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INEQUALITIES RELATED TO THE ZEROS OF SOLUTIONS OF CERTAIN SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper we establish some new inequalities related to the zeros of the solutions of certain second order differential equations by using elementary analysis. The inequalities obtained here can be used as handy tools in the study of qualitative behavior of solutions of the associated equations.

1. Introduction

In this paper we consider the following nonlinear second order differential equations of the forms:

$$\Big(r(t)|y'(t)|^{\alpha-1}y'(t)\Big)' + q(t)|y(t)|^{\beta-1}y(t) = 0\,, \tag{A}$$

$$\Big(r(t)|y(t)|^{p}|y'(t)|^{k-2}y'(t)\Big)' + q(t)|y(t)|^{p+k-2}y(t) = 0\,, \tag{B}$$

where $t \in I = [t_0, +\infty)$, $t_0 \geq 0$ and I contains the points a and b ($a < b$), $\alpha \geq 1$, $\beta \geq 1$, $p \geq 0$, $k \geq 2$ are real constants and $k > p$, the function $r\colon I \to \mathbb{R} = (-\infty, +\infty)$ is $\mathbb{C}^1$ – smooth and $r > 0$, and the function $q\colon I \to \mathbb{R}$ is continuous. Much has been written about the special versions of equations (A) and (B). The general versions of equations (A) and (B) have been recently dealt with in [1]–[3], [6], [7], [9]–[12] and results on existence, uniqueness and other properties of the solutions are established. The object of this paper is to derive some new inequalities which not only relates points a and b in I at which the solutions of (A), (B) have zeros but also any point $c \in (a, b)$) where the solutions of (A), (B) are maximized. The inequalities that we propose here can be used as handy tools in the study of the qualitative nature of the solutions of equations (A), (B). Here we give some such applications to convey the importance of our results to the literature. Finally, we give in brief the further extensions of our results to the higher order differential equations.

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2. Main Results

Our main results are established in the following theorems.

Theorem 1. *Let $y(t)$ be a solution of* (A) *with $y(a) = y(b) = 0$ and $y(t) \neq 0$ for $t \in (a, b)$. Let $|y((t)|$ be maximized in a point $c \in (a, b)$. Then*

$$1 \leq M^{\beta-\alpha}\left(\int_a^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_a^b |q(s)|\,ds\right), \tag{1}$$

$$1 \leq 2^{\alpha+1}M^{\beta-\alpha}\left(\int_a^c r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_a^c |q(s)|\,ds\right), \tag{2}$$

$$1 \leq 2^{\alpha+1}M^{\beta-\alpha}\left(\int_c^b r^{-(1/\alpha)}(s)\,ds\right)^{\alpha}\left(\int_c^b |q(s)|\,ds\right), \tag{3}$$

where $M = \max |y(t)| = |y(c)|$, $c \in (a, b)$.

Proof. Let $M = |y(c)|$, $c \in (a, b))$. From the hypotheses $y(a) = y(b) = 0$, we have

$$M^2 = 2\int_a^c y(s)y'(s)\,ds\,, \tag{4}$$

$$M^2 = -2\int_c^b y(s)y'(s)\,ds\,. \tag{5}$$

From (4) and (5) we observe that

$$\begin{aligned} M^2 &\leq \int_a^b |y(s)||y'(s)|\,ds \\ &= \int_a^b \left(r^{-(1/(\alpha+1))}(s)|y(s)|\right)\left(r^{(1/(\alpha+1))}(s)|y'(s)|\right)ds\,. \end{aligned} \tag{6}$$

Now by using the Hölder's inequality on the right side of (6) with indices $(\alpha + 1)/\alpha$, $\alpha + 1$, performing integration by parts and using the fact that $y(t)$ is a solution of (A) such that $y(a) = y(b) = 0$, we observe that

$$\begin{aligned} M^2 &\leq \left(\int_a^b r^{-1/\alpha}(s)|y(s)|^{(\alpha+1)/\alpha}\,ds\right)^{\alpha/(\alpha+1)} \\ &\quad\times\left(\int_a^b r(s)|y'(s)|^{\alpha+1}\,ds\right)^{1/(\alpha+1)} \\ &\leq \left(\int_a^b r^{-1/\alpha}(s)|y(s)|^{(\alpha+1)/\alpha}\,ds\right)^{\alpha/(\alpha+1)} \\ &\quad\times\left(\int_a^b |q(s)||y(s)|^{\beta+1}\,ds\right)^{1/(\alpha+1)}, \end{aligned}$$

because of

$$\begin{aligned}\int_a^b r(s)|y'(s)|^{\alpha+1}\,ds &= \int_a^b \Big(r(s)|y'(s)|^{\alpha-1}y'(s)\Big)y'(s)\,ds\\ &= -\int_a^b \Big(r(s)|y'(s)|^{\alpha-1}y'(s)\Big)'y(s)\,ds\\ &= \int_a^b q(s)|y(s)|^{\beta-1}y(s)y(s)\,ds\\ &\le \int_a^b |q(s)||y(s)|^{\beta+1}\,ds\,.\end{aligned}$$

Finally,

$$\begin{aligned}M^2 \le M\,M^{(\beta+1)/(\alpha+1)}&\left(\int_a^b r^{-1/\alpha}(s)\,ds\right)^{\alpha/(\alpha+1)}\\ &\times\left(\int_a^b |q(s)|\,ds\right)^{1/(\alpha+1)}.\end{aligned} \tag{7}$$

Dividing both sides of (7) by M^2 and then raising the power $\alpha+1$ to both sides of the resulting inequality we get the required inequality in (1).

From (4), (5) we have

$$M^2 \le 2\int_a^c |y(s)||y'(s)|\,ds\,, \tag{8}$$

$$M^2 \le 2\int_c^b |y(s)||y'(s)|\,ds\,. \tag{9}$$

Inequalities in (2) and (3) follows in similar way by rewriting the integrand on the right sides in (8) and (9) as in (6) and using Hölder's inequality with indices $(\alpha+1)/\alpha$, $\alpha+1$. Performing the integration by parts, the fact that $y(t)$ is a solution of equation (A) such that $y(a)=y(b)=0$, $y'(c)=0$ and following the last arguments as in the proof of inequality (1) given above. The proof is complete. $\square$

Theorem 2. *Let* $y(t)$ *be a solution of* (B) *with* $y(a)=y(b)=0$ *and* $y(t)\neq 0$

for $t \in (a,b)$. Let $y(t)$ be maximized in a point $c \in (a,b)$. Then

$$1 \le \left(\int_a^b r^{-1/(k-1)}(s)\,ds\right)^{k-1}\left(\int_a^b |q(s)|\,ds\right), \tag{10}$$

$$1 \le 2^k\left(\int_a^c r^{-1/(k-1)}(s)\,ds\right)^{k-1}\left(\int_a^c |q(s)|\,ds\right), \tag{11}$$

$$1 \le 2^k\left(\int_c^b r^{-1/(k-1)}(s)\,ds\right)^{k-1}\left(\int_c^b |q(s)|\,ds\right). \tag{12}$$

Proof. By following the proof of Theorem 1, from the hypotheses we have (4) and (5). From (4) and (5) we have

$$\begin{aligned} M^2 &\le \int_a^b |y(s)||y(s)|\,ds \\ &= \int_a^b \left(r^{-1/k}(s)|y(s)|^{1-p/k}\right)\left(r^{1/k}(s)|y(s)|^{p/k}|y'(s)|\right)ds\,. \end{aligned} \tag{13}$$

Now by applying the Hölder's ineqquality on the right side of (13) with indices $k/(k-1)$, k, performing integration by parts and using the fact that $y(t)$ is a solution of (B) such that $y(a) = y(b) = 0$, we have

$$\begin{aligned} M^2 &\le \left(\int_a^b r^{-1/(k-1)}(s)|y(s)|^{(k-p)/(k-1)}\,ds\right)^{(k-1)/k} \\ &\quad\times\left(\int_a^b r(s)|y(s)|^p|y'(s)|^k\,ds\right)^{1/k} \\ &\le \left(\int_a^b r^{-1/(k-1)}(s)|y(s)|^{(k-p)/(k-1)}\,ds\right)^{(k-1)/k} \\ &\quad\times\left(\int_a^b |q(s)||y(s)|^{p+k}\,ds\right)^{1/k}, \end{aligned}$$

because of

$$\begin{aligned} \int_a^b r(s)|y(s)|^p|y'(s)|^k\,ds &= \int_a^b \left(r(s)|y(s)|^p|y'(s)|^{k-2}y'(s)\right)y'(s)\,ds \\ &= -\int_a^b \left(r(s)|y(s)|^p|y'(s)|^{k-2}y'(s)\right)'y(s)\,ds \\ &= \int_a^b q(s)|y(s)|^{p+k-2}y(s)y(s)\,ds \\ &\le \int_a^b |q(s)||y(s)|^{p+k}\,ds\,. \end{aligned}$$

Finally,

$$M^2 \le M^{(k-p)/k} M^{(p+k)/k} \left(\int_a^b r^{-1/(k-1)}(s)\, ds \right)^{(k-1)/k} \times \left(\int_a^b |q(s)|\, ds \right)^{1/k} . \tag{14}$$

Dividing both sides of (14) by M^2 and then raising the power k to both sides of the resulting inequality we get the required inequality in (10).

The inequalities (11), (12) follows in similar fashion as mentioned in the case of inequalities (2), (3) with suitable modifications. The proof is complete. □

3. Some Applications

We next establish the following theorems which deals with the applications of some of our inequalities given in Theorems 1 and 2.

Theorem 3. (i) *If*

$$\int^{+\infty} r^{-1/\alpha}(s)\, ds < +\infty \;, \quad \int^{+\infty} |q(s)|\, ds < +\infty \,, \tag{15}$$

then every oscillatory solution of (A) *is bounded on* I.

(ii) *If*

$$\int^{+\infty} r^{-1/(k-1)}(s)\, ds < +\infty \;, \quad \int^{+\infty} |q(s)|\, ds < +\infty \,, \tag{16}$$

then every oscillatory solution of (B) *is bounded on* I.

Proof. Here we will prove (i) only. The proof of (ii) can be completed similarly. Let $y(t)$ be an oscillatory solution of equation (A) on I. Suppose to the contrary that $\limsup |y(t)| = +\infty$. Indeed, since $y(t)$ is oscillatory, there exists an interval (t_1, t_2) such that $y(t_1) = y(t_2) = 0$, $|y(t)| > 0$ on (t_1, t_2) and $M = \max\big\{|y(t)| \,:\, t_1 \le t \le t_2\big\}$. Choose c in (t_1, t_2) such that $|y(c)| = M$. Because of (15), we can choose $T \ge t_0$ large enough so that for every $t_1 \ge T$,

$$\int_{t_1}^{+\infty} r^{-1/\alpha}(s)\, ds < M^{-(\beta-\alpha)/\alpha} \;, \quad \int_{t_1}^{+\infty} |q(s)|\, ds < 1 \,. \tag{17}$$

Clearly, the inequality (1) in Theorem 1 is true on the interval (t_1, t_2) and we have

$$1 \le M^{\beta-\alpha}\left(\int_{t_1}^{t_2} r^{-1/\alpha}(s)\,ds\right)^{\alpha}\left(\int_{t_1}^{t_2} |q(s)|\,ds\right). \tag{18}$$

From (18) and using (17) we have

$$1 \le M^{\beta-\alpha}\left(\int_{t_1}^{+\infty} r^{-1/\alpha}(s)\,ds\right)^{\alpha}\left(\int_{t_1}^{+\infty} |q(s)|\,ds\right) < 1\,.$$

This contradiction shows that the solution $y(t)$ of equation (A) is bounded. This completes the proof. □

Theorem 4. *Suppose that* $|q(t)| \in \mathbb{L}^{\mu}[t_0, +\infty)$, $1 \le \mu < +\infty$. *If* $y(t)$ *is any oscillatory solution of* (A) *with* $r(t) = 1$ (*and respectively of* (B) *with* $r(t) = 1$), *then the distance between conscutive zeros of* $y(t)$ *tends to infinity as* $t \to +\infty$.

Proof. Here we will give the proof concerning the equation (B). The proof concerning the equation (A) can be completed similarly. We first assume that $y(t)$ is any oscillatory solution of (B) with $r(t) = 1$ and the conclusion is not true. Then there exists a solution $y(t)$ with its sequence of zeros $\{t_n\}$ having a subsequence $\{t_{n_m}\}$ such that $|t_{n_{m+1}} - t_{n_m}| \le N < +\infty$ for all m. Let s_{n_m} be a point in $(t_{n_m}, t_{n_{m+1}})$ at which $|y(t)|$ is maximized. Then $|s_{n_m} - t_{n_m}| < N$ for all m. Let μ' be the index conjugate with μ, namely $1/\mu + 1/\mu' = 1$. Suppose $|q(t)| \in \mathbb{L}^{\mu}[t_0, +\infty)$, $1 \le \mu < +\infty$, for m large enough we can write

$$\left(\int_{t_{n_m}}^{+\infty} |q(s)|^{\mu}\,ds\right)^{1/\mu} \le 2^{-k} N^{-(k-1+1/\mu')}\,. \tag{19}$$

By using the inequality in (11) with $r(t) = 1$, we have

$$1 \le 2^k \big(s_{n_m} - t_{n_m}\big)^{k-1}\left(\int_{t_{n_m}}^{s_{n_m}} |q(s)|\,ds\right). \tag{20}$$

Now, using the Hölder's inequality with indices μ, μ' on the right side of (20), and then making use of the inequality (19), we get

$$\begin{aligned} 1 &\le 2^k \big(s_{n_m} - t_{n_m}\big)^{k-1}\left(\int_{t_{n_m}}^{s_{n_m}} |q(s)|^{\mu}\,ds\right)^{1/\mu}\big(s_{n_m} - t_{n_m}\big)^{1/\mu'} \\ &\le 2^k \big(s_{n_m} - t_{n_m}\big)^{k-1+1/\mu'}\left(\int_{t_{n_m}}^{+\infty} |q(s)|^{\mu}\,ds\right)^{1/\mu} \\ &< 2^k N^{k-1+1/\mu'} 2^{-k} N^{-(k-1+1/\mu')} = 1\,. \end{aligned}$$

This is a contradiction and the conclusion is true. This completes the proof. $\square$

4. Further Extensions

In this section we indicate in brief the further extensions of our results given in Theorem 1 and 2 to the following higher order differential equations of the forms:

$$\Big(r(t)|y'(t)|^{\alpha-1}y'(t)\Big)^{(n-1)} + q(t)|y(t)|^{\beta-1}y(t) = 0\,, \tag{21}$$

$$\Big(r(t)|y(t)|^{p}|y'(t)|^{k-2}y'(t)\Big)^{(n-1)} + q(t)|y(t)|^{p+k-2}y(t) = 0\,, \tag{22}$$

where α, β, p, k, $r(t)$, $q(t)$ are the same as in equations (A) and (B), except that γ is $\mathbb{C}^{(n-2)}$–smooth, and $n \geq 3$. We use the following notation for simplification of details of presentation. For $t \in I$ and some function $h(t)$, $t \in I$, we set

$$E\big(t, h(t)\big) = \int_t^{\gamma_1}\int_{s_2}^{\gamma_2}\cdots\int_{s_{n-2}}^{\gamma_{n-2}} h(s)\,ds ds_{n-2}\cdots ds_3 ds_2\,, \tag{*}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_{n-2}$ are suitable points in I. We denote by $\bar{E}\big(t, H(t)\big)$ the integral on the right side of (*) with the upper limits $\gamma_1, \gamma_2, \ldots, \gamma_{n-2}$ of integrals are all replaced by the largest γ_i, $i = 1, 2, \ldots, n-2$.

Theorem 5. *Let $\gamma_1 > \gamma_2 > \cdots > \gamma_{n-2}$ be respectively zeros of*

$$\Big(r(t)|y'(t)|^{\alpha-1}y'(t)\Big)'\,,\quad \Big(r(t)|y'(t)|^{\alpha-1}y'(t)\Big)''\,,\ \ldots\,,$$
$$\Big(r(t)|y'(t)|^{\alpha-1}y'(t)\Big)^{(n-2)},$$

where $y(t)$ is a solution of (21), let $a < \gamma_{n-2}$ and $b > \gamma_1$ be zeros of $y(t)$ and $|y(t)|$ is maximized in $c \in (a,b)$. Then

$$1 \leq M^{\beta-\alpha}\left(\int_a^b r^{-1/\alpha}(s)\,ds\right)^{\alpha}\left(\int_a^b \bar{E}\big(s_1, |q(s)|\big)\,ds_1\right), \tag{23}$$

$$1 \leq 2^{\alpha+1}M^{\beta-\alpha}\left(\int_a^c r^{-1/\alpha}(s)\,ds\right)^{\alpha}\left(\int_a^c \bar{E}\big(s_1, |q(s)|\big)\,ds_1\right), \tag{24}$$

$$1 \leq 2^{\alpha+1}M^{\beta-\alpha}\left(\int_c^b r^{-1/\alpha}(s)\,ds\right)^{\alpha}\left(\int_c^b \bar{E}\big(s_1, |q(s)|\big)\,ds_1\right), \tag{25}$$

where $M = \max|y(t)| = |y(c)|$, $c \in (a,b)$.

Theorem 6. *Let $\gamma_1 > \gamma_2 > \cdots > \gamma_{n-2}$ be respectively zeros of*

$$\Big(r(t)|y(t)|^p|y'(t)|^{k-2}y'(t)\Big)', \quad \Big(r(t)|y(t)|^p|y'(t)|^{k-2}y'(t)\Big)'', \ \dots, \\ \Big(r(t)|y(t)|^p|y'(t)|^{k-2}y'(t)\Big)^{(n-2)},$$

where $y(t)$ is a solution of (22), let $a < \gamma_{n-2}$ and $b > \gamma_1$ be zeros of $y(t)$ and $|y(t)|$ is maximized in $c \in (a,b)$. Then

$$1 \le \left(\int_a^b r^{-1/(k-1)}(s)\,ds\right)^{k-1}\left(\int_a^b \bar{E}\big(s_1, |q(s)|\big)\,ds_1\right), \tag{26}$$

$$1 \le 2^k\left(\int_a^c r^{-1/(k-1)}(s)\,ds\right)^{k-1}\left(\int_a^c \bar{E}\big(s_1, |q(s)|\big)\,ds_1\right), \tag{27}$$

$$1 \le 2^k\left(\int_c^b r^{-1/(k-1)}(s)\,ds\right)^{k-1}\left(\int_c^b \bar{E}\big(s_1, |q(s)|\big)\,ds_1\right). \tag{28}$$

Integrating $n-2$ times the equations (21), (22), by using the hypotheses, we get

$$(-1)^{n-2}\Big(r(t)|y'(t)|^{\alpha-1}y'(t)\Big)' + E\big(t, q(s)|y(s)|^{\beta-1}y(s)\big) = 0, \tag{29}$$

$$(-1)^{n-2}\Big(r(t)|y(t)|^p|y'(t)|^{k-2}y'(t)\Big)' + E\big(t, q(s)|y(s)|^{p+k-2}y(s)\big) = 0. \tag{30}$$

By following the proofs of Theorems 1 and 2 with suitable modifications and using the facts that, by hypotheses, the solution $y(t)$ of (21) or (22) satisfies the equivalent integral equations (29) or (30) such that $y(a) = y(b) = 0$, we get the desired inequalities in (23)–(25) and (26)–(28).

Finally, we note that our results in Theorems 1 and 2 can be very easily extended to the following more general equations of the forms:

$$\Big(r(t)|y'(t)|^{\alpha-1}y'(t)\Big)' + q(t)|y(t)|^{\beta-1}y(t)f\big(t, y(t)\big) = 0, \tag{31}$$

$$\Big(r(t)|y(t)|^p|y'(t)|^{k-2}y'(t)\Big)' + q(t)|y(t)|^{p+k-2}y(t)f\big(t, y(t)\big) = 0, \tag{32}$$

and also to the equations of the forms:

$$\Big(r(t)|y'(t)|^{\alpha-1}y'(t)\Big)^{(n-1)} + q(t)|y(t)|^{\beta-1}y(t)f\big(t, y(t)\big) = 0, \tag{33}$$

$$\Big(r(t)|y(t)|^p|y'(t)|^{k-2}y'(t)\Big)^{(n-1)} + q(t)|y(t)|^{p+k-2}y(t)f\big(t, y(t)\big) = 0, \tag{34}$$

where α, β, p, k, $r(t)$, $q(t)$ are as defined in equations (A) and (B), $n \geq 3$ and the function $f: I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the condition $|f(t,y)| \leq w(t,|y|)$, the function $w: I \times \mathbb{R}_+ \to \mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty)$, is continuous and $w(t,u) \leq w(t,v)$ for $0 \leq u \leq v$. For similar results, see [4], [5], [8], [14].

In concluding this paper we note that the results analogous to those of given in Theorems 3 and 4 can be obtained to the equations (21), (22) as well as to the equations (31), (32) and (33), (34). The precise formulations of such results is similar to that of our results given in Theorems 3 and 4 and closely looking at the results given in [3]–[5], [8], [11]–[14] with suitable modifications and hence we do not discuss the details.

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