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WEAK PROBABILITY POLYADIC ALGEBRAS

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Abstract. We are motivated by the connection between Boolean algebra and propositional logic, cylindric algebra and predicate logic, Keisler's $L(V, \nu, \mathfrak{m})$ logic and polyadic algebra [2], to introduce an algebra calling weak probability polyadic algebra which 'corresponds' to the weak probability logic with infinitely predicates $L(V, \nu, \mathfrak{m}, R)$, introduced in [4].

Structure $(B, +, \cdot, -, 0, 1, C_{(K)}, C^r_{(K)}, S_\tau)$, where $(B, +, \cdot, -, 0, 1)$ is a Boolean algebra, $C_{(K)}$, $C^r_{(K)}$ and S_τ are unary operations on B, for each sequence (K) of ordinals from β of length α, $\overline{\overline{\alpha}} \leq \overline{\overline{\beta}}$, each $r \in R$ (R is as in [4]) and $\tau \in \beta^\beta$, *is a weak probability polyadic algebra of dimension* β, briefly WPP_β, if the following postulates hold:

(WPP_1) $(B, +, \cdot, -, 0, 1, C_{(K)}, S_\tau)$ is a polyadic algebra of dimension β,

(WPP_2) (i) $C^r_{(\emptyset)}x = x, \quad r > 0,$

(ii) $C^r_{(K)}0 = 0, \quad r > 0,$

(WPP_3) $C^0_{(K)}x = 1,$

(WPP_4) $C^r_{(K)}x \leq C^s_{(K)}x, \quad r \geq s,$

(WPP_5) $C^r_{(K)}C^s_{(K)}x = C^s_{(K)}x, \quad r > 0,$

(WPP_6) (i) $-C^r_{(K)}x \cdot C^{1-s}_{(K)} - y \leq -C^{\min\{1,r+s\}}_{(K)}(x + y),$

(ii) $C^r_{(K)}x \cdot C^s_{(K)}y \cdot C^1_{(K)} - (x \cdot y) \leq C^{\min\{1,r+s\}}_{(K)}(x + y),$

(WPP_7) (i) $C^r_{(K)} - x \geq -C^{1-r}_{(K)}x,$

(ii) $C^s_{(K)} - x \leq -C^{1-r}_{(K)}x, \quad s > r,$

(iii) $-C^{1-s}_{(K)} - x \leq C^{s^+}_{(K)}x$, where $s^+ = \min\{r \in R \mid r > s\}$,

(WPP_8) $S_\sigma C^r_{(K)}x = S_\tau C^r_{(K)}x, \quad$ if $\sigma \restriction (\beta \setminus \operatorname{rang} K) = \tau \restriction (\beta \setminus \operatorname{rang} K)$,

(WPP_9) $S_\sigma C^r_{\sigma^{-1}(K)}x = C^r_{(K)}S_\sigma x, \quad$ if $\sigma \restriction \sigma^{-1}(K)$ is 1–1 function,

(WPP_{10}) (i) $C^r_{(K)}x \leq C_{(K)}x, \quad r > 0,$

(ii) $C_{(K)}C^r_{(K_1)}x = C^r_{(K_1)}x, \quad \operatorname{rang} K \subset \operatorname{rang} K_1$

(iii) $C^r_{(K)}C_{(K_1)}x = C_{(K_1)}x, \quad r > 0, \ \operatorname{rang} K \subset \operatorname{rang} K_1.$

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Theorem 1. *If A is WPP_β algebra, then*

(1) $C^r_{(K)}1 = 1$,

(2) *If* $x \le y$, *then* $C^r_{(K)}x \le C^r_{(K)}y$,

(3) $C^r_{(K)}x + C^r_{(K)}y \le C^r_{(K)}(x+y)$,

(4) $C^r_{(K)}(x \cdot y) \le C^r_{(K)}x \cdot C^r_{(K)}y$,

(5) $C^1_{(K)}x = x$ *iff* $C_{(K)}x = x$,

(6) $S_\tau 1 = 1$.

The proof is similar to that one in [6].

Let V_1 be a set of new variables such that $V_1 \cap V = \emptyset$, $V_1 \cap P = \emptyset$ and $\overline{\overline{V_1}} = \overline{\overline{V}}$. Let be $V^* = V \cup V_1$, $\tau_0 : V^* \xrightarrow{1-1} V_1$, G the set of all sentences in L^*, $H = \{\Phi \mid V_f(\Phi) \subset V,\ \Phi \in \mathrm{Form}_{L^*}\}$ and $\Gamma \subset G$. For each $\Phi, \Psi \in H$ we define $\Phi/\Gamma = \{\Psi \mid \Gamma \vdash_{L^*} \Psi \leftrightarrow \Phi,\ \Psi \in H\}$, $H_\Gamma = \{\Phi/\Gamma \mid \Phi \in H\}, 0_\Gamma = \bot/\Gamma, 1_\Gamma = \top/\Gamma$ and operations $+_\Gamma, -_\Gamma, \cdot_\Gamma$ on H_Γ as usual. It is well known the next result:

Lemma 1. *The set of sentences Γ is consistent in L^* iff $(H_\Gamma, +_\Gamma, \cdot_\Gamma, -_\Gamma, 1_\Gamma, 0_\Gamma)$ is notrivial Boolean algebra.*

For each sequence x of different variables from V, $r \in R$, $\tau \in V^V$ and $\Phi \in H$ we define:

$$\begin{aligned}(\exists^\Gamma x)(\Phi/\Gamma) &= \big((\exists x)\Phi\big)/\Gamma,\\ (P^\Gamma x \ge r)(\Phi/\Gamma) &= \big((Px \ge r)\Phi\big)/\Gamma,\\ S^\Gamma(\tau)(\Phi/\Gamma) &= \big(S_f(\tau)S_f(\tau_0^{-1})S(\tau_0)\Phi\big)/\Gamma.\end{aligned}$$

Lemma 2. (i) *If Φ is atomic formula and $\tau \in V^V$, then*

$$S^\Gamma(\tau)(\Phi/\Gamma) = \big(S_f(\tau)\Phi\big)/\Gamma = (S(\tau)\Phi)/\Gamma.$$

(ii) *If $\tau \in (V \setminus V_b(\Phi))^V$, then $S^\Gamma(\tau)(\Phi/\Gamma) = \big(S_f(\tau)\Phi\big)/\Gamma$.*

(iii) *If $\tau \in V^V$ is 1–1, then $S^\Gamma(\tau)(\Phi/\Gamma) = (S(\tau)\Phi)/\Gamma$.*

Lemma 3. *If $\mathfrak{m} = \overline{\overline{V}}^{\,+}$ and Γ is consistent in L^*, then*

$$\mathbb{H}_\Gamma = \big(H_\Gamma, +_\Gamma, \cdot_\Gamma, -_\Gamma, 1_\Gamma, 0_\Gamma, \exists^\Gamma x, P^\Gamma x \ge r, S^\Gamma(\tau)\big),$$

where $\tau \in V^V$, $r \in R$ and x is a sequence of different variables from V of length α, $\overline{\overline{\alpha}} < \mathfrak{m}$, is a weak probability polyadic algebra of dimension V.

Proof. Let be $\Phi^\Gamma = \Phi/\Gamma$. It follows from Lemma 1 $(H_\Gamma, +_\Gamma, \cdot_\Gamma, -_\Gamma, 1_\Gamma, 0_\Gamma)$ is a Boolean algebra. The operations $\exists^\Gamma x$, $P^\Gamma x \ge r$ and $S^\Gamma(\tau)$ are well defined and the set H_Γ is closed for them. It is easy to verify (by using the axioms and the rules of inference of L, see [4]) that all axioms $(WPP_1) - (WPP_{10})$ hold in $\mathbb{H}_\Gamma$. We shall only prove (WPP_{10}):

(i)

$$
\begin{aligned}
(P^\Gamma x \geq r)\Phi^\Gamma &= \big((Px \geq r)\Phi\big)^\Gamma \\
&\leq_\Gamma \big((Px > 0)\Phi\big)^\Gamma \quad \text{for } r > 0, \text{ by } A_{10} \\
&\leq_\Gamma \big((\exists x)\Phi\big)^\Gamma \quad \text{by contraposition } A_{12} \\
&= (\exists^\Gamma x)\Phi^\Gamma.
\end{aligned}
$$

(ii)

$$
(\exists^\Gamma x)(P^\Gamma y \geq r)\Phi^\Gamma = \big((\exists x)(Py \geq r)\Phi\big)^\Gamma \leq_\Gamma \big((Py \geq r)\Phi\big)^\Gamma = (P^\Gamma y \geq r)\Phi^\Gamma,
$$

by supposing $\operatorname{rang} x \subset \operatorname{rang} y$. The above inequality follows from

$$
\Gamma \vdash_{L^*} (\exists x)(Py \geq r)\Phi \rightarrow (Py \geq r)\Phi
$$

that we obtain from

$$
\Gamma \vdash_{L^*} (\forall x)\big((Py < r)\Phi \rightarrow (Py < r)\Phi\big)
$$

by using A_2.

(iii)

$$
\begin{aligned}
(\exists^\Gamma y)\Phi^\Gamma &= \big((\exists y)\Phi\big)^\Gamma \\
&\leq_\Gamma \big((Px \geq r)(\exists y)\Phi\big)^\Gamma \quad \text{for } V_f\big((\exists y)\Phi\big) \cap \operatorname{rang} x = \emptyset, \text{ by } R_3 \text{ and } A_4 \\
&= (P^\Gamma x \geq r)(\exists^\Gamma y)\Phi^\Gamma.
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
(P^\Gamma x \geq r)(\exists^\Gamma y)\Phi^\Gamma &= \big((Px \geq r)(\exists y)\Phi\big)^\Gamma \\
&\leq_\Gamma \big((Px > 0)(\exists y)\Phi\big)^\Gamma \quad \text{for } r > 0, \text{ by } A_{10} \\
&\leq_\Gamma \big((\exists x)(\exists y)\Phi\big)^\Gamma \quad \text{by } A_{12} \\
&= \big((\exists y)\Phi\big)^\Gamma \quad \text{from } A_2, \text{ since } \operatorname{rang} x \subset \operatorname{rang} y \\
&= (\exists^\Gamma y)\Phi^\Gamma. \qquad \square
\end{aligned}
$$

Let $\mathfrak{A} = (A, R_p, \mu_\alpha)_{p \in P, \overline{\overline{\alpha}} < \overline{\overline{V}}^+}$ be a weak probability structure for L and

$$
V = \beta, \quad \Phi^{\mathfrak{A}} = \{a \in A^\beta \mid \models_{\mathfrak{A}} \Phi[a \restriction V^*]\} \quad \text{and} \quad \mathcal{A} = \{\Phi^{\mathfrak{A}} \mid \Phi \in H\}.
$$

We shall define for each sequence (K) of ordinals from β of length $\alpha, \overline{\overline{\alpha}} \leq \overline{\overline{\beta}}$, each $r \in R$ and each $\sigma \in \beta^\beta$:

$$
\begin{aligned}
c_{(K)}(\Phi^{\mathfrak{A}}) &= \big\{a \in A^\beta \mid b \restriction (\beta \setminus \operatorname{rang}(K)) = a \restriction (\beta \setminus \operatorname{rang}(K)), \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } \ b \in \Phi^{\mathfrak{A}}\big\}, \\
c^r_{(K)}(\Phi^{\mathfrak{A}}) &= \{a \in A^\beta \mid \mu_\alpha\{b \circ (K) \mid b \in \Phi^{\mathfrak{A}}, \\
&\qquad\qquad b \restriction (\beta \setminus \operatorname{rang}(K)) = a \restriction (\beta \setminus \operatorname{rang}(K))\} \geq r\}, \\
s_\sigma(\Phi^{\mathfrak{A}}) &= \Big\{a \in A^\beta \mid a_\sigma \in \big(S_f(\tau_0^{-1})S(\tau_0)\Phi\big)^{\mathfrak{A}}\Big\},
\end{aligned}
$$

where $a_\sigma = \big(a_{\sigma(\alpha)}\big)_{\alpha<\beta}$ for $a = (a_\alpha)_{\alpha<\beta}$.

It is easy to verify that the above operations are well defined and that the sets from definition of $c^r_{(K)}$ are measurable. We may show that $\mathcal{A}$ is closed for these operations.

$$
\begin{aligned}
c_{(K)}(\Phi^{\mathfrak{A}}) &= \big\{a \in A^\beta \mid b \restriction (\beta \setminus \operatorname{rang}(K)) = a \restriction (\beta \setminus \operatorname{rang}(K)), \\
&\qquad\qquad \text{for some } \ b \in \Phi^{\mathfrak{A}}\big\} \\
&= \{a \in A^\beta \mid b \restriction (\beta \setminus \operatorname{rang}(K)) = a \restriction (\beta \setminus \operatorname{rang}(K)), \\
&\qquad\qquad \text{for some } b \text{ such that } \models_{\mathfrak{A}} \Phi[b \restriction V^*]\} \\
&= \{a \in A^\beta \mid \models_{\mathfrak{A}} (\exists x)\Phi[a \restriction V^*]\} \\
&= ((\exists x)\,\Phi)^{\mathfrak{A}} \in \mathcal{A}. \\
c^r_{(K)}(\Phi^{\mathfrak{A}}) &= \{a \in A^\beta \mid \mu_\alpha\{b \circ K \mid b \in \Phi^{\mathfrak{A}}, \\
&\qquad\qquad b \restriction (\beta \setminus \operatorname{rang}(K)) = a \restriction (\beta \setminus \operatorname{rang}(K))\} \geq r\} \\
&= \{a \in A^\beta \mid \mu_\alpha\{b \circ K \mid \models_{\mathfrak{A}} \Phi[b \restriction V^*], \\
&\qquad\qquad b \restriction (\beta \setminus \operatorname{rang}(K)) = a \restriction (\beta \setminus \operatorname{rang}(K))\} \geq r\} \\
&= \big\{a \in A^\beta \mid \models_{\mathfrak{A}} (Px \geq r)\Phi[a \restriction V^*]\big\} \\
&= ((Px \geq r)\Phi)^{\mathfrak{A}} \in \mathcal{A}. \\
s_\sigma(\Phi^{\mathfrak{A}}) &= \Big\{a \in A^\beta \mid a \circ \sigma \in \big(S_f(\tau_0^{-1})S(\tau_0)\Phi\big)^{\mathfrak{A}}\Big\} \\
&= \big\{a \in A^\beta \mid \models_{\mathfrak{A}} S_f(\tau_0^{-1})S(\tau_0)\Phi[a \circ (\sigma \restriction V^*)]\big\} \\
&= \big\{a \in A^\beta \mid \models_{\mathfrak{A}} S_f(\sigma)S_f(\tau_0^{-1})S(\tau_0)\Phi[a \restriction V^*]\big\} \\
&= \big(S_f(\sigma)S_f(\tau_0^{-1})S(\tau_0)\Phi\big)^{\mathfrak{A}} \in \mathcal{A}.
\end{aligned}
$$

The algebra $(\mathcal{A}, \cup, \cap, \sim, \emptyset, A^\beta, c_{(K)}, c^r_{(K)}, s_\sigma)$ will be called *weak probability polyadic set algebra.*

Lemma 4. *Let $f : H_\Gamma \to \mathcal{A}$ be defined by $f(\Phi^\Gamma) = \Phi^{\mathfrak{A}}$. Then, if $\models_{\mathfrak{A}} \Gamma$ then f is homomorphism $\mathbb{H}_\Gamma$ onto $\mathcal{A}$ and $\mathcal{A}$ is a weak probability polyadic algebra.*

Theorem 2. *Let $\mathbb{B} = (B, +, \cdot, -, 0, 1, C_{(K)}, C^r_{(K)}, S_\sigma)$ be a weak probability polyadic algebra of infinite dimension β. Let be $V = \beta$, $P = \{p \mid p \in B\}$, $\mathfrak{m} = \overline{\overline{\beta}}^+$ and $\overline{\overline{\nu(p)}} = \overline{\overline{\beta}}$ for each $p \in P$. Then there exists a set of sentences Γ in logic $L = L(V, \nu, \mathfrak{m}, R)$ such that $\mathbb{H}_\Gamma \cong \mathbb{B}$.*

Proof. We shall put $V = \beta$, $P = B$, $\nu(p) = \beta$ for each $p \in P$ and $x \in \beta^\beta$ "1–1" and "onto". Let Γ be the next set of sentences:

$$
\begin{aligned}
&(\forall x)\left((-p)(x) \leftrightarrow \neg\left(p(x)\right)\right), && (\forall x)\left((S_\tau p)(x) \leftrightarrow S(\tau)p(x)\right),\\
&(\forall x)\left((p+q)(x) \leftrightarrow p(x) \vee q(x)\right), && (\forall x)\left((C_{(K)}p)(x) \leftrightarrow (\exists y)p(x)\right),\\
&(\forall x)\left((p \cdot q)(x) \leftrightarrow p(x) \wedge q(x)\right), && (\forall x)\Big((C^r_{(K)}p)(x) \leftrightarrow (Py \geq r)p(x)\Big),
\end{aligned}
$$

where $p, q \in P$, $\tau \in V^V$, $r \in R$ and $(K) = y$ is a sequence of different variables of length α, $\overline{\overline{\alpha}} < \mathfrak{m}$. First, we shall show that $H_\Gamma = \Big\{(p(x))^\Gamma \mid p \in P\Big\}$. For each formula $\Phi \in H$ there exists formula $\Psi = S_f(\tau_0^{-1})S(\tau_0)\Phi \in H$ such that $\Phi^\Gamma = \Psi^\Gamma$, $V_b(\Psi) \subset V_1$ and $V_f(\Psi) \subset V$. Really,

$$\Psi^\Gamma = S_f(id)S_f(\tau_0)S(\tau_0)\Phi^\Gamma = S^\Gamma(id)\Phi^\Gamma = \Phi^\Gamma.$$

We shall prove that for each formula Φ from H, with free variables in V and bound in V_1, there exists formula $\Psi \in F$ such that $\Phi^\Gamma = \Psi^\Gamma$. Let be $\tau_1 : V^* \xrightarrow[\text{na}]{1\text{–}1} V$. Then

$$\Gamma \vdash_{L^*} S(\tau_1)\Phi \leftrightarrow S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi.$$

By (R_3), we have

$$\Gamma \vdash_{L^*} S(\tau_1^{-1})S(\tau_1)\Phi \leftrightarrow S(\tau_1^{-1})S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi.$$

Since $V_b(S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi) \subset V_1$, it follows

$$
\begin{aligned}
S(\tau_1^{-1})S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi &= S_f(\tau_1^{-1})S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi\\
&= S_f(\tau_0^{-1})S(\tau_0)S_f(\tau_1^{-1})S(\tau_1)\Phi.
\end{aligned}
$$

So,

$$\Gamma \vdash_{L^*} \Phi \leftrightarrow S_f(\tau_0^{-1})S(\tau_0)S_f(\tau_1^{-1})S(\tau_1)\Phi.$$

Putting $\Psi = S_f(\tau_1^{-1})S(\tau_1)\Phi$, we have

$$\Gamma \vdash_{L^*} \Phi \leftrightarrow \Psi, \quad \Psi \in F.$$

It follows that it is enough to prove that for each formula $\Phi \in F$ there exists $p \in P$ such that $\Phi^\Gamma = p(x)^\Gamma$. We can prove it by induction of complexity of Φ.

Now, we shall define a function $g : B \longrightarrow H_\Gamma$ on the following way:

$$g(p) = (p(x))^\Gamma .$$

It is easy to verify that g is well defined, onto and homomorphism. To prove that it is one to one we shall define $h : F \to B$ such that, for $\Phi \in F$ holds:

$$\text{if } \ \Gamma \vdash_{L^*} \Phi \ \text{ then } \ h(\Phi) = 1.$$

For formulas from $F' = \{\Phi \in F \mid V_b(\Phi) \cap V_f(\Phi) = \emptyset\}$ we define the mapping h by induction:

$$\begin{array}{ll}
h\left(p(x)\right) = p, & h(\Phi \vee \Psi) = h(\Phi) + h(\Psi), \\
h\left(p(\tau \circ y)\right) = S_\tau p, & h\left((\exists y)\Phi\right) = C_{(K)} h(\Phi), \\
h(\neg\Phi) = -h(\Phi), & h\left((Py \geq r)\Phi\right) = C^r_{(K)} h(\Phi).
\end{array}$$

For $\Phi \in F \setminus F'$, let be

$$h(\Phi) = S_{(\tau_1^{-1} \restriction V_f(\Psi)) \restriction V} h(\Psi),$$

where $\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi$ and τ_1 is a fixed injection from V^* onto V.

First, we shall prove that for each function $\sigma : V_f(\Phi) \longrightarrow V \setminus V_b(\Phi)$ and every formula $\Phi \in F$ we have

$$(*) \qquad h(S_f(\sigma)\Phi) = S_{\sigma \restriction V} h(\Phi).$$

For $\Phi \in F'$ the proof is by induction on the complexity of formula.

If $\Phi = p(x)$, then $h(S_f(\sigma)\Phi) = h(p(\sigma \circ x)) = S_\sigma p = S_{\sigma\restriction V} h(\Phi)$.

If $\Phi = p(y),\ y = \tau \circ x,\ \tau \in V^V$, we have

$$\begin{aligned} h\,(S_f(\sigma)p(y)) &= h\,(p(\sigma \circ \tau \circ x)) \quad \text{by definition } S_f \\ &= S_{\sigma\circ\tau}p \quad \text{by definition } h \\ &= S_{\sigma\restriction V} S_\tau p \quad \text{by } (P_9) \text{ (see [1])} \\ &= S_{\sigma\restriction V} h\,(p(y)) \quad \text{by definition } h. \end{aligned}$$

If $\Phi = \neg\Psi$ or $\Phi = \Psi \vee \Theta$ it is easy to verify.

If $\Phi = (Py \geq r)\Psi$, then

$$\begin{aligned} h\,(S_f(\sigma)\Phi) &= h\,(S_f(\sigma)(Py \geq r)\Psi) \\ &= h((Py \geq r)S_f(\sigma_1)\Psi \quad \text{where } \sigma_1 = \sigma \restriction V_f(\Psi) \\ &= C^r_{(K)} S_{\sigma_1\restriction V} h(\Psi) \\ &= S_{\sigma_1\restriction V} C^r_{\sigma_1^{-1}\circ(K)} h(\Psi) \\ &= S_{\sigma\restriction V} C^r_{(K)} h(\Psi) \quad \text{since } \sigma \restriction (K) = id \\ &= S_{\sigma\restriction V} h((Py \geq r)\Psi). \end{aligned}$$

To show that $(*)$ holds for each formula from $F \setminus F'$, we shall prove, first, that

$$(**) \qquad h(S(\sigma)\Phi) = S_{(\sigma\restriction V_f(\Phi))\restriction V} h(\Phi),$$

for each formula $\Phi \in F$ and each $\sigma : V(\Phi) \xrightarrow{1-1} V$.

First, let be $\Phi \in F'$.

If $\Phi = p(x)$, then $h(S(\sigma)p(x)) = h(p(\sigma \circ x)) = S_\sigma p = S_{\sigma\restriction V_f(\Phi)} h(\Phi)$.

If $\Phi = (\exists y)\Psi$, we have

$$\begin{aligned} h(S(\sigma)\Phi) &= h((\exists \sigma \circ y)S(\sigma)\Psi) \\ &= C_{\sigma\circ(K)} h(S(\sigma)\Psi) \\ &= C_{\sigma\circ(K)} S_{(\sigma\restriction V_f(\Psi))\restriction V} h(\Psi) \quad \text{by induction hypothesis} \\ &= S_{(\sigma\restriction V_f(\Psi))\restriction V} C_{(K)} h(\Psi) \quad \text{by } (P_{12}) \\ &= S_{(\sigma\restriction V_f(\Phi))\restriction V} C_{(K)} h(\Psi) \quad \text{by } (P_{11}) \\ &= S_{(\sigma\restriction V_f(\Phi))\restriction V} h((\exists y)\Psi) = S_{(\sigma\restriction V_f(\Phi))\restriction V} h(\Phi). \end{aligned}$$

If $\Phi \in F \setminus F'$ and $\sigma : V(\Phi) \xrightarrow{1-1} V$, let be $\Theta = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)S(\sigma)\Phi$, $\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi$ and $\sigma_1 : V(\Psi) \longrightarrow V$ defined by $\sigma_1 \restriction V_b(\Psi) = \tau_1 \circ \tau_0 \circ \sigma \circ \tau_0^{-1} \circ \tau_1^{-1} \restriction V_b(\Psi)$ and $\sigma_1 \restriction V_f(\Psi) = \tau_1 \circ \sigma \circ \tau_1^{-1} \restriction V_f(\Psi)$. Then

$$\begin{aligned}
h(S(\sigma)\Phi) &= S_{(\tau_1^{-1}\restriction V_f(\Theta))\restriction V} h(\Theta) = S_{(\tau_1^{-1}\restriction V_f(\Theta))\restriction V} h(S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)S(\sigma)\Phi) \\
&= S_{(\tau_1^{-1}\restriction V_f(\Theta))\restriction V} h(S(\sigma_1)S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi) \\
&= S_{(\tau_1^{-1}\restriction V_f(\Theta))\restriction V} h(S(\sigma_1)\Psi \\
&= S_{(\tau_1^{-1}\restriction V_f(\Theta))\restriction V} S_{(\sigma_1\restriction V_f(\Psi))\restriction V} h(\Psi) \\
&= S_{(\tau_1^{-1}\restriction V_f(\Theta))\restriction V} S_{(\tau_1\circ\sigma\circ\tau_1^{-1}\restriction V_f(\Psi))\restriction V} h(\Psi) \\
&= S_{(\tau_1^{-1}\restriction V_f(\Theta))\restriction V\circ(\tau_1\circ\sigma\circ\tau_1^{-1}\restriction V_f(\Psi))\restriction V} h(\Psi) \\
&= S_{(\sigma\restriction V_f(\Phi))\restriction V} S_{\tau_1^{-1}\restriction V_f(\Psi))\restriction V} h(\Psi) = S_{(\sigma\restriction V_f(\Phi))\restriction V} h(\Phi).
\end{aligned}$$

By using $(**)$ it is easy to verify that for each formula $\Phi \in F'$ holds

$$h(\Phi) = S_{(\tau_1^{-1}\restriction V_f(\Psi))\restriction V} h(\Psi),$$

where

$$\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_o)\Phi.$$

Now, we shall prove that $(*)$ is valid for each formula Φ from $F\setminus F'$. Let be $\Theta = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)S_f(\sigma)\Phi$ and $\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi$. Then

$$\begin{aligned}
h(S_f(\sigma)\Phi) &= S_{(\tau_1^{-1}\restriction V_f(\Theta))\restriction V} h(\Theta) \\
&= S_{(\tau_1^{-1}\restriction V_f\Theta))\restriction V} h(S_f(\tau_1\circ\sigma\circ\tau_1^{-1})S(\tau_1)S_f(\tau_0^{-1})S(\tau_0))\Phi) \\
&= S_{(\tau_1^{-1}\restriction V_f(\Theta))\restriction V} S_{(\tau_1\circ\sigma\circ\tau_1^{-1}\restriction V_f(\Psi))\restriction V} h(\Psi) \\
&= S_{\sigma\restriction V} S_{(\tau_1^{-1}\restriction V_f(\Psi))\restriction V} h(\Psi) = S_{\sigma\restriction V} h(\Phi).
\end{aligned}$$

Now, we shall prove that $\Gamma \vdash_{L^*} \Phi$ implies $h(\Phi) = 1$. It follows from $\Gamma \subset F$, $\Phi \in F$, that $\Gamma \vdash_{L^*} \Phi$ iff $\Gamma \vdash_L \Phi$. So, it is enough to show that from $\Gamma \vdash_L \Phi$ follows $h(\Phi) = 1$. At the beginning, we shall prove that each axiom maps to 1. First, let the axiom be from F'.

(A_1) It is sufficient to show that $h(\Phi) \neq 1$ implies Φ is not tautology. Let I be maximal ideal of Boolean algebra $(B, +, \cdot, -, 0, 1)$ which contains $h(\Phi)$ and π natural homomorphism from B onto two-elements Boolean algebra B/I. From $h(\Phi) \neq 1$ it follows $\pi \circ h(\Phi) = 0$, i.e. Φ is not tautology.

(A_2)

$$
\begin{aligned}
& h\left((\forall y)(\Phi \to \Psi) \to (\Phi \to (\forall y)\Psi)\right) \\
= {} & h\left(\neg(\exists y)\neg(\Phi \to \Psi) \to (\Phi \to \neg(\exists y)\neg\Psi)\right) \\
= {} & C_{(K)}\left(-h(\neg\Phi \vee \Psi)\right) + -h(\Phi) + -C_{(K)}\left(-h(\Psi)\right) \\
= {} & C_{(K)}\left(h(\Phi) \cdot -h(\Psi)\right) + -h(\Phi) + -C_{(K)}\left(-h(\Psi)\right) \\
= {} & C_{(K)}h(\Phi) \cdot C_{(K)}\left(-h(\Psi)\right) + -h(\Phi) + -C_{(K)}\left(-h(\Psi)\right) \\
\geq {} & h(\Phi) \cdot C_{(K)}\left(-h(\Psi)\right) + -h(\Phi) + -C_{(K)}\left(-h(\Psi)\right) \\
= {} & -\big(-h(\Phi) - C_{(K)}\left(-h(\Psi)\right)\big) \cdot h(\Phi) \cdot C_{(K)}\left(-h(\Psi)\right) \\
= {} & -\big(-h(\Phi) \cdot h(\Phi) \cdot C_{(K)}\left(-h(\Psi)\right) + -C_{(K)}\left(-h(\Psi)\right) \cdot h(\Phi) \cdot C_{(K)}\left(-h(\Psi)\right)\big) \\
= {} & -(0+0) = 1,
\end{aligned}
$$

with condition $V_f(\Phi) \cap \operatorname{rang} y = \emptyset$.

(A_3)

$$
\begin{aligned}
& h\left((\forall y)\Phi \to S_f(\tau)\Phi\right), \quad \text{where } \tau : \operatorname{rang} y \longrightarrow V \setminus V_b(\Phi) \\
= {} & h\left((\exists y)\neg\Phi \vee S_f(\tau)\Phi\right) \\
= {} & C_{(K)}\left(-h(\Phi)\right) + S_{\tau'}h(\Phi), \quad \text{where } \tau' = \tau \restriction V \\
= {} & S_{\tau'}C_{(K)}\left(-h(\Phi)\right) + S_{\tau'}h(\Phi) \quad \text{by } (P_6) \text{ since} \\
& \qquad \tau' \restriction (V \setminus \operatorname{rang} y) = id \restriction (V \setminus \operatorname{rang} y) \\
= {} & S_{\tau'}\big(C_{(K)}\left(-h(\Phi)\right) + h(\Phi)\big) \\
\geq {} & S_{\tau'}\left(-h(\Phi) + h(\Phi)\right) \\
= {} & S_{\tau'}1 = 1.
\end{aligned}
$$

(A_4)

$$
h\left((\forall y)\Phi \to (\forall \tau \circ y)\right)\Phi = C_{(K)} - h(\Phi) + -C_{(K)} - h(\Phi) = 1
$$

by (P_{13}) supposing $\tau : y \xrightarrow[\text{onto}]{1\text{–}1} y$.

(A_5)

$$
h\left((Py \geq 0)\Phi\right) = C^0_{(K)}h(\Phi) = 1 \quad \text{from } (WPP_2).
$$

(A_6)

$$
\begin{aligned}
& h\left((Py \geq r)\Phi \to (Py \geq s)\Phi\right) = -C^r_{(K)}h(\Phi) + C^s_{(K)}h(\Phi) \\
\geq {} & -C^r_{(K)}h(\Phi) + C^r_{(K)}h(\Phi) = 1, \text{ for } r \geq s, \text{ by } (WPP_4).
\end{aligned}
$$

(A_7)

$$\begin{aligned}
&\quad h\left((Py \geq r)\Phi \wedge (Py \geq s)\Psi \wedge (Py \leq 0)(\Phi \wedge \Psi)\right.\\
&\rightarrow \left.(Py \geq \min\{1, r+s\})(\Phi \vee \Psi)\right)\\
&= -\Big(C^{r}_{(K)} h(\Phi) \cdot C^{s}_{(K)} h(\Psi) \cdot C^{1}_{(K)} - (h(\Phi) \cdot h(\Psi))\Big)\\
&+ C^{\min\{1,r+s\}}_{(K)} (h(\Phi) + h(\Psi))\\
&\geq -C^{\min\{1,r+s\}}_{(K)} (h(\Phi) + h(\Psi)) + C^{\min\{1,r+s\}}_{(K)} (h(\Phi) + h(\Psi)) = 1.
\end{aligned}$$

(A_8)

$$\begin{aligned}
&\quad h\left((Py \leq r)\Phi \wedge (Py < s)\Psi \rightarrow (Py < \max\{0, r+s-1\})(\Phi \wedge \Psi)\right)\\
&= -\Big(C^{1-r}_{(K)} - h(\Phi) \cdot -C^{s}_{(K)} h(\Psi)\Big) + -C^{\max\{0,r+s-1\}}_{(K)} (h(\Phi) \cdot h(\Psi))\\
&\geq C^{\max\{0,r+s-1\}}_{(K)} (h(\Phi) \cdot h(\Psi)) + -C^{\max\{0,r+s-1\}}_{(K)} (h(\Phi) \cdot h(\Psi)) = 1.
\end{aligned}$$

(A_9)

$$\begin{aligned}
&h\left((Py < s)\Phi \rightarrow (Py \leq s)\Phi\right) = -C^{s}_{(K)} h(\Phi) + C^{1-s}_{(K)} - h(\Phi)\\
&\geq C^{s}_{(K)} h(\Phi) + -C^{s}_{(K)} h(\Phi) = 1, \text{ by } (WPP_8).
\end{aligned}$$

(A_{10})

$$\begin{aligned}
&h\left((Py \geq s)\Phi \rightarrow (Py > r)\Phi\right) = -C^{s}_{(K)} h(\Phi) + C^{r}_{(K)} h(\Phi)\\
&\geq -C^{s}_{(K)} h(\Phi) + C^{s}_{(K)} h(\Phi) = 1, \text{ by } (WPP_8), \text{ for } s > r.
\end{aligned}$$

(A_{11})

$$\begin{aligned}
&h\big((Py > s)\Phi \rightarrow (Py \geq s^{+})\Phi\big) = C^{1-s}_{(K)} - h(\Phi) + C^{s^{+}}_{(K)} h(\Phi)\\
&\geq -C^{s^{+}}_{(K)} h(\Phi) + C^{s^{+}}_{(K)} h(\Phi) = 1, \quad \text{by } (WPP_8).
\end{aligned}$$

(A_{12})

$$\begin{aligned}
&h\left((\forall y)\Phi \rightarrow (Py \geq 1)\Phi\right) = C_{(K)} - h(\Phi) + C^{1}_{(K)} h(\Phi)\\
&\geq C_{(K)} - h(\Phi) + -C^{0^{+}}_{(K)} - h(\Phi) \quad \text{by } (WPP_7) \text{ (iii)}\\
&\geq C_{(K)} - h(\Phi) + -C_{(K)} - h(\Phi) \quad \text{from } (WPP_{10}) \text{ (ii)}\\
&= 1.
\end{aligned}$$

Now, let Φ be axiom from $F \setminus F'$. Then $\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi$ is an axiom from F'. It follows $h(\Psi) = 1$. Then

$$h(\Phi) = S_{(\tau_1^{-1}\restriction V_f(\Psi))\restriction V} h(\Psi) = S_{(\tau_1^{-1}\restriction V_f(\Psi))\restriction V} 1 = 1.$$

It is easy to verify that all formulas from Γ map to 1.

Now, we shall show that from formula mapped to 1, by any rule of inference, we get the formula which maps to 1, i.e. that all theorems map to 1. It is only nontrivial for (R_3) and (R_4).

(R_3) Let be $\Phi \vdash_L S(\sigma)\Phi$, where $\sigma : V(\Phi) \xrightarrow{1\text{–}1} V$, $h(\Phi) = 1$. Then

$$h\left(S(\sigma)\Phi\right) = S_{(\sigma\restriction V_f(\Phi))\restriction V} h(\Phi) = S_{(\sigma\restriction V_f(\Phi))\restriction V} 1 = 1.$$

(R_4) Let be $S_f(\sigma)\Phi \vdash_L \Phi$, where $\sigma : V_f(\Phi) \xrightarrow{1\text{–}1} V \setminus V_b(\Phi)$, $h\left(S_f(\sigma)\Phi\right) = 1$.

If $\Phi \in F'$, we have $\Phi = S_f(\sigma^{-1})S_f(\sigma)\Phi$. It follows

$$h(\Phi) = S_{\sigma^{-1}\restriction V} h(S_f(\sigma)\Phi) = S_{\sigma^{-1}\restriction V} 1 = 1.$$

If $\Phi \in F \setminus F'$, let be $\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi$ and $\sigma_1 : V(S_f(\sigma)\Phi) \longrightarrow V$ defined by $\sigma_1 \restriction V_b(S_f(\sigma)\Phi) = \tau_1 \circ \tau_0$ and $\sigma_1 \restriction V_f(S_f(\sigma)\Phi) = \tau_1 \circ \sigma^{-1}$. It is clear that σ_1 is "1–1" function. Then

$$\begin{aligned} h(\Phi) &= S_{(\tau_1^{-1}\restriction V_f(\Psi))\restriction V} h(\Psi) = S_{(\tau_1^{-1}\restriction V_f(\Psi))\restriction V} h(S(\sigma_1)S_f(\sigma)\Phi) \\ &= S_{(\tau_1^{-1}\restriction V_f(\Psi))\restriction V} S_{(\sigma_1\restriction V_f(S_f(\sigma)\Phi))\restriction V} h(S_f(\sigma)\Psi) \\ &= S_{(\tau_1^{-1}\restriction V_f(\Psi))\restriction V} S_{(\sigma_1\restriction V_f(S_f(\sigma)\Phi))\restriction V} 1 = 1. \end{aligned}$$

Finally, we shall show that $g : B \to H_\Gamma$, $g(p) = (p(x))^\Gamma$, is "1–1". If $g(p) = g(q)$, then $\Gamma \vdash_{L^*} p(x) \leftrightarrow q(x)$. It follows $h(p(x) \leftrightarrow q(x)) = 1$, and hence $p = q$. $\square$

Theorem 3. *If $\mathbb{B}$ is a weak probability polyadic algebra with infinite dimension β, then there exists homomorphism of $\mathbb{B}$ to some weak probability polyadic set algebra.*

Proof. By the previous theorem there exists the set of sentences Γ in logic $L = L(\beta, \nu, \overline{\overline{\beta}}^+, R)$ such that $\mathbb{H}_\Gamma \cong \mathbb{B}$. Since $\overline{\overline{B}} > 1$, it follows that Γ is consistent in L. By the completeness theorem, Γ has a weak model $\mathfrak{A}$. By Lemma 4 $\mathbb{A} = (\mathcal{A}, \cup, \cap, \sim, A^\beta, \emptyset, c_{(K)}, c^r_{(K)}, s_\tau)$ is a weak probability polyadic set algebra and there exists homomorphism from $\mathbb{H}_\Gamma$ onto $\mathbb{A}$. It means that there exists homomorphism from $\mathbb{B}$ onto $\mathbb{A}$. $\square$

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