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THE CONVERSE THEOREM OF APPROXIMATION BY ANGLE IN VARIOUS METRICS FOR NON-PERIODIC FUNCTIONS

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Abstract. The modulus of smoothness and the best approximation by angle of the derivative of a non-periodic function in the metric of L_q are estimated by the best approximations by angle from entire functions of exponential type in the metric L_p, $1 \le p \le q < +\infty$. In addition, the representation theorem of the derivative of a function is proved. The corresponding results for periodic functions are given in [4].

1. Introduction and Preliminaries

In this paper results from [4] are extended to the case of non–periodic functions defined on the space $\mathbb{R}^n$. The results that were obtained can be used to establish embedding theorems of the space of functions which are defined by the modulus of smoothness, or by the best approximations.

In this section we will give the notations and assertions which are used to obtain the fundamental results of the paper. The entire functions of exponential type are the basic tools used to obtain the results.

Let $\mathbb{R}^n$ denote an n–dimensional Euclidian space and let $\boldsymbol{x} = (x_1, \dots, x_n)$ denote its points. For a real–valued function $f(x)$ it is supposed that $|f(\boldsymbol{x})|^p$ is integrable with norm

$$\|f\| = \|f\|_p = \Big(\int |f(\boldsymbol{x})|^p \, d\boldsymbol{x}\Big)^{1/p} , \quad 1 \le p < +\infty , \quad \int = \int_{\mathbb{R}^n} .$$

Denote the set of all functions f for which $\|f\| < +\infty$ by $L_p = L_p(\mathbb{R}^n)$.

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Denote the mixed modulus of smoothness of order k_{i_j} with respect to the variable x_{i_j} of function f, $1 \le i_j \le n$, $1 \le j \le m \le n$ (see [3], [4]), by

$$\omega_{k_{i_1},\dots,k_{i_m}}(f, \delta_{i_1}, \dots, \delta_{i_m})_p\,,$$

where $k_{i_j} \in \mathbb{N}$ and $\delta_{i_j} \ge 0$.

Let $g_{\nu_i} = g_{\nu_i}(x_1, \dots, x_n) \in L_p$ be an entire function of exponential type ν_i with respect to the variable x_i, but with respect to all other variables $g_{\nu_i} \in L_p$ it follows that $g_{\nu_i} = 0$ for $\nu_i = 0$, $1 \le p < +\infty$ (see [2], 3.1, 3.2.2).

The best approximation by an m–dimensional angle of the function $f \in L_p$ with respect to the variables $x_{i_1}, \dots, x_{i_m}$, $1 \le i_j \le n$, $1 \le j \le m \le n$, is the quantity (see [3])

$$Y_{\nu_{i_1}\dots\nu_{i_m}}(f)_p = \inf_g \Big\| f - \sum_{j=1}^{m} g_{\nu_{i_j}} \Big\|_p\,, \quad \nu_i \ge 0\,. \tag{1.1}$$

Like in the paper [3] we will be using the Fejer general integral, which for a function $f(x)$ of one variable x is given by the following equality

$$K_\lambda f = K_\lambda f(\boldsymbol{x}) = \frac{\lambda}{2} \int f(\boldsymbol{x} - \boldsymbol{t})\, \Phi\Big(\frac{\lambda}{2}\, \boldsymbol{t}\Big)\, d\boldsymbol{t}\,, \quad \lambda > 0\,, \qquad \int = \int_{-\infty}^{+\infty}\,, \tag{1.2}$$

where (see [1], 106; [2], 8.6)

$$\Phi(u) = \frac{\cos u - \cos 2u}{\pi u^2}\,, \quad \|\Phi\|_1 < +\infty\,. \tag{1.3}$$

For a function of two variables $f(x, y) \in L_p(\mathbb{R}^2)$ we form the integrals

$$\begin{aligned} K_{\lambda\infty} f &= K_{\lambda\infty} f(x, y) = \frac{\lambda}{2} \int f(x - t, y)\, \Phi\Big(\frac{\lambda}{2}\, t\Big)\, dt\,, \\ K_{\infty\mu} f &= K_{\infty\mu} f(x, y) = \frac{\mu}{2} \int f(x, y - u)\, \Phi\Big(\frac{\mu}{2}\, u\Big)\, du\,, \\ K_{\lambda\mu} f &= K_{\lambda\mu} f(x, y) = K_{\lambda\infty} K_{\infty\mu} f(x, y)\,, \end{aligned} \tag{1.4}$$

where $\lambda, \mu > 0$, $\displaystyle\int = \int_{-\infty}^{+\infty}$.

If we denote

$$\Theta_\lambda(t) = \frac{\lambda}{2}\, \Phi\Big(\frac{\lambda}{2}\, t\Big)\,, \tag{1.5}$$

$$W_{\lambda\mu}(t,u) = \Theta_\lambda(t)\Theta_\mu(u)\,, \tag{1.6}$$

then

$$K_{\lambda\mu}f(x,y) = \int_{\mathbb{R}_2} f(x-t,y-u)\,W_{\lambda\mu}(t,u)\,dtdu = W_{\lambda\mu} * f\,. \tag{1.7}$$

Therefore, the function $K_{\lambda\mu}f(x,y)$ is the convolution of the functions $W_{\lambda\mu}$ and f. The function $W_{\lambda\mu}$ belongs to the space $L_1(\mathbb{R}^2)$ and $\|W_{\lambda\mu}\|_1 \le M$ independent of $\lambda > 0$, $\mu > 0$.

We define

$$K_{0\infty}f = 0\ , \quad K_{\infty 0}f = 0\ , \quad K_{00}f = 0\ , \quad K_{\lambda 0}f = 0\ , \quad K_{0\mu}f = 0\,, \tag{1.8}$$

for $\lambda > 0$, $\mu > 0$.

The function $K_{\lambda\infty}f(x,y)$ is entire of exponential type λ with respect to x, and function $K_{\infty\mu}f(x,y)$ is entire of exponential type with respect to y. Function $K_{\lambda\mu}f(x,y)$ is entire of exponential type λ with respect to x and of type μ with respect to y (see [1], [2]).

The two–dimensional angle from the Fejer integrals is the function

$$\chi_{\lambda\mu}f = K_{2\lambda\infty}f + K_{\infty 2\mu}f - K_{2\lambda 2\mu}f\,. \tag{1.9}$$

The following result holds:

Lemma 1.1 ([3, Lemma 1])**.** *For a function $f(x,y) \in L_p(\mathbb{R}^2)$, $1 \le p < +\infty$, and $\lambda \ge 0$, $\mu \ge 0$ the following holds*

$$\left\|f - \chi_{\lambda\mu}f\right\|_p \le CY_{\lambda\mu}(f)_p\,, \tag{1.10}$$

where C is an absolute constant.

The best approximation of the function $f(x,y)$ by a one–dimensional angle is denoted by

$$Y_{\lambda\infty}(f)_p = Y_\lambda(f)_p\ , \quad Y_{\infty\mu}(f)_p = Y_\mu(f)_p\,. \tag{1.11}$$

It holds

$$Y_{\lambda\infty}(f)_p = Y_{\lambda 0}(f)_p\ , \quad Y_{\infty\mu}(f)_p = Y_{0\mu}(f)_p\,. \tag{1.12}$$

Note that

$$Y_{0\infty}(f)_p = Y_{\infty 0}(f)_p = Y_{00}(f)_p = \|f\|_p\,, \quad 1 \le p < +\infty\,. \tag{1.13}$$

We define the entire functions $g_{ij} = g_{ij}f$, $i,j = 0,1,2,\dots$ as following

$$\begin{aligned} g_{00} &= K_{22}f\,,\\ g_{0j} &= K_{22^{j+1}}f - K_{22^j}f\,,\\ g_{i0} &= K_{2^{i+1}2}f - K_{2^i 2}f\,,\\ g_{ij} &= K_{2^{i+1}2^{j+1}}f - K_{2^{i+1}2^j}f - K_{2^i 2^{j+1}}f + K_{2^i 2^j}f\\ &\quad = -\big\{\chi_{2^i 2^j}f - \chi_{2^i 2^{j-1}}f - \chi_{2^{i-1}2^j}f + \chi_{2^{i-1}2^{j-1}}f\big\}\,, \end{aligned} \tag{1.14}$$

where $i,j = 1,2,\dots$.

The function $g_{ij}f$ is entire of type 2^{i+1} with respect to x and of type 2^{j+1} with respect to y. From Lemma 1.1 we deduce that the following holds

$$\|g_{ij}f\|_p \ll Y_{[2^{i-1}][2^{j-1}]}(f)_p\,, \quad i,j = 0,1,2,\dots\,, \tag{1.15}$$

where $\left[2^{k-1}\right] = 0$ for $k = 0$, $\left[2^{k-1}\right] = 2^{k-1}$ for $k =\ge 1$.

Instead of $a \le Cb$ we use $a \ll b$, where C is a constant which does not depend on a and b, $a \ge 0$, $b \ge 0$.

The function $f(x,y)$ can be represented by the entire function $g_{ij}f = g_{ij}$. It holds:

Lemma 1.2 ([3, Thm. 2]). *For a function $f(x,y) \in L_p(\mathbb{R}^2)$, $1 \le p < +\infty$, in the sense of L_p, the following equality holds*

$$f(x,y) \overset{(p)}{=} \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} g_{ij}f\,. \tag{1.16}$$

We also need the following result:

Lemma 1.3. *For singular integrals $K_{2^{\lambda+1}\infty}f$, $K_{\infty 2^{\mu+1}}f$ of a function f, $f(x,y) \in L_p(\mathbb{R}^2)$, $1 \le p < +\infty$, in the sense of L_p, the following equalities hold*

$$K_{2^{\lambda+1}\infty}f \overset{(p)}{=} \sum_{i=0}^{\lambda}\sum_{j=0}^{+\infty} g_{ij}f\,, \quad \lambda = 0,1,2,\dots\,, \tag{1.17}$$

$$K_{\infty 2^{\mu+1}}f \overset{(p)}{=} \sum_{i=0}^{+\infty}\sum_{j=0}^{\mu} g_{ij}f\,, \quad \mu = 0,1,2,\dots\,. \tag{1.18}$$

Proof. For a fixed number λ note the sequence

$$G_{\lambda N} = G_{\lambda N} f = \sum_{i=0}^{\lambda} \sum_{j=0}^{N} g_{ij} f \ , \quad N = 0, 1, 2, \dots \ . \tag{1.19}$$

In view of (1.14), we deduce that

$$G_{\lambda N} f = K_{2^{\lambda+1} 2^{N+1}} f \, . \tag{1.20}$$

Since

$$K_{2^{\lambda+1}\infty} f - K_{2^{\lambda+1} 2^{N+1}} f = K_{2^{\lambda+1}\infty} \big(f - K_{\infty 2^{N+1}} f \big),$$

then using (1.20) and Lemma 1.1, we get

$$\big\| K_{2^{\lambda+1}\infty} f - G_{\lambda N} f \big\|_p \ll Y_{\infty 2^N}(f)_p \, . \tag{1.21}$$

Since $Y_{\infty 2^N}(f)_p \to 0$ as $N \to +\infty$ (see Theorem 1 in [3]), then in view of (1.21) and (1.19) we deduce that (1.17) holds.

In the same way we prove equality (1.18).

Thus, Lemma 1.3 has been proved. $\square$

To prove Lemma 1.4, which is greatly used to obtain the fundamental results of the paper, we will use methods and facts concerning the space which is wider than the space L_p. That space is the space of generalized functions (distributions).

Denote the set of all finite functions which are infinitely differentiable in $\mathbb{R}^n$ by $\mathbb{D} = \mathbb{D}(\mathbb{R}^n)$. The convergence in $\mathbb{D}$ is defined, so $\mathbb{D}$ is called the space of test functions.

A linear continuous functional on the space of test functions $\mathbb{D}$ is called a generalized function. The set of all generalized functions is denoted by $\mathbb{D}' = \mathbb{D}'(\mathbb{R}^n)$.

Every function $f(\boldsymbol{x})$ which is locally integrable on $\mathbb{R}^n$ defines the generalized function by equality

$$(f, \varphi) = \int f(\boldsymbol{x}) \varphi(\boldsymbol{x}) \, d\boldsymbol{x} \ , \quad \varphi \in \mathbb{D} \, ,$$

and it is called regular generalized function. In that sense the inclusion $L_p \subset \mathbb{D}'$ holds.

The convolution $f*g$ of generalized functions f and g is defined in [5, 7.4]. The following assertion holds ([5, 7.5(c)]): *If the convolution $f*g$ exists, then so do the convolutions $D^{\boldsymbol{\alpha}} f * g$ and $f * D^{\boldsymbol{\alpha}} g$ with*

$$D^{\boldsymbol{\alpha}}(f * g) = D^{\boldsymbol{\alpha}} f * g = f * D^{\boldsymbol{\alpha}} g$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ is a vector with non–negative integer components α_j and $D^{\boldsymbol{\alpha}} f$ denotes the derivative

$$D^{\boldsymbol{\alpha}} f(\boldsymbol{x}) = \frac{\partial^{|\boldsymbol{\alpha}|} f(x_1, \dots, x_n)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \;, \quad |\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n \,.$$

Now we can prove the following result:

Lemma 1.4. *Let $f(x,y) \in L_p(\mathbb{R}^2)$, $1 \le p < +\infty$, and let r_1, r_2 be non–negative integers. In the function $f(x,y)$ has the derivative*

$$f^{(r_1,r_2)} = \frac{\partial^{r_1+r_2} f}{\partial x^{r_1} \partial y^{r_2}} \in L_p(\mathbb{R}^2) \,,$$

then the convolution $K_{\lambda\mu} f = W_{\lambda\mu} * f$ has the derivative $\big(K_{\lambda\mu} f\big)^{(r_1,r_2)} \in L_p$ and almost everywhere the equality

$$\big(K_{\lambda\mu} f\big)^{(r_1,r_2)} = K_{\lambda\mu} f^{(r_1,r_2)} \;, \quad \lambda, \mu \ge 0 \,, \tag{1.22}$$

holds.

Proof. In view of the assertion above (see [5]), we deduce that

$$\Big(\big(K_{\lambda\mu} f\big)^{(r_1,r_2)}, \varphi\Big) = \big(K_{\lambda\mu} f^{(r_1,r_2)}, \varphi\big) \;, \quad \varphi \in \mathbb{D} \,, \tag{1.23}$$

holds.

Since $f^{(r_1,r_2)} \in L_p$ and $W_{\lambda\mu} \in L_1$ we have $K_{\lambda\mu} f^{(r_1,r_2)} \in L_p$. Therefore from equality (1.23) the equality (1.22) follows (Lemma of Du Bois Reymond, [5, 5.6]).

Lemma 1.4 has been proved. $\square$

2. On Representation of the Derivative of a Function

In this section we derive the theorem which is used to represent the derivatives of a function and derivatives of Fejer singular integrals of the function using the series whose terms are entire functions. In fact, we generalize Lemmas 1.2 and 1.3.

Theorem 2.1. *Let $f(x,y) \in L_p(\mathbb{R}^2)$, $1 \le p \le q < +\infty$. Let r_i be non–negative integers,*

$$\sigma_i = r_i + \frac{1}{p} - \frac{1}{q}\ , \quad i = 1,2\,,$$

and

$$\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}(i+1)^{\sigma_1 q-1}(j+1)^{\sigma_2 q-1}Y_{ij}^{q}(f)_p < +\infty\,. \tag{2.1}$$

Then functions $f(x,y)$, $K_{2^{\lambda+1}\infty}f$, $K_{\infty 2^{\mu+1}}f$ have derivatives

$$f^{(r_1,r_2)}\ , \quad \big(K_{2^{\lambda+1}\infty}f\big)^{(r_1,r_2)}\ , \quad \big(K_{\infty 2^{\mu+1}}f\big)^{(r_1,r_2)}$$

belonging to the space L_q and the following equalities hold (in the sense of L_q):

$$\big(K_{2^{\lambda+1}\infty}f\big)^{(r_1,r_2)} \overset{(q)}{=} \sum_{i=0}^{\lambda}\sum_{j=0}^{+\infty} g_{ij}^{(r_1,r_2)}\ , \quad \lambda = 0,1,2,\dots\,, \tag{2.2}$$

$$\big(K_{\infty 2^{\mu+1}}f\big)^{(r_1,r_2)} \overset{(q)}{=} \sum_{i=0}^{+\infty}\sum_{j=0}^{\mu} g_{ij}^{(r_1,r_2)}\ , \quad \mu = 0,1,2,\dots\,, \tag{2.3}$$

$$f^{(r_1,r_2)} \overset{(q)}{=} \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} g_{ij}^{(r_1,r_2)}\,. \tag{2.4}$$

Proof. Taking into account Lemmas 1.2 and 1.3 and the fact that g_{ij} are entire functions of exponential type 2^{i+1} with respect to x and 2^{j+1} with respect to y, and that for the entire functions and trigonometric polynomials the same inequalities used in [4] hold, we can apply the method used to prove the corresponding theorem for periodic functions (see [4, Thm. 3.1]). Therefore in the proof of this theorem we will only give the main results using the results obtained in [4].

We will prove that equality (1.17) from Lemma 1.3 holds in L_q. As in [4] we denote

$$G^P_{\lambda N} = \sum_{i=0}^{\lambda} \sum_{j=N+1}^{P} g_{ij} = G_{\lambda P} - G_{\lambda N} \ , \quad P > N+1 \, , \tag{2.5}$$

$$A = \left\| G^P_{\lambda N} \right\|_q^q . \tag{2.6}$$

Then (see [4, Eqs. (3.14)–(3.54)])

$$A \ll \sum_{i=0}^{\lambda} \sum_{j=N+1}^{P} 2^{(i+j)q\left(\frac{1}{p}-\frac{1}{q}\right)} Y^q_{[2^{i-1}][2^{j-1}]}(f)_p \, . \tag{2.7}$$

Using (2.7) and condition (2.1) we deduce that in the sense of L_q equality (1.7) holds, i.e.

$$K_{2^{\lambda+1}\infty} f \overset{(q)}{=} \sum_{i=0}^{\lambda} \sum_{j=0}^{+\infty} g_{ij} \, . \tag{2.8}$$

In the following step from equality (2.8) we derive equality (2.2). To obtain that we use the method which was used in [4] to obtain equality (3.75) from equality (3.58) and the corresponding inequality for entire functions of exponential type.

Similarly equalities (2.3) and (2.4) are established.

Thus, Theorem 2.1 has been proved. □

Theorem 2.2. *Let conditions of Theorem* 2.1 *hold for a function* $f(x,y)$. *Then*

$$\left(K_{2^{\lambda+1}\infty} f\right)^{(r_1,r_2)} = K_{2^{\lambda+1}\infty} f^{(r_1,r_2)} \, , \tag{2.9}$$

$$\left(K_{\infty 2^{\mu+1}} f\right)^{(r_1,r_2)} = K_{\infty 2^{\mu+1}} f^{(r_1,r_2)} \, , \tag{2.10}$$

$$\left(\chi_{2^\lambda 2^\mu}\right)^{(r_1,r_2)} = \chi_{2^\lambda 2^\mu} f^{(r_1,r_2)} \, , \tag{2.11}$$

where $\lambda, \mu = 0, 1, 2, \ldots$.

Proof. In view of equality (1.14), Theorem 2.1 and Lemma 1.4 we deduce that equality

$$\left(g_{ij} f\right)^{(r_1,r_2)} = g_{ij} f^{(r_1,r_2)} \tag{2.12}$$

holds.

Now, from (2.2) in view of (2.12) it follows

$$\big(K_{2^{\lambda+1}\infty}f\big)^{(r_1,r_2)} \overset{(q)}{=} \sum_{i=0}^{\lambda}\sum_{j=0}^{+\infty} g_{ij}f^{(r_1,r_2)}\,. \tag{2.13}$$

Since $f^{(r_1,r_2)} \in L_q(\mathbb{R}^2)$ (Theorem 2.1), we can apply Lemma 1.3 and obtain

$$K_{2^{\lambda+1}\infty}f^{(r_1,r_2)} \overset{(q)}{=} \sum_{i=0}^{\lambda}\sum_{j=0}^{+\infty} g_{ij}f^{(r_1,r_2)}\,. \tag{2.14}$$

Equalities (2.13) and (2.14) yield equality (2.9).

Similarly, we prove equality (2.10). Equality (3.11) is the consequence of equalities (1.9), (2.10), (2.11), Theorem 2.1 and Lemma 11.4.

Thus, Theorem 2.2 has been proved. $\square$

Corollary 2.3. *The equality* (2.4) *implies*

$$\big\|f^{(r_1,r_2)}\big\| \ll \left\{\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}(i+1)^{\sigma_1 q-1}(j+1)^{\sigma_2 q-1}Y_{ij}^{q}(f)_p\right\}^{1/q}. \tag{2.15}$$

Putting $p=q$, from Theorem 2.1 we obtain the following statement:

Corollary 2.4. *Let $f(x,y)\in L_p(\mathbb{R}^2)$, $1\le p<+\infty$. Let r_1 be non–negative integers, and*

$$\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}(i+1)^{r_1 p-1}(j+1)^{r_2 p-1}Y_{ij}^{p}(f)_p < +\infty\,. \tag{2.16}$$

Then functions $f(x,y)$, $K_{2^{\lambda+1}\infty}f$, $K_{\infty 2^{\mu+1}}f$ have derivatives

$$f^{(r_1,r_2)}\,,\quad \big(K_{2^{\lambda+1}\infty}f\big)^{(r_1,r_2)}\,,\quad \big(K_{\infty 2^{\mu+1}}f\big)^{(r_1,r_2)}$$

belonging to the space L_p and equalities (2.2), (2.3), (2.4) *hold* (*in the sense of L_p*).

3. On Approximation by Angle of the Derivative of a Function

Theorems 2.1 and 2.2 give a possibility to estimate the best approximation by angle of the derivative of a function in the norm of L_q by best approximations by angle of the function in the norm of L_p.

Theorem 3.1. *Let conditions of Theorem* 2.1 *be satisfied for a function* $f(x,y) \in L_p(\mathbb{R}^2)$, $1 \le p \le q < +\infty$. *Then*

$$Y_{2^\lambda 2^\mu}\left(f^{(r_1,r_2)}\right)_q \ll \left\{\sum_{i=\lambda}^{+\infty}\sum_{j=\mu}^{+\infty} 2^{iq\sigma_1}2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\right\}^{1/q}, \tag{3.1}$$

$$Y_{2^\lambda \infty}\left(f^{(r_1,r_2)}\right)_q \ll \left\{\sum_{i=\lambda}^{+\infty}\sum_{j=\mu}^{+\infty} 2^{iq\sigma_1}2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\right\}^{1/q}, \tag{3.2}$$

$$Y_{\infty 2^\mu}\left(f^{(r_1,r_2)}\right)_q \ll \left\{\sum_{i=0}^{+\infty}\sum_{j=\mu}^{+\infty} 2^{iq\sigma_1}2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\right\}^{1/q}, \tag{3.3}$$

for $\lambda, \mu = 0, 1, 2, \ldots$.

Proof. By definition of the best approximation by angle and equality (1.9) we deduce that

$$Y_{2^\lambda 2^\mu}\left(f^{(r_1,r_2)}\right)_q \le \left\|f^{(r_1,r_2)} - \chi_{[2^{\lambda}-1][2^{\mu}-1]}f^{(r_1,r_2)}\right\|_q \tag{3.4}$$

from which, using theorems 2.2 and 2.1, it follows

$$Y_{2^\lambda 2^\mu}\left(f^{(r_1,r_2)}\right)_q \le \left\|\sum_{i=\lambda}^{+\infty}\sum_{j=\mu}^{+\infty} g_{ij}^{(r_1,r_2)}\right\|_q . \tag{3.4}$$

Inequality (3.5) yields (3.1) (see (2.5), (2.6) and (2.7)).

Similarly, inequalities (3.2) and (3.3) are proved.

Thus, Theorem 3.1 has been proved. □

4. The Converse Theorem of Approximation by Angle

In this section we establish the converse theorem of approximation by angle in various metrics for non-periodic functions using the results of Sections 2 and 3. This way we obtain the generality of Theorem 3 in [3] for $1 \le p < +\infty$.

Theorem 4.1. *Let $f(x_1,\dots,x_n)\in L_p(\mathbb{R}^n)$, $1\le p\le q<+\infty$, $k_i,l_i\in\mathbb{N}$, r_i nonnegative integers, $\sigma_i=r_i+\frac{1}{p}-\frac{1}{q}$, $i=1,\dots,n$, and let*

$$\sum_{\nu_1=0}^{+\infty}\cdots\sum_{\nu_n=0}^{+\infty}\prod_{i=1}^{n}(\nu_i+1)^{q\sigma_i-1}Y^q_{\nu_1\dots\nu_n}(f)_p<+\infty\,. \tag{4.1}$$

Then for the mixed modulus of smoothness ω of the derivative $f^{(r_1,\dots,r_n)}$, for any set of indices $\{i_1,\dots,i_n\}$, $1\le i_j\le n$, $1\le j\le m\le n$, the following holds

$$\begin{aligned}\omega_{k_{i_1}\dots k_{i_m}}&\Big(f^{(r_1,\dots,r_n)},\frac{1}{l_{i_1}},\dots,\frac{1}{l_{i_m}}\Big)_q\\
\le& C\sum_{\{i_1,\dots,i_s\}}\prod_{j=1}^{s}l_{i_j}^{-k_{i_j}}\\
&+\Big\{\sum_{\nu_{i_1}=0}^{l_{i_1}}\cdots\sum_{\nu_{i_s}=0}^{l_{i_s}}\prod_{j=1}^{s}(\nu_{i_j}+1)^{q(k_{i_j}+\sigma_{i_j})-1}\\
&\times\sum_{\nu_{i_{s+1}}=l_{i_{s+1}}+1}^{+\infty}\cdots\sum_{\nu_{i_m}=l_{i_m}+1}^{+\infty}\sum_{\nu_{i_{m+1}}=0}^{+\infty}\cdots\sum_{\nu_{i_n}=0}^{+\infty}\\
&\prod_{j=s+1}^{n}(\nu_{i_j}+1)^{q\sigma_{i_j}-1}Y^q_{\nu_1\dots\nu_n}(f)_p\Big\}^{1/q}\\
&+\Big\{\sum_{\nu_{i_1}=l_{i_1}+1}^{+\infty}\cdots\sum_{\nu_{i_m}=l_{i_m}+1}^{+\infty}\sum_{\nu_{i_{m+1}}=0}^{+\infty}\cdots\sum_{\nu_{i_n}=0}^{+\infty}\\
&\prod_{j=1}^{n}(\nu_{i_j}+1)^{q\sigma_{i_j}-1}Y^q_{\nu_1\dots\nu_n}(f)_p\Big\}^{1/q},\end{aligned}\tag{4.2}$$

where summing is over all $\{i_1,\dots,i_s\}\subset\{i_1,\dots,i_m\}$, and constant C does not depend on neither f nor $l_i=1,2,\dots$.

For $n=2$ the following inequalities are contained in formula (4.2):

$$\omega_{k_1k_2}\big(f^{(r_1,r_2)},1/l_1,1/l_2\big)_q\ \ll\ A, \tag{4.3}$$

$$\omega_{k_1}\big(f^{(r_1,r_2)},1/l_1\big)_q\ \ll\ B, \tag{4.4}$$

$$\omega_{k_2}\big(f^{(r_1,r_2)},1/l_2\big)_q\ \ll\ C, \tag{4.5}$$

where

$$
\begin{aligned}
A \;=\;& l_1^{-k_1} l_2^{-k_2} \Big\{ \sum_{i=0}^{l_1} \sum_{j=0}^{l_2} (i+1)^{q(k_1+\sigma_1)-1} (j+1)^{q(k_2+\sigma_2)-1} Y_{ij}^q(f)_p \Big\}^{1/q} \\
&+ l_1^{-k_1} \Big\{ \sum_{i=0}^{l_1} (i+1)^{q(k_1+\sigma_1)-1} \sum_{j=l_2+1}^{+\infty} j^{q\sigma_2-1} Y_{ij}^q(f)_p \Big\}^{1/q} \\
&+ l_2^{-k_2} \Big\{ \sum_{j=0}^{l_2} (j+1)^{q(k_2+\sigma_2)-1} \sum_{i=l_1+1}^{+\infty} i^{q\sigma_1-1} Y_{ij}^q(f)_p \Big\}^{1/q} \\
&+ \Big\{ \sum_{i=l_1+1}^{+\infty} \sum_{j=l_2+1}^{+\infty} i^{q\sigma_1-1} j^{q\sigma_2-1} Y_{ij}^q(f)_p \Big\}^{1/q},
\end{aligned}
$$

$$
\begin{aligned}
B \;=\;& l_1^{-k_1} \Big\{ \sum_{i=0}^{l_1} (i+1)^{q(k_1+\sigma_1)-1} \sum_{j=0}^{+\infty} (j+1)^{q\sigma_2-1} Y_{ij}^q(f)_p \Big\}^{1/q} \\
&+ \Big\{ \sum_{i=l_1+1}^{+\infty} \sum_{j=0}^{+\infty} i^{q\sigma_1-1} (j+1)^{q\sigma_2-1} Y_{ij}^q(f)_p \Big\}^{1/q},
\end{aligned}
$$

$$
\begin{aligned}
C \;=\;& l_2^{-k_2} \Big\{ \sum_{j=0}^{l_2} (j+1)^{q(k_2+\sigma_2)-1} \sum_{i=0}^{+\infty} (i+1)^{q\sigma_1-1} Y_{ij}^q(f)_p \Big\}^{1/q} \\
&+ \Big\{ \sum_{i=0}^{+\infty} \sum_{j=l_2+1}^{+\infty} (i+1)^{q\sigma_1-1} j^{q\sigma_2-1} Y_{ij}^q(f)_p \Big\}^{1/q},
\end{aligned}
$$

for $l_1, l_2 = 1, 2, \ldots$.

Proof. We will prove inequalities (4.3), (4.4) and (4.5). As in [4] (see [4, Proof of Thm. 5.1]), we have

$$
\begin{aligned}
(4.6)\quad \omega_{k_1k_2}\big(f^{(r_1,r_2)}, 1/l_1, 1/l_2\big)_q &\le \omega_{k_1k_2}\big(f^{(r_1,r_2)} - \chi_{2^\lambda 2^\mu} f^{(r_1,r_2)}, 1/l_1, 1/l_2\big)_q \\
&\quad + \omega_{k_1k_2}\big(\chi_{2^\lambda 2^\mu} f^{(r_1,r_2)}, 1/l_1, 1/l_2\big)_q = I_1 + I_2\,.
\end{aligned}
$$

For I_1, by virtue of the property of the modulus and Lemma 1.1, the following inequality

$$
(4.7)\qquad I_1 \ll Y_{2^\lambda 2^\mu}\big(f^{(r_1,r_2)}\big)_q
$$

holds.

Using Theorem 2.2 for quantity I_2 we have

$$I_2 = \omega_{k_1 k_2}\Big(\big(\chi_{2^\lambda 2^\mu} f\big)^{(r_1,r_2)}, 1/l_1, 1/l_2\Big)_q . \tag{4.8}$$

Since $\chi_{00} f = 0$ for $f \in L_p$, $1 \le p < +\infty$, the equality

$$\chi_{2^\lambda 2^\mu} f = \sum_{i=0}^{\lambda} \psi_i + \sum_{j=0}^{\mu} \eta_j - \sum_{i=0}^{\lambda}\sum_{j=0}^{\mu} g_{ij} \tag{4.9}$$

holds, where

$$\psi_i = \chi_{2^i 2^\mu} f - \chi_{[2^{i-1}]2^\mu} f \ , \quad \eta_j = \chi_{2^\lambda 2^j} f - \chi_{2^\lambda [2^{j-1}]} f \, . \tag{4.10}$$

Since

$$\sum_{i=0}^{\lambda} \psi_i = K_{2^{\lambda+1}\infty} f - K_{2^{\lambda+1}2^{\mu+1}} f$$

by virtue of the equalities (1.17), (1.19), (1.20) (Lemma 1.3), we get

$$\sum_{i=0}^{\lambda} \psi_i \overset{(p)}{=} \sum_{i=0}^{\lambda} \sum_{j=\mu+1}^{+\infty} g_{ij} f \, . \tag{4.11}$$

From equality (4.11) we derive equality

$$\left(\sum_{i=0}^{\lambda} \psi_i\right)^{r_1+k_1, r_2+k_2)} \overset{(q)}{=} \sum_{i=0}^{\lambda} \sum_{j=\mu+1}^{+\infty} g_{ij}^{(r_1+k_1, r_2+k_2)} \tag{4.12}$$

with

$$\left\| \left(\sum_{i=0}^{\lambda} \psi_i\right)^{(r_1+k_1)} \right\|_q \ll \left\{ \sum_{i=0}^{\lambda} \sum_{j=\mu+1}^{+\infty} 2^{iq(k_1+\sigma_1)} 2^{jq_2\sigma_2} Y_{[2^{i-1}][2^{j-1}]}(f)_p \right\}^{1/q} . \tag{4.13}$$

In view of the results that we have obtained ((4.6)–(4.13)) and inequality (3.1), applying the method used to get the corresponding inequality in [4]

((5.19)), we deduce that

$$
\begin{aligned}
(4.14)\quad & \omega_{k_1k_2}\big(f^{(r_1,r_2)}, 1/l_1, 1/l_2\big)_q \\
&\ll l_1^{-k_1} l_2^{-k_2}\Big\{\sum_{i=0}^{\lambda}\sum_{j=0}^{\mu} 2^{iq(k_1+\sigma_1)}2^{jq(k_2+\sigma_2)}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\Big\}^{1/q} \\
&+ l_1^{-k_1}\Big\{\sum_{i=0}^{\lambda} 2^{iq(k_1+\sigma_1)}\sum_{j=\mu+1}^{+\infty} 2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\Big\}^{1/q} \\
&+ l_2^{-k_2}\Big\{\sum_{j=0}^{\mu} 2^{jq(k_2+\sigma_2)}\sum_{i=\lambda+1}^{+\infty} 2^{iq\sigma_1}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\Big\}^{1/q} \\
&+ \Big\{\sum_{i=\lambda+1}^{+\infty}\sum_{j=\mu+1}^{+\infty} 2^{iq\sigma_1}2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\Big\}^{1/q}.
\end{aligned}
$$

Choosing λ and μ so that $2^{\lambda-1}\le l_1$, $2^{\mu-1}\le l_2 < 2^{\mu}$, (4.14) implies (4.3).

Now we will prove inequality (4.4). We have

$$
\begin{aligned}
(4.15)\quad \omega_{k_1}\big(f^{(r_1,r_2)}, 1/l_1\big)_1 &\le \omega_{k_1}\big(f^{(r_1,r_2)} - K_{2^{\lambda+1}\infty}f^{(r_1,r_2)}, 1/l_1\big)_q \\
&+ \omega_{k_1}\big(K_{2^{\lambda+1}\infty}f^{(r_1,r_2)}, 1/l_1\big)_q = I_3 + I_4\,.
\end{aligned}
$$

Using the property of modulus and Lemma 1.1 we obtain

$$
(4.16)\qquad I_3 \ll Y_{2^{\lambda}\infty}\big(f^{(r_1,r_2)}\big)_q\,.
$$

In order to estimate the quantity I_4 we will use the equality

$$
\begin{aligned}
(4.17)\qquad K_{2^{\lambda+1}\infty}f^{(r_1,r_2)} &= K_{2^{\lambda+1}\infty}f^{(r_1,r_2)} - K_{2^{\lambda+1}2^{t+1}}f^{(r_1,r_2)} \\
&+ K_{2^{\lambda+1}2^{t+1}}f^{(r_1,r_2)}\,,
\end{aligned}
$$

where t is an arbitrary natural number.

In view of equalities (1.19) and (1.20) and Lemma 1.4, we get

$$
(4.18)\qquad K_{2^{\lambda+1}2^{t+1}}f^{(r_1,r_2)} = \sum_{i=0}^{\lambda}\sum_{j=0}^{t}(g_{ij})^{(r_1,r_2)}\,.
$$

Now we derive

$$\omega_{k_1}\big(K_{2^{\lambda+1}2^{t+1}}f^{(r_1,r_2)}\big)_q \ll l_1^{-k_1}\Big\|\sum_{i=0}^{\lambda}\sum_{j=0}^{t}(g_{ij}f)^{(r_1+k_1,r_2)}\Big\|_q \tag{4.19}$$

$$\ll l_1^{-k_1}\Big\{\sum_{i=0}^{\lambda}2^{iq(k_1+\sigma_1)}\sum_{j=0}^{t}2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\Big\}^{1/q}$$

(see the procedure for estimation of quantity B in Theorem 3.1 in [4]). Also,

$$\omega_{k_1}\big(K_{2^{\lambda+1}\infty}f^{(r_1,r_2)} - K_{2^{\lambda+1}2^{t+1}}f^{(r_1,r_2)}, 1/l_1\big)_q \ll \big\|f^{(r_1,r_2)} - K_{\infty 2^{t+1}}f^{(r_1,r_2)}\big\|_q \ll Y_{\infty 2^t}\big(f^{(r_1,r_2)}\big)_q. \tag{4.20}$$

Using (4.17), (4.19) and (4.20) we obtain

$$I_4 \ll Y_{\infty 2^t}\big(f^{(r_1,r_2)}\big)_q + l_1^{-k_1}\Big\{\sum_{i=0}^{\lambda}2^{iq(k_1+\sigma_1)}\sum_{j=0}^{t}2^{jq\sigma_2}Y_{[2^{i-1}[2^{j-1}]}(f)_p\Big\}^{1/q}. \tag{4.21}$$

By (3.2) and as $t\to+\infty$ in view of (4.15), (4.16) and (4.21) we conclude that

$$\omega_{k_1}\big(f^{(r_1,r_2)}, 1/l_1\big) \ll l_1^{-k_1}\Big\{\sum_{i=0}^{\lambda}2^{iq(k_1+\sigma_1)}\sum_{j=0}^{+\infty}2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\Big\}^{1/q} + \Big\{\sum_{i=\lambda}^{+\infty}\sum_{j=0}^{+\infty}2^{iq\sigma_1}2^{jq\sigma_2}Y^q_{[2^{i-1}][2^{j-1}]}(f)_p\Big\}^{1/q}. \tag{4.22}$$

Choosing λ so that $2^{\lambda-1}\le l_1 < 2^{\lambda}$ from (4.22) we obtain (4.4). Similarly we establish (4.5). $\square$

Remark. Theorem 4.1 can be interpreted geometrically for $n=1,2,3$, which was done in [4, Thm. 5.1].

Corollary 4.2. *For $n=1$ we have $Y=E$ and formula (4.2) contains only the following inequality:*

$$\omega_k\big(f^{(r)}, 1/l\big)_q \ll l^{-k}\Big\{\sum_{i=0}^{l}(i+1)^{q(k+\sigma)-1}E_i^q(f)_p\Big\}^{1/q} + \Big\{\sum_{i=l+1}^{+\infty}i^{q\sigma_1}E_i^q(f)_p\Big\}^{1/q},$$

where

$$\sum_{i=0}^{+\infty}(i+1)^{q\sigma-1}E_i^q(f)_p < +\infty\,, \tag{4.23}$$

$k \in \mathbb{N}$, r *non–negative integer,* $\sigma = r + \frac{1}{p} - \frac{1}{q}$, $l = 1, 2, \ldots$.

If (4.23) *holds then* $f^{(r)} \in L_q(\mathbb{R})$ *and the corresponding theorem of representation holds.*

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