every Cauchy sequence in X is a improper Cauchy sequence;*

2° *There are improper Cauchy sequences in each nontrivial normed space X which are not Cauchy sequences.*

Proof. The part 1° is clear. In order to prove 2° we consider the sequences

$$a, 0, a, 0, a, 0, \ldots \quad \text{and} \quad a, -a, a, -a, \ldots \quad (a \in X, a \neq 0).$$

Clearly, they are not Cauchy sequences. Further, for all even p and $k_i = i$ $(i = 0, 1, 2, \ldots)$, for both sequences we have

$$\sum_{i=0}^{p-1} (x_{n+i} - x_{m+i}) = 0 \quad (n, m, = 1, 2, \ldots),$$

which shows, because of (5.1), that these sequences are improper Cauchy sequences. □

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