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CONDITIONS FOR CONVEXITY OF A DERIVATIVE AND APPLICATIONS TO THE GAMMA AND DIGAMMA FUNCTION*

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Abstract. We consider necessary and sufficient conditions for convexity of a function $x \mapsto f'(x)$ in terms of some properties of the associated function of two variables $F(x,y) = (f(y) - f(x))/(y - x)$. These results are applied to the theory of the Gamma function.

1. Introduction

This paper follows the ideas presented in [13] and [14] and discusses the topic of bounds for a ratio of Gamma functions, which has been researched by many authors. Our basic goal is to show that these and many other bounds and asymptotic expansions can be derived as simple consequences of logarithmic convexity of the Gamma function and hence to study the topic in a systematic and unified way. In a sense, we follow Artin's approach [3] to explanation of main properties of the real Gamma function via logarithmic convexity.

In the paper [13] we proposed a method that produces sharp bounds for the ratio $\Gamma(x + \beta)/\Gamma(x)$, where $x > 0$ and $\beta \in (0, 1)$. In [14], we presented an application of logarithmic convexity to bounds for the same ratio that involve the Digamma function. In this paper we start with three necessary and sufficient conditions for convexity of a derivative and apply these conditions to obtain further bounds for the Gamma and the Digamma function. Since in the theory of the Gamma function we deal with smooth functions, the Theorem 1 below is formulated for the functions with continuous third

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derivative, which enables the proof via Taylor's expansions and consequently, an analysis of error terms. A more general case is handled in [15].

2. Necessary and Sufficient Conditions for Convexity

Let f be a function defined on an interval I and let the derivative f' exist. Define the function F of two variables by

$$F(x,y) = \frac{f(y)-f(x)}{y-x} \quad (x \neq y), \qquad F(x,x) = f'(x), \tag{2.1}$$

where $(x,y) \in I^2$. Among other properties, we shall consider Schur-convexity of F. This notion can be formulated for a function in any number of variables, but we need a simplest case of two variables. For a convenience, recall that a symmetric function $(x,y) \mapsto g(x,y)$ is Schur-convex on I^2 if and only if

$$g(x,y) \leq g(x-\varepsilon, y+\varepsilon)$$

for every $x, y \in I$, $\varepsilon > 0$ such that $x-\varepsilon, y+\varepsilon \in I$. A function g is Schur-concave if $-g$ is Schur-convex.

Let $(x,y) \mapsto g(x,y)$ be a symmetric and continuous function on I^2. Suppose that both partial derivatives exist and are continuous on the set $\{(x,y) \in I^2 \mid x \neq y\}$. Then g is Schur-convex on I^2 if and only if ([12, 3.A.4] and [15])

$$\frac{\partial g(x,y)}{\partial y} - \frac{\partial g(x,y)}{\partial x} \geq 0 \quad \text{for } (x,y) \in I^2 \text{ such that } x < y. \tag{2.2}$$

For further details on Schur-convexity see [12].

Let us now consider the following statements:

(A) $\quad f'$ is convex on I,

(B) $\quad F(x,y) \leq \dfrac{f'(x)+f'(y)}{2}$ for all $x, y \in I$,

(C) $\quad f'\left(\dfrac{x+y}{2}\right) \leq F(x,y)$ for all $x, y \in I$,

(D) $\quad F$ is Schur-convex on I^2,

and

(A') f' is concave on I.

(B') $F(x,y) \geq \dfrac{f'(x)+f'(y)}{2}$ for all $x, y \in I$,

(C') $f'\left(\dfrac{x+y}{2}\right) \geq F(x,y)$ for all $x, y \in I$,

(D') F is Schur-concave on I^2.

Theorem 2.1. *If $x \mapsto f'''(x)$ is continuous on I then the conditions $(A)-(D)$ are equivalent and the conditions $(A')-(D')$ are equivalent.*

Proof. $(A) \Rightarrow (C)$: If (A) holds, then $f'''(t) \geq 0$ on I. Let x, y $(x < y)$ be arbitrary points in I. By Taylor's expansion around $c = (x+y)/2$ we have

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + R_1 \tag{2.3}$$

$$f(y) = f(c) + f'(c)(y-c) + \frac{f''(c)}{2}(y-c)^2 + R_2, \tag{2.4}$$

where $R_1 = f'''(\xi_1)(x-c)^3 \leq 0$ and similarly $R_2 = f'''(\xi_2)(y-c)^3 \geq 0$. Then from (2.3) and (2.4) we get

$$f(y) - f(x) = f'(c)(y-x) + R_2 - R_1 \geq f'(c)(y-x),$$

wherefrom (C) follows.

$(C) \Rightarrow (A)$: Suppose that (C) holds and that (A) does not hold. Therefore, there exists $c \in I$ with $f'''(c) < 0$. By continuity of f''', there is an interval $I^* \in I$ so that $f'''(t) < 0$ for $t \in I^*$. Then the function $-f'$ is convex on I^* and by the above proof of $(A) \Rightarrow (C)$, we conclude that (C) holds for $-f$ on $I^* \subset I$, hence (C) does not hold for f, which is a contradiction.

$(A) \Rightarrow (B)$: Let f' be convex on I and let $x, y \in I$, $x < y$. Then for each $t \in [x, y]$ we have

$$t = \lambda(t)x + (1-\lambda(t))y, \quad \text{where} \quad \lambda(t) = \frac{y-t}{y-x}.$$

An application of Jensen's inequality and the integration over $t \in [x, y]$ yields

$$\begin{aligned} \int_x^y f'(t)\,\mathrm{d}t &\leq f(x)\int_x^y \lambda(t)\,\mathrm{d}t + f(y)\int_x^y (1-\lambda(t))\,\mathrm{d}t \\ &= \frac{f(x)+f(y)}{2(y-x)}, \end{aligned}$$

which is equivalent to (B).

$(B) \iff (D)$: Since

$$\frac{\partial F(x,y)}{\partial x} = \frac{-f'(x)(y-x) + (f(y) - f(x))}{(y-x)^2},$$

$$\frac{\partial F(x,y)}{\partial y} = \frac{f'(y)(y-x) - (f(y) - f(x))}{(y-x)^2},$$

we see that the partial derivatives are continuous in each point $(x, y) \in I^2$, $x \neq y$. Then the condition (2.2) for Schur-convexity of F is equivalent to (B).

$(D) \Rightarrow (C)$: Suppose that F is Schur-convex. Then for sufficiently small $\varepsilon > 0$ it follows that

$$F\left(\frac{x+y}{2} - \varepsilon, \frac{x+y}{2} + \varepsilon\right) \leq F(x,y).$$

Letting $\varepsilon \to 0$, we get

$$f'\left(\frac{x+y}{2}\right) \leq F(x,y),$$

which is the statement (C).

This ends the proof of equivalence of conditions $(A) - (D)$. The second part follows upon replacing f by $-f$. $\square$

Let us note that (B) and (C) are differential forms of Hadamard's inequalities for convex functions. It is known that their integral form is equivalent to convexity [11]; more precisely, a continuous function g is convex on I if and only if [17, p.15]

$$g(u) \leq \frac{1}{2h} \int_{u-h}^{u+h} g(t)\, \mathrm{d}t$$

for all $u \in I$ and $h > 0$ such that $u \pm h \in I$ and if and only if ([16, p.39] or [17, p.15])

$$\frac{1}{y-x} \int_x^y g(t)\, \mathrm{d}t \leq \frac{g(x) + g(y)}{2}$$

for all $x, y \in I$. Letting here $g = f'$, we get the equivalence of (A) with (B) and (C). However, the presented proof yields the following corollary.

Corollary 2.1. *Suppose that $x \mapsto f'(x)$ is a convex (concave) function for positive large enough values of x and also suppose that $f'''(x) \to 0$ as $x \to +\infty$. Then the difference between left and right side in (B) and (C) (in (B') and (C')) converges to zero as $x \to +\infty$ and $y - x = \mathrm{const}$.*

Proof. Let $y = x + h$, where $h > 0$ is fixed and let f be convex. By the proof of $(A) \Rightarrow (B)$, the difference between the right and left hand side in (B) is expressed in the form

$$\int_x^{x+h} r(x,t)\,\mathrm{d}t,$$

where

$$r(x,t) = \lambda(t) f'(x) + (1 - \lambda(t)) f'(y) - f'(t), \quad \lambda(t) = \frac{y-t}{y-x}\ .$$

Two successive applications of the mean value theorem yield

$$\begin{aligned} r(x,t) &= \lambda(t)(f'(x) - f'(t)) + (1-\lambda(t))(f'(y) - f'(t)) \\ &= \lambda(t)(x-t) f''(\xi_1) + (1-\lambda(t))(y-t) f''(\xi_2) \\ &= (y-t)(t-x)(f''(\xi_2) - f''(\xi_1)) = (y-t)(t-x)(\xi_2 - \xi_1) f'''(\xi), \end{aligned}$$

where $\xi_1 \in (x,t)$; $\xi_2 \in (t,y)$, $\xi \in (x,y)$. If $y = x + h$ then

$$0 \le r(x,t) \le h^3 \max_{t \in (x,x+h)} f'''(t), \quad \text{and so}$$

$$0 \le \int_x^{x+h} r(x,t)\,\mathrm{d}t \le h^4 \max_{t \in (x,x+h)} f'''(t),$$

Therefore, if $f'''(x) \to 0$ as $x \to +\infty$, then the difference between the right and the left side in (B) converges to zero as $x \to +\infty$ and $y = x + h$. The corresponding statement for (C) follows directly from the proof of $(A) \Rightarrow (C)$. Statements for a concave function f can be obtained from the above proof with the convex function $-f$. $\square$

Theorem 2.1 can serve as a tool for producing inequalities and Corollary 2.1 is a starting point for asymptotic expansions. This is especially suitable for the Gamma function or in a more general context, for all convex or concave solutions of the functional equation $f(x+1) - f(x) = g(x)$, where g is a sum of a convex and a concave function on $x > 0$. For the theory of this functional equation, see [9] or [10].

3. Applications to the Gamma and Digamma Function

Inequalities for the ratio $Q(x,\beta) = \Gamma(x+\beta)/\Gamma(x)$ with $x > 0$ and usually $\beta \in (0,1)$, have been studied by many authors (see [2, 13] and references therein). Bounds for the ratio Q that involve the function Ψ and its derivatives have been investigated in [2, 4, 5, 6, 8, 14], using a variety of methods. In this section we present some new inequalities of this type, starting from $(A') - (D')$ with $f = \log\Gamma$ and $I = (0, +\infty)$. It is well known that $f'''(x) = \Psi''(x) \to 0$ as $x \to +\infty$, so Corollary 2.1 applies.

Firstly, (B') and (C') give

$$\frac{1}{2}(\Psi(x) + \Psi(y)) \le \frac{\log\Gamma(y) - \log\Gamma(x)}{y - x} \le \Psi\left(\frac{x+y}{2}\right). \tag{3.1}$$

Letting $y = x + \beta$, $\beta > 0$, we get

$$\exp\left(\beta\frac{\Psi(x) + \Psi(x+\beta)}{2}\right) \le Q(x, \beta) \le \exp(\beta\Psi(x + \beta/2)). \tag{3.2}$$

The upper bound in (3.2) was also obtained in [8] by other means. In [14] we showed that the lower bound in (3.2) is closer than a lower bound in [8].

Applying the recurrence relation for the Gamma function, one obtains

$$Q(x, \beta) = \Pi(x, n, \beta)Q(x + n, \beta),$$

where

$$\Pi(x, n, \beta) = \frac{x(x+1)\cdots(x+n-1)}{(x+\beta)(x+\beta+1)\cdots(x+\beta+n-1)}.$$

Therefore, replacing x by $x + n$ in (3.2), we get

$$\begin{aligned} \Pi(x, n, \beta)\ \exp\left(\beta\frac{\Psi(x+n) + \Psi(x+n+\beta)}{2}\right) &\le Q(x, \beta) \\ &\le \Pi(x, n, \beta)\exp(\beta\Psi(x + n + \beta/2)). \end{aligned} \tag{3.3}$$

By Corollary 2.1, inequalities in (3.3) are asymptotically exact, i.e., both bounds in (3.3) converge to the ratio $Q(x, \beta)$ as $n \to +\infty$.

The condition (D') implies

$$\frac{\log\Gamma(y) - \log\Gamma(x)}{y - x} \ge \frac{\log\Gamma(y+\varepsilon) - \log\Gamma(x-\varepsilon)}{y - x + 2\varepsilon}$$

for $0 < x < y$ and $0 < \varepsilon < x$. In particular, replacing x by $x + \beta$ and letting $y = x + 2\beta$ and $\varepsilon = \beta$, we obtain

$$\frac{\Gamma(x+3\beta)}{\Gamma(x)} \le \left(\frac{\Gamma(x+2\beta)}{\Gamma(x+\beta)}\right)^2, \qquad x > 0, \beta > 0. \tag{3.4}$$

Finally, let us derive bounds and expansions for the Digamma function Ψ. Letting $y = x + 1$ in (3.1), we get

$$\Psi(x) + \frac{1}{2x} \le \log x \le \Psi\left(x + \frac{1}{2}\right),$$

wherefrom it follows

$$\log\left(x - \frac{1}{2}\right) \le \Psi(x) \le \log x - \frac{1}{2x}, \qquad x > 1/2. \tag{3.5}$$

Now replacing x by $x + n$ and using the recurrence relation for the Digamma function, we get

$$\begin{aligned} \log\left(x + n - \frac{1}{2}\right) - \sum_{k=0}^{n-1} \frac{1}{x+k} &\le \Psi(x) \\ &\le \log(x+n) - \frac{1}{2(x+n)} - \sum_{k=0}^{n-1} \frac{1}{x+k} . \end{aligned} \tag{3.6}$$

By Corollary 2.1, both bounds in (3.6) are asymptotic expansions for $\Psi(x)$; it is not difficult to see that they are asymptotically equivalent to the well known expansion [1]

$$\Psi(x) = -\gamma + \sum_{k=1}^{+\infty} \frac{x-1}{k(k+x-1)} .$$

However, the advantage of (3.6) is that it gives inequalities.

More applications and examples will be presented in our forthcoming papers.

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