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AN INEQUALITY OF OSTROWSKI TYPE IN n INDEPENDENT VARIABLES

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Abstract. In the present note we establish a new integral inequality of Ostrowski type in n independent variables. The discrete analogue of the main result is also given.

1. Introduction

In 1938 A. Ostrowski ([2, p. 468]) proved the folllowing interesting integral inequality.

Let f be differentiable function on (a, b) and $|f'(x)| \leq M$ on (a, b). Then for every $x \in (a, b)$

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M\,. \tag{1}$$

In the years thereafter, numerious generalizations, extensions, discretizations and variants of inequality (1) have appeared in the literature (see [1], [2], [3]). The purpose of the present note is to establish a new inequality of Ostrowski type in n independent variables. The discrete version of the main result is also given.

2. Main Result

Let $\mathbb{R}^n$ denotes the n–dimensional Euclidean space and S be the set of all–valued functions $u(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^n$, which are continuous on a subset $B = \prod_{i=1}^{n} [a_i, b_i]$ of $\mathbb{R}^n$. For $u \in S$ we denote by $\int_B u(\boldsymbol{x})\,d\boldsymbol{x}$ the n–fold integral $\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} u(x_1, \ldots, x_n)\,dx_1 \cdots dx_n$. The notation $\frac{\partial}{\partial x_i} u(x_1, \ldots, x_n)$ for $i = 1, 2, \ldots, n$, we mean the partial derivatives of $u(x_1, \ldots, x_n)$.

Our main result is given in the following theorem.

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Theorem 1. *Let $u(\boldsymbol{x})$ be a real–valued continuous function defined on B and*

$$v_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) := u(\boldsymbol{x})\,\big|_{x_i=a_i} + u(\boldsymbol{x})\,\big|_{x_i=b_i} \qquad (i=1,\dots,n),$$

then

$$(2)\qquad \left|\int_B u(\boldsymbol{x})\,d\boldsymbol{x} - \frac{1}{2n}\sum_{i=1}^{n}\int_B v_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\,d\boldsymbol{x}\right| \le \frac{1}{2n}\left(\sum_{i=1}^{n}(b_i-a_i)\int_B\left|\frac{\partial}{\partial x_i}u(\boldsymbol{x})\right|d\boldsymbol{x}\right).$$

Proof. For $\boldsymbol{x}\in B$ it is easy to observe that the following identities hold (see [4]):

$$(3)\qquad nu(\boldsymbol{x}) = u(a_1,x_2,\dots,x_n)+\dots+u(x_1,\dots,x_{n-1},a_n) + \int_{a_1}^{x_1}\frac{\partial}{\partial t_1}u(t_1,x_2,\dots,x_n)\,dt_1+\dots+\int_{a_n}^{x_n}\frac{\partial}{\partial t_n}u(x_1,\dots,x_{n-1},t_n)\,dt_n\,,$$

$$(4)\qquad nu(\boldsymbol{x}) = u(b_1,x_2,\dots,x_n)+\dots+u(x_1,\dots,x_{n-1},b_n) - \int_{x_1}^{b_1}\frac{\partial}{\partial t_1}u(t_1,x_2,\dots,x_n)\,dt_1-\dots-\int_{x_n}^{b_n}\frac{\partial}{\partial t_n}u(x_1,\dots,x_{n-1},t_n)\,dt_n\,.$$

From (3) and (4) we get

$$u(\boldsymbol{x}) = \frac{1}{2n}\Big\{\big[u(a_1,x_2,\dots,x_n)+u(b_1,x_2,\dots,x_n)\big]+\cdots + \big[u(x_1,\dots,x_{n-1},a_n)+u(x_1,\dots,x_{n-1},b_n)\big] + \left[\int_{a_1}^{x_1}\frac{\partial}{\partial t_1}u(t_1,x_2,\dots,x_n)\,dt_1 - \int_{x_1}^{b_1}\frac{\partial}{\partial t_1}u(t_1,x_2,\dots,x_n)\,dt_1\right]+\cdots + \left[\int_{a_n}^{x_n}\frac{\partial}{\partial t_n}u(x_1,\dots,x_{n-1},t_n)\,dt_n - \int_{x_n}^{b_n}\frac{\partial}{\partial t_n}u(x_1,\dots,x_{n-1},t_n)\,dt_n\right]\Big\}.$$

Integrating both sides of (5) on B and by making elementary calculations we get the desired inequality in (2) and the proof is complete. □

Remark 1. We note that in the special case when $n=1$, $u(x_1)=u(t)$, $t\in\mathbb{R}$, $a_1=a$, $b_1=b$, the inequality (2) reduces to

$$(6)\qquad \left|\int_a^b u(t)\,dt - \frac{1}{2}(b-a)\big[u(a)+u(b)\big]\right| \le \frac{1}{2}(b-a)\int_a^b |u'(t)|\,dt\,.$$

3. Discrete Analogue

Let $\mathbb{N}$ denotes the set of natural numbers, $A_1 = \{1, 2, \dots, a_1 + 1\}$, $A_2 = \{1, 2, \dots, a_2 + 1\}$, $\dots$, $A_n = \{1, 2, \dots, a_n + 1\}$ for $a_1, a_2, \dots, a_n$ in $\mathbb{N}$ and $A = A_1 \times A_2 \times \cdots \times A_n$. For a real–valued function w defined on $\mathbb{N}^n$ we define the difference operators as

$$\begin{aligned}\Delta_1 w(x_1, x_2, \dots, x_n) &= w(x_1 + 1, x_2, \dots, x_n) - w(x_1, x_2, \dots, x_n)\,,\\ &\ \ \vdots\\ \Delta_n w(x_1, x_2, \dots, x_n) &= w(x_1, \dots, x_{n-1}, x_n + 1) - w(x_1, x_2, \dots, x_n)\,.\end{aligned}$$

For any real–valued function $w(\boldsymbol{x})$ defined on A we use the following notation

$$\sum_A w(\boldsymbol{x}) = \sum_{x_1=1}^{a_1} \cdots \sum_{x_n=1}^{a_n} w(\boldsymbol{x})\,.$$

The discrete version of Theorem 1 is established in the following theorem.

Theorem 2. *Let $u(\boldsymbol{x})$ be a real–valued function defined on A, then*

$$\begin{aligned}(7)\quad &\Bigg| \int_B u(\boldsymbol{x})\, d\boldsymbol{x} - \frac{1}{2n} \Bigg\{ a_1 \sum_{x_2=1}^{a_2} \cdots \sum_{x_n=1}^{a_n} \Big[u(1, x_2, \dots, x_n) + u(a_1 + 1, x_2, \dots, x_n) \Big] \\ &+ \cdots + a_n \sum_{x_1=1}^{a_1} \cdots \sum_{x_{n-1}=1}^{a_{n-1}} \Big[u(x_1, \dots, x_{n-1}, 1) + u(x_1, \dots, x_{n-1}, a_n + 1) \Big] \Bigg\} \Bigg| \\ &\le \frac{1}{2n} \left(\sum_{i=1}^{n} a_i \sum_A \big| \Delta_i u(x) \big| \right).\end{aligned}$$

Proof. For $\boldsymbol{x} = (x_1, \dots, x_n)$ in A it is easy to observe that the following identities hold (see [5])

$$\begin{aligned}(8)\quad nu(\boldsymbol{x}) &= u(1, x_2, \dots, x_n) + \cdots + u(x_1, \dots, x_{n-1}, 1) \\ &+ \sum_{y_1=1}^{x_1-1} \Delta_1 u(y_1, x_2, \dots, x_n) + \cdots + \sum_{y_n=1}^{x_n-1} \Delta_n u(x_1, \dots, x_{n-1}, y_n)\,,\end{aligned}$$

$$\begin{aligned}(9)\quad nu(\boldsymbol{x}) &= u(a_1 + 1, x_2, \dots, x_n) + \cdots + u(x_1, \dots, x_{n-1}, a_n + 1) \\ &- \sum_{y_1=x_1}^{a_1} \Delta_1 u(y_1, x_2, \dots, x_n) - \cdots - \sum_{y_n=x_n}^{a_n} \Delta_n u(x_1, \dots, x_{n-1}, y_n)\, dt_n\,.\end{aligned}$$

From (8) and (9) we get

$$
\begin{aligned}
(10)\ u(\boldsymbol{x}) = \frac{1}{2n}\Big\{ & \big[u(1, x_2, \dots, x_n) + u(a_1 + 1, x_2, \dots, x_n)\big] + \cdots \\
& + \big[u(x_1, \dots, x_{n-1}, 1) + u(x_1, \dots, x_{n-1}, a_n + 1)\big] \\
& + \Bigg[\sum_{y_1=1}^{x_1-1} \Delta_1 u(y_1, x_2, \dots, x_n) - \sum_{y_1=x_1}^{a_1} \Delta_1 u(y_1, x_2, \dots, x_n)\Bigg] + \cdots \\
& + \Bigg[\sum_{y_n=1}^{x_n-1} \Delta_n u(x_1, \dots, x_{n-1}, y_n) - \sum_{y_n=x_n}^{a_n} \Delta_n u(x_1, \dots, x_{n-1}, y_n)\Bigg]\Big\}.
\end{aligned}
$$

Summing both sides of (10) on A and by making elementary calculations we get the desired inequality in (7). The proof is complete. □

Remark 2. In the special case when $n = 1$, $u(x_1) = u(t)$, $t \in \mathbb{N}$, $a_1 = a$, $\Delta_1 = \Delta$, the inequality (7) reduces to

$$
\left|\sum_{t=1}^{a} u(t) - \frac{1}{2} a\big[u(1) + u(a+1)\big]\right| \le \frac{1}{2}\, a \sum_{t=1}^{a} |\Delta u(t)| \,. \tag{11}
$$

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