FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 16 (2001), 25–33

GRONWALL–BELLMAN TYPE INTEGRAL INEQUALITIES FOR MULTI–DISTRIBUTIONS

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Abstract. The object of this paper is to establish a new Gronwall-Bellman type integral inequalities for multi-distributions. These inequalities generalize some results of Zhihong and Yongquing obtained in [5].

1. Introduction

The origin of the results obtained in this paper is the Gronwall–Bellman inequality which plays an important role in the study of the properties of solutions of differential and integral equations (see for example [1] and the references cited therein). Due to various motivations, many linear, nonlinear and discrete generalizations of Gronwall–Bellman type inequalities have been obtained and applied extensively (see for example [1]).

The purpose of this paper is to further investigate the Gronwall type inequalities for multi–distributions and to extend some of the results obtained in [5] and where necessary to obtain improved apriori bounds than those given in [5].

The results obtained in this paper are in the sense of Lebesgue–Stieltjes integral for functions of bounded variation. Throughout this paper, we shall assume that the functions $u_j(t)$ is right continuous at $t = 0$, $j = 1, \dots, m$. We shall let BV(I) denote the set of all functions of bounded variation defined on $I \subset \mathbb{R}$ and taking values in $\mathbb{R}$.

1. Main Results

The following results will be needed in the proof of our main results.

Received January 29, 1999.
2000 *Mathematics Subject Classification.* Primary 26D15, 26D10.

Lemma 2.1 ([5]). *Let f and g be two real-valued functions on the real line $\mathbb{R}$ such that both are of bounded variation on every compact subinterval of $\mathbb{R}$. Then fg defines a distribution, and the derivative of fg in the sense of the distribution is equal to the locally summable function $(fg)'$ given by*

$$f'(x)g(x) + f(x)g'(x)$$

for almost all x. That is $D(fg) = (Df)g + f(Dg)$, where Df and Dg denote the derivatives of the functions f and g respectively in the sense of the distributions.

Theorem 2.1. *Suppose that for $j = 1, \dots, m$ and $t, s \in [0, T]$:*

$1°$ *$Q_j(t, s) \geq 0$, $y(t) \geq 0$ and $Q_j(t, s), y(t), f(t) \in BV[0, T]$.*

$2°$ *$u_j(t)$ are nondecreasing in t.*

$3°$ *$Q_j(t, s)$ and its partial derivatives $\frac{\partial}{\partial t}Q_j(t, s)$ are continuous and nondecreasing in its first variable and that $Q_j(t, s)$ and $\frac{\partial}{\partial t}Q_j(t, s)$ are nonnegative and integrable with respect to $u_j(t)$ and if the following inequality holds*

$$y(t) \leq f(t) + \sum_{j=1}^{m} \int_0^t Q_j(t, s) y(s) du_j(s)\,. \tag{1}$$

Then

$$\begin{aligned} y(t) \leq A_m(f) + A_m(1) \int_0^t \left(Q_m(s, s) A_m(f) + \int_0^s \frac{\partial}{\partial s} Q_m(s, \tau) A_m(f) \right) \\ \times \exp\left(\int_s^t Q_m(s, \tau) A_m(1) du_m(\tau) \right) du_m(s). \end{aligned} \tag{2}$$

for all $t, s, \tau \in [0, T]$, and where $A_k(v)$ is defined inductively as follows

$$\begin{aligned} A_1(v) &= v\,, \\ A_{k+1}(v) &= A_k(v) + \int_0^t \Big(A_k\left(Q_k(s, s)\right) A_k(v) \\ &\quad + \int_0^s \frac{\partial}{\partial s} A_k\left(Q_k(s, \tau)\right) A_k(v) du_k(\tau) \Big) \\ &\quad \times \exp\left(\int_s^t A_k\left(Q_k(s, \tau)\right) du_k(\tau) \right) du_k(s). \end{aligned} \tag{3}$$

Proof. Let

$$x_i(t) = \int_0^t Q_i(t,s)y(s)du_i(s)\,, \quad t,s\in[0,T]\,, \quad i=1,\dots,m\,. \tag{4}$$

Clearly $x_i(t)$ are functions of bounded variation. We also observed that $x_i(0)=0$. Hence, in view of (4), inequality (1) becomes

$$y(t) \le f(t) + \sum_{j=0}^{m} x_j(t)\,. \tag{5}$$

Thus

$$Dx_i(t) = Q_i(t,t)y(t)Du_i(t) + \int_0^t \frac{\partial}{\partial t}Q_i(t,s)y(s)Du_i(s)\,.$$

If we put $i=1$ in (5) and (6), we obtain

$$\begin{aligned} Dx_1(t) &= Q_1(t,t)y(t)Du_1(t) + \int_0^t \frac{\partial}{\partial t}Q_1(t,s)y(s)Du_1(s) \\ &\le \left(Q_1(t,t)\left[f(t)+\sum_{j=1}^{m}x_j(t)\right]\right. \\ &\quad \left.+\int_0^t \frac{\partial}{\partial t}Q_1(t,s)\left[f(s)+\sum_{j=1}^{m}x_j(s)\right]\right)Du_1(t)\,. \end{aligned}$$

That is

$$\begin{aligned} Dx_1(t)-&\left(Q_1(t,t)x_1(t)+\int_0^t \frac{\partial}{\partial t}Q_1(t,s)x_1(s)\right)Du_1(t) \\ &\le \left(Q_1(t,t)\left[f(t)+\sum_{j=2}^{m}x_j(t)\right]\right) \\ &\quad +\int_0^t \frac{\partial}{\partial t}Q_1(t,s)\left[f(s)+\sum_{j=2}^{m}x_j(s)\right]\Bigg)Du_1(t)\,. \end{aligned} \tag{7}$$

Multiply both sides of (7) by $\exp\left(-\int_0^t Q_1(t,s)du_1(s)\right)$ we have

$$\begin{aligned} &\left[Dx_1(t)-\left(Q_1(t,t)x_1(t)+\int_0^t \frac{\partial}{\partial t}Q_1(t,s)x_1(s)\right)Du_1(t)\right] \\ &\qquad\qquad \times\ \exp\left(-\int_0^t Q_1(t,s)du_1(s)\right) \end{aligned}$$

$$\leq \left(Q_1(t,t)\left[f(t)+\sum_{j=2}^{m} x_j(t)\right] \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\left[f(s)+\sum_{j=2}^{m} x_j(s)\right]\right) \times \exp\left(-\int_0^t Q_1(t,s)du_1(s)\right) Du_1(t)\,.$$

By Lemma 2.1, we have

$$D\left(x_1(t)\exp\left(-\int_0^t Q_1(t,s)du_1(s)\right)\right) \leq \left(Q_1(t,t)\left[f(t)+\sum_{j=2}^{m} x_j(t)\right] + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\left[f(s)\sum_{j=2}^{m} x_j(s)\right]\right) \times \exp\left(-\int_0^t Q_1(t,s)du_1(s)\right) Du_1(t)\,.$$

Integrate with respect to t from 0 to t, we have

$$(x_1(t)-x_1(0))\exp\left(-\int_0^t Q_1(t,s)du_1(s)\right) \leq \int_0^t \left(Q_1(s,s)\left[f(s)+\sum_{j=2}^{m} x_j(s)\right] + \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)\left[f(\tau)+\sum_{j=2}^{m} x_j(\tau)\right]\right) \times \exp\left(-\int_0^s Q_1(s,\tau)du_1(\tau)\right) du_1(s)\,.$$

Since $x_1(0)=0$, we obtain

$$(8)\quad x_1(t) \leq \int_0^t \left(Q_1(s,s)\left[f(s)+\sum_{j=2}^{m} x_j(s)\right] + \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)\left[f(\tau)+\sum_{j=2}^{m} x_j(\tau)\right]\right) \exp\left(-\int_s^t Q_1(s,\tau)du_1(\tau)\right) du_1(s)\,.$$

If we put (8) into (5) and using the fact that $X_j(t)$ are nondecreasing, we obtain

$$
\begin{aligned}
(9) \qquad y(t) \le f(t) &+ \int_0^t \biggl(Q_1(s,s)\Bigl[f(s) + \sum_{j=2}^m x_j(s)\Bigr] \\
&+ \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)\Bigl[f(\tau) + \sum_{j=2}^m x_j(\tau)\Bigr]\biggr) \\
&\times \ \exp\left(-\int_s^t Q_1(s,\tau)du_1(\tau)\right) du_1(s) + \sum_{j=2}^m x_j(t) \\
\le f(t) &+ \int_0^t \left(Q_1(s,s)f(s) + \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)f(\tau)\right) \\
&\times \ \exp\left(-\int_0^s Q_1(s,\tau)du_1(\tau)\right) du_1(s) \\
&+ \sum_{j=2}^m \biggl[1 + \int_0^t \left(Q_1(s,s) + \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)\right) \\
&\times \ \exp\left(-\int_s^t Q_1(s,\tau)du_1(\tau)\right) du_1(s)\biggr] x_j(t),
\end{aligned}
$$

i.e.,

$$
y(t) \le A_2(f) + \sum_{j=2}^m A_2(1)x_j(t)\,,
$$

where $A_2(f)$ and $A_2(1)$ are as defined in (3).

When $i = 2$, inequalities (6) and (9) gives

$$
\begin{aligned}
Dx_2(t) &= Q_2(t,t)y(t)Du_2(t) + \int_0^t \frac{\partial}{\partial t} Q_2(t,s)y(s)Du_2(s) \\
&\le \biggl(Q_2(t,t)\Bigl[A_2(f) + \sum_{j=2}^m A_2(1)x_j(t)\Bigr] \\
&\qquad + \int_0^t \frac{\partial}{\partial t} Q_2(t,s)\Bigl[A_2(f) + \sum_{j=2}^m A_2(f)x_j(s)\Bigr]\biggr) Du_2(t)\,,
\end{aligned}
$$

i.e.,

$$Dx_2(t) - \left(Q_2(t,t)A_2(1)x_2(t) + \int_0^t \frac{\partial}{\partial t}Q_2(t,s)A_2(1)x_2(s)\right)Du_2(t) \leq \left(Q_2(t,t)\left[A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(t)\right] + \int_0^t \frac{\partial}{\partial t}Q_2(t,s)\left[A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(s)\right]\right)Du_2(t)\,. \tag{10}$$

Multiply both sides of (10) by $\exp\left(-\int_0^t Q_1(t,s)du_1(s)\right)$ we have

$$\left[Dx_2(t) - \left(Q_2(t,t)A_2(1)x_2(t) + \int_0^t \frac{\partial}{\partial t}Q_2(t,s)A_2(1)x_2(t)\right)Du_2(t)\right] \times \exp\left(-\int_0^t Q_2(t,s)A_2(1)du_2(s)\right) \leq \left(Q_2(t,t)\left[A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(t)\right] + \int_0^t \frac{\partial}{\partial t}Q_2(t,s)\left[A_2(1) + \sum_{j=3}^{m} A_2(1)x_j(s)\right]\right) \times \exp\left(-\int_0^t Q_2(t,s)A_2(1)du_2(s)\right)Du_2(t)\,.$$

By Lemma 2.1, we have

$$D\left(x_2(t)\exp\left(-\int_0^t Q_2(t,s)A_2(1)\,du_2(s)\right)\right) \leq \left(Q_2(t,t)\left[A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(t)\right] + \int_0^t \frac{\partial}{\partial t}Q_2(t,s)\left[A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(s)\right]\right) \times \exp\left(-\int_0^t Q_2(t,s)A_2(1)du_2(s)\right)Du_2(t)\,.$$

Integrate with respect to t from 0 to t and noting that $x_2(0) = 0$ we have

$$x_2(t) \le \int_0^t \bigg(Q_2(s,s)\Big[A_2(f) + \sum_{j=3}^m A_2(1)x_j(s)\Big] + \int_0^s \frac{\partial}{\partial s} Q_2(s,\tau)\Big[A_2(f) + \sum_{j=3}^m A_2(1)x_j(\tau)\Big]\bigg) \times \exp\bigg(-\int_s^t Q_2(s,\tau)A_2(1)du_2(\tau)\bigg)\,du_2(s)\,. \tag{11}$$

On putting (11) into (9) and using the fact that $x_j(t)$ are nondecreasing, we obtain

$$\begin{aligned} y(t) &\le A_2(f) + A_2(1)\int_0^t \bigg(Q_2(s,s)\Big[A_2(f) + \sum_{j=3}^m A_2(1)x_j(s)\Big] \\ &\quad + \int_0^s \frac{\partial}{\partial s} Q_2(s,\tau)\Big[A_2(f) + \sum_{j=3}^m A_2(1)x_j(\tau)\Big]\bigg) \\ &\quad \times \exp\bigg(-\int_s^t Q_2(s,\tau)A_2(1)du_2(\tau)\bigg)\,du_2(s) + \sum_{j=3}^m A_2(1)x_j(t) \\ &\le A_2(f) + A_2(1)\int_0^t \bigg(Q_2(s,s)A_2(f) + \int_0^s \frac{\partial}{\partial s} Q_2(s,\tau)A_2(f)\bigg) \\ &\quad \times \exp\bigg(-\int_0^s Q_2(s,\tau)A_2(1)du_2(\tau)\bigg)\,du_2(s) \\ &\quad + \sum_{j=1}^m \bigg[A_2(1) + A_2(1)\int_0^t \bigg(Q_2(s,s)A_2(1) + \int_0^s \frac{\partial}{\partial s} Q_2(s,\tau)A_2(1)\bigg) \\ &\quad \times \exp\bigg(-\int_0^s Q_2(s,\tau)A_2(1)du_2(\tau)\bigg)\,du_2(s)\bigg]\,x_j(t) \\ &= A_3(f) + \sum_{j=3}^m A_3(1)x_j(t)\,, \end{aligned} \tag{12}$$

where $A_3(f)$ and $A_3(1)$ are as defined in (3).

If we set $i = m-1$, then we easily obtain

$$y(t) \le A_m(f) + A_m(1)x_m(t)\,.$$

Next, suppose $i = m$, then (6) and (11) implies

$$\begin{aligned} Dx_m(t) &= Q_m(t,t)y(t)Du_m(t) + \int_0^t \frac{\partial}{\partial t} Q_m(t,s)y(s)Du_m(s) \\ &\le \Big(Q_m(t,t)\Big[A_m(f) + A_m(1)x_m(t)\Big] \\ &\quad + \int_0^t \frac{\partial}{\partial t} Q_m(t,s)\Big[A_m(f) + A_m(1)x_m(s)\Big]\Big) Du_m(t)\,. \end{aligned}$$

Thus,

$$\begin{aligned} (14)\quad & Dx_m(t) - \Big(Q_m(t,t)A_m(1)x_m(t) + \int_0^t \frac{\partial}{\partial t} Q_m(t,s)A_m(1)x_m(s)\Big) Du_m(t) \\ &\le \left(Q_m(t,t)A_m(f) + \int_0^t \frac{\partial}{\partial t} Q_m(t,s)A_m(f)\right) Du_m(t)\,. \end{aligned}$$

Multiply both sides of (10) by $\exp\Big(-\int_0^t Q_m(t,s)A_m(1)du_m(s)\Big)$ and integrate with respect to t from 0 to t and noting that $x_m(0) = 0$, we have

$$\begin{aligned} (15)\qquad x_m(t) &\le \int_0^t \left(Q_m(s,s)A_m(f) + \int_0^s \frac{\partial}{\partial s} Q_m(s,\tau)A_m(f)\right) \\ &\quad \times \exp\left(-\int_s^t Q_m(t,\tau)A_m(1)du_m(\tau)\right) du_m(s)\,. \end{aligned}$$

Substituting (14) into (13) and noting that $x_m(t)$ is nondecreasing, we obtain

$$\begin{aligned} y(t) &\le A_m(f) + A_m(1) \int_0^t \left(Q_m(s,s)A_m(f) + \int_0^s \frac{\partial}{\partial s} Q_m(s,\tau)A_m(f)\right) \\ &\quad \times \exp\left(-\int_s^t Q_m(t,\tau)A_m(1)du_m(\tau)\right) du_m(s)\,. \end{aligned}$$

This completes the proof of the theorem. □

As an immediate consequence of our result, we observe that if we set $Q_j(t,s) = g_j(t)h_j(s)$, $j = 1, 2, \ldots, m$, then Theorem 2.1 becomes

Corollary 2.1. *Suppose that for* $j = 1, \dots, m$ *and* $t, s \in [0, T]$,

1° $g_j(t) \geq 0$, $y(t) \geq 0$ *and* $g_j(t), y(t), f(t) \in BV[0, T]$.

2° $u_j(t)$ *are nondecreasing in* t.

3° $h_j(t)$ *are nonnegative and integrable with respect to* $u_j(t)$ *and if the following inequality holds*

$$y(t) \leq f(t) + \sum_{j=1}^{m} g_j(t) \int_0^t h_j(s) y(s) du_j(s)\,. \tag{16}$$

Then

$$y(t) \leq A_m(f) + A_m(1) \int_0^t g_m(s) h_m(s) A_m(f) \times \exp\left(\int_s^t g_m(s) h_m(\tau) du_m(\tau) \right) du_m(s)\,. \tag{17}$$

Remark 2.1. Corollary 2.1 is essentially not the same as Theorem 2.1 in [5] in the sense that Corollary 2.1 contains Theorem 2.1 in [5] as a special case. Indeed Corollary 2.1 is more general than Theorem 2.1 in [5].

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