WEAK PROBABILITY POLYADIC ALGEBRAS

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Abstract. We are motivated by the connection between Boolean algebra and propositional logic, cylindric algebra and predicate logic, Keisler’s $L(V,\nu,m)$ logic and polyadic algebra [2], to introduce an algebra calling weak probability polyadic algebra which ‘corresponds’ to the weak probability logic with infinitely predicates $L(V,\nu,m,R)$, introduced in [4].

Structure $(B,+,\cdot,-,0,1,C^\tau(K),C^\tau(K)_S,\tau)$, where $(B,+,\cdot,-,0,1)$ is a Boolean algebra, $C^\tau(K)$, $C^\tau(K)_S$ and $\tau$ are unary operations on $B$, for each sequence $(K)$ of ordinals from $\alpha$, $\bar{\alpha} \leq \bar{\beta}$, each $r \in R$ ($R$ is as in [4]) and $\tau \in \beta^\beta$, is a weak probability polyadic algebra of dimension $\beta$, briefly $WPP_\beta$, if the following postulates hold:

(WPP1) $(B,+,\cdot,-,0,1,C^\tau(K),S_\tau)$ is a polyadic algebra of dimension $\beta$,

(WPP2) (i) $C^\tau_{(\emptyset)}x = x$, $r > 0$,

(ii) $C^\tau_{(K)}0 = 0$, $r > 0$,

(WPP3) $C^0_{(K)}x = 1$,

(WPP4) $C^s_{(K)}x \leq C^s_{(K)}x$, $r \geq s$,

(WPP5) $C^s_{(K)}C^s_{(K)}x = C^s_{(K)}x$, $r > 0$,

(WPP6) (i) $-C^\tau_{(K)}x \cdot C^{1-s}_{(K)}y \leq -C^{\min\{1,r+s\}}_{(K)}(x \cdot y)$,

(ii) $C^\tau_{(K)}x \cdot C^s_{(K)}y \cdot C^1_{(K)} - (x \cdot y) \leq C^{\min\{1,r+s\}}_{(K)}(x \cdot y)$,

(WPP7) (i) $C^r_{(K)}x - x \geq C^{1-r}_{(K)}x$,

(ii) $C^s_{(K)}x - x \leq C^{1-s}_{(K)}x$, $s > r$,

(iii) $-C^{1-s}_{(K)}x - x \leq C^{s+}_{(K)}x$, where $s^+ = \min\{r \in R \mid r > s\}$,

(WPP8) $S_\sigma C^\tau_{(K)}x = S_\tau C^\tau_{(K)}x$, if $\sigma \upharpoonright (\beta \setminus \text{rang } K) = \tau \upharpoonright (\beta \setminus \text{rang } K)$,

(WPP9) $S_\sigma C^\tau_{\sigma^{-1}(K)}x = C^\tau_{(K)}S_\sigma x$, if $\sigma \cdot \sigma^{-1}(K)$ is 1–1 function,

(WPP10) (i) $C^r_{(K)}x \leq C^r_{(K)}x$, $r > 0$,

(ii) $C_{(K)}C^r_{(K_1)}x = C^r_{(K_1)}x$, $\text{rang } K \subset \text{rang } K_1$

(iii) $C^r_{(K)}C_{(K_1)}x = C^r_{(K_1)}x$, $r > 0$, $\text{rang } K \subset \text{rang } K_1$.
Theorem 1. If \( A \) is \( \text{WPP}_{\beta} \) algebra, then

1. \( C_{(K)}^{\top} 1 = 1 \),
2. If \( x \leq y \), then \( C_{(K)}^{\top} x \leq C_{(K)}^{\top} y \),
3. \( C_{(K)}^{\top} x + C_{(K)}^{\top} y \leq C_{(K)}^{\top} (x + y) \),
4. \( C_{(K)}^{\top}(x \cdot y) \leq C_{(K)}^{\top} x \cdot C_{(K)}^{\top} y \),
5. \( C_{(K)}^{1}(x) = x \) iff \( C_{(K)} x = x \),
6. \( S_{x} 1 = 1 \).

The proof is similar to that one in [6].

Let \( V_{1} \) be a set of new variables such that \( V_{1} \cap V = \emptyset \), \( V_{1} \cap P = \emptyset \) and \( \overline{V} = \overline{V} \). Let be \( V^{\ast} = V \cup V_{1}, \tau_{0} : V^{\ast} \setminus V_{1} \). Then \( V \cap P \) is closed for them. It is easy to verify (by using the axioms and the next result: for each \( \tau \in V^{V} \), we define:

\[
(\exists F x)(\Phi/\Gamma) = ((\exists x)\Phi)/\Gamma,
\]

\[
(P F x \geq r)(\Phi/\Gamma) = ((P x \geq r)\Phi)/\Gamma,
\]

\[
S^{F}(\tau)(\Phi/\Gamma) = (S_{f}(\tau)S_{f}((\tau_{0}^{-1})S(\tau_{0})\Phi)/\Gamma.
\]

Lemma 2. (i) If \( \Phi \) is atomic formula and \( \tau \in V^{V} \), then

\[
S^{F}(\tau)(\Phi/\Gamma) = (S_{f}(\tau)\Phi)/\Gamma = (S(\tau)\Phi)/\Gamma.
\]

(ii) If \( \tau \in (V \setminus V_{0}(\Phi))^{V} \), then \( S^{F}(\tau)(\Phi/\Gamma) = (S_{f}(\tau)\Phi)/\Gamma. \)

(iii) If \( \tau \in V^{V} \) is 1–1, then \( S^{F}(\tau)(\Phi/\Gamma) = (S(\tau)\Phi)/\Gamma. \)

Lemma 3. If \( m = \overline{V}^{+} \), then \( \overline{H}_{\Gamma} = (H_{\Gamma}, +, \cdot, \top, -, 1_{\Gamma}, 0_{\Gamma}, \exists x, P^{F} x \geq r, S^{F}(\tau)) \), where \( \tau \in V^{V} \), \( r \in R \) and \( x \) is a sequence of different variables from \( V \) of length \( \alpha \), \( \overline{\alpha} \prec m \), is a weak probability polyadic algebra of dimension \( V \).

Proof. Let be \( \Phi^{F} = \Phi/\Gamma \). It follows from Lemma 1 \( (H_{\Gamma}, +, \cdot, -\), \( 1_{\Gamma}, 0_{\Gamma}) \) is a Boolean algebra. The operations \( \exists F x, P^{F} x \geq r \) and \( S^{F}(\tau) \) are well defined and the set \( H_{\Gamma} \) is closed for them. It is easy to verify (by using the axioms and the rules of inference of \( L \), see [4]) that all axioms \( (\text{WPP}_{1}) - (\text{WPP}_{10}) \) hold in \( \overline{H}_{\Gamma} \).

We shall only prove \( (\text{WPP}_{10}) \):
Weak Probability Polyadic Algebras

(i) 

\[ (P^r x \geq r) \phi^\Gamma = ((P x \geq r) \phi)^\Gamma \]
\[ \leq^\Gamma ((P x > 0) \phi)^\Gamma \quad \text{for } r > 0, \text{ by } A_{10} \]
\[ \leq^\Gamma ((\exists x) \phi)^\Gamma \quad \text{by contraposition } A_{12} \]
\[ = (\exists^r x) \phi^\Gamma. \]

(ii) 

\[ (\exists^r x)(P^r y \geq r) \phi^\Gamma = ((\exists x)(P y \geq r) \phi)^\Gamma \leq^\Gamma ((P y \geq r) \phi)^\Gamma = (P^r y \geq r) \phi^\Gamma, \]
by supposing \( \text{rang } x \subset \text{rang } y \). The above inequality follows from

\[ \Gamma \vdash_{L^*} (\exists x)(P y \geq r) \Phi \rightarrow (P y \geq r) \Phi \]
that we obtain from

\[ \Gamma \vdash_{L^*} (\forall x)((P y < r) \Phi \rightarrow (P y < r) \Phi) \]
by using \( A_2 \).

(iii) 

\[ (\exists^r y) \phi^\Gamma = ((\exists y) \phi)^\Gamma \]
\[ \leq^\Gamma ((P x \geq r) (\exists y) \phi)^\Gamma \quad \text{for } V_f((\exists y) \phi) \cap \text{rang } x = \emptyset, \text{ by } R_3 \text{ and } A_4 \]
\[ = (P^r x \geq r)(\exists^r y) \phi^\Gamma. \]

Conversely,

\[ (P^r x \geq r)(\exists^r y) \phi^\Gamma = ((P x \geq r)(\exists y) \phi)^\Gamma \]
\[ \leq^\Gamma ((P x > 0)(\exists y) \phi)^\Gamma \quad \text{for } r > 0, \text{ by } A_{10} \]
\[ \leq^\Gamma ((\exists x)(\exists y) \phi)^\Gamma \quad \text{by } A_{12} \]
\[ = ((\exists y) \phi)^\Gamma \quad \text{from } A_2, \text{ since } \text{rang } x \subset \text{rang } y \]
\[ = (\exists^r y) \phi^\Gamma. \quad \square \]

Let \( \mathcal{A} = (A, R_p, \mu_\alpha)_{p \in \overline{\mathbb{R}_+}} \) be a weak probability structure for \( L \) and

\[ V = \beta, \quad \phi^\mathcal{A} = \{ a \in A^\beta \mid \models_{\mathcal{A}} \Phi[a \mid V^+] \} \quad \text{and} \quad \mathcal{A} = \{ \phi^\mathcal{A} \mid \Phi \in H \}. \]
We shall define for each sequence \((K)\) of ordinals from \(\beta\) of length \(\alpha, \overline{\alpha} \leq \overline{\beta}\), each \(r \in R\) and each \(\sigma \in \beta^\beta\):

\[
c_{(K)}(\Phi^\alpha) = \begin{cases} a \in A^\beta \mid b \upharpoonright (\beta \setminus \text{rang}(K)) = a \upharpoonright (\beta \setminus \text{rang}(K)), \\
\text{for some } b \in \Phi^\alpha \end{cases},
\]

\[
c'_{(K)}(\Phi^\alpha) = \begin{cases} a \in A^\beta \mid \mu_\alpha \{b \circ (K) \mid b \in \Phi^\alpha, \\
b \upharpoonright (\beta \setminus \text{rang}(K)) = a \upharpoonright (\beta \setminus \text{rang}(K)) \geq r \} \\
\end{cases},
\]

\[
s_\sigma(\Phi^\alpha) = \begin{cases} a \in A^\beta \mid a_\sigma \in (S_f(\tau_0^{-1})S(\tau_0)\Phi)^\alpha \\
\end{cases},
\]

where \(a_\sigma = (a_{\sigma(\alpha)})_{\alpha < \beta}\) for \(a = (a_\alpha)_{\alpha < \beta}\).

It is easy to verify that the above operations are well defined and that the sets from definition of \(c'_{(K)}\) are measurable. We may show that \(\mathcal{A}\) is closed for these operations.

\[
c_{(K)}(\Phi^\alpha) = \begin{cases} a \in A^\beta \mid b \upharpoonright (\beta \setminus \text{rang}(K)) = a \upharpoonright (\beta \setminus \text{rang}(K)), \\
\text{for some } b \in \Phi^\alpha \end{cases},
\]

\[
c'_{(K)}(\Phi^\alpha) = \begin{cases} a \in A^\beta \mid \mu_\alpha \{b \circ (K) \mid b \in \Phi^\alpha, \\
b \upharpoonright (\beta \setminus \text{rang}(K)) = a \upharpoonright (\beta \setminus \text{rang}(K)) \geq r \} \\
\end{cases},
\]

\[
s_\sigma(\Phi^\alpha) = \begin{cases} a \in A^\beta \mid a_\sigma \in (S_f(\tau_0^{-1})S(\tau_0)\Phi)^\alpha \\
\end{cases},
\]

The algebra \((\mathcal{A}, \cup, \cap, \sim, \emptyset, A^\beta, c_{(K)}, c'_{(K)}, s_\sigma)\) will be called \textit{weak probability polyadic set algebra}. 
Lemma 4. Let $f : H_\Gamma \to \mathcal{A}$ be defined by $f(\Phi^\Gamma) = \Phi^\mathcal{A}$. Then, if $\models_{\mathcal{A}} \Gamma$ then $f$ is homomorphism $\mathbb{H}_\Gamma$ onto $\mathcal{A}$ and $\mathcal{A}$ is a weak probability polyadic algebra.

Theorem 2. Let $\mathbb{B} = (B, +, , - , 0, 1, C_K, C_{K}^r, S_\alpha)$ be a weak probability polyadic algebra of infinite dimension $\beta$. Let be $V = \beta$, $P = \{ p \mid p \in B \}$, $m = \beta^+$ and $\nu(p) = \beta$ for each $p \in P$. Then there exists a set of sentences $\Gamma$ in logic $L = L(V, \nu, m, R)$ such that $\mathbb{H}_\Gamma \cong \mathbb{B}$.

Proof. We shall put $V = \beta$, $P = B$, $\nu(p) = \beta$ for each $p \in P$ and $x \in \beta^+$ “1–1” and “onto”. Let $\Gamma$ be the next set of sentences:

\[
\begin{align*}
(\forall x)((-p)(x) & \leftrightarrow \neg(p(x))) , \\
(\forall x)((S_r p)(x) & \leftrightarrow S_\tau(p x)) , \\
(\forall x)((p + q)(x) & \leftrightarrow p(x) \lor q(x)) , \\
(\forall x)((C_K p)(x) & \leftrightarrow (\exists y)p(x)) , \\
(\forall x)((p \cdot q)(x) & \leftrightarrow p(x) \land q(x)) , \\
(\forall x)((C^r_K p)(x) & \leftrightarrow (Py \geq r)p(x)) ,
\end{align*}
\]

where $p, q \in P$, $\tau \in V^\beta$, $r \in R$ and $(K) = y$ is a sequence of different variables of length $\alpha$, $\beta < m$. First, we shall show that $H_\Gamma = \{(p(x))^\Gamma \mid p \in P\}$.

For each formula $\Phi \in H$ there exists formula $\Psi = S_f(\tau_0^{-1})S(\tau_0)\Phi \in H$ such that $\Phi^\Gamma = \Psi^\Gamma$, $V_\Gamma(\Psi) \in V_1$ and $V_f(\Psi) \subseteq V$. Really,

\[
\Psi^\Gamma = S_f(id)S_f(\tau_0)S(\tau_0)\Phi^\Gamma = S^\Gamma(id)\Phi^\Gamma = \Phi^\Gamma.
\]

We shall prove that for each formula $\Phi$ from $H$, with free variables in $V$ and bound in $V_1$, there exists formula $\Psi \in F$ such that $\Phi^\Gamma = \Psi^\Gamma$. Let be $\tau_1 : V^* \xrightarrow{\text{na}} V$. Then

\[
\Gamma \vdash_{L^*} S(\tau_1)\Phi \leftrightarrow S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi.
\]

By $(R_3)$, we have

\[
\Gamma \vdash_{L^*} S(\tau_1^{-1})S(\tau_1)\Phi \leftrightarrow S(\tau_1^{-1})S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi.
\]

Since $V_\Gamma(S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi) \subseteq V_1$, it follows

\[
S(\tau_1^{-1})S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi = S_f(\tau_1^{-1})S_f(\tau_0^{-1})S(\tau_0)S(\tau_1)\Phi
\]

\[
= S_f(\tau_1^{-1})S(\tau_0)S_f(\tau_1^{-1})S(\tau_1)\Phi.
\]
\[ \Gamma \vdash_{L^*} \Phi \leftrightarrow S_f(\tau_0^{-1})S(\tau_0)S_f(\tau_1^{-1})S(\tau_1)\Phi. \]

Putting \( \Psi = S_f(\tau_1^{-1})S(\tau_1)\Phi \), we have

\[ \Gamma \vdash_{L^*} \Phi \leftrightarrow \Psi, \quad \Psi \in F. \]

It follows that it is enough to prove that for each formula \( \Phi \in F \) there exists \( p \in P \) such that \( \Phi^p = p(x)^\Gamma \). We can prove it by induction of complexity of \( \Phi \).

Now, we shall define a function \( g : B \rightarrow H_\Gamma \) on the following way:

\[ g(p) = (p(x))^\Gamma. \]

It is easy to verify that \( g \) is well defined, onto and homomorphism. To prove that it is one to one we shall define \( h : F \rightarrow B \) such that, for \( \Phi \in F \) holds:

- if \( \Gamma \vdash_{L^*} \Phi \) then \( h(\Phi) = 1 \).

For formulas from \( F' = \{ \Phi \in F \mid V_b(\Phi) \cap V_f(\Phi) = \emptyset \} \) we define the mapping \( h \) by induction:

\[
\begin{align*}
  h(p(x)) &= p, & h(\Phi \vee \Psi) &= h(\Phi) + h(\Psi), \\
  h(p(\tau \circ y)) &= S_\tau p, & h(\exists y \Phi) &= C_{(K)} h(\Phi), \\
  h(-\Phi) &= -h(\Phi), & h((Py \geq r)\Phi) &= C'_{(K)} h(\Phi).
\end{align*}
\]

For \( \Phi \in F \setminus F' \), let be

\[ h(\Phi) = S((\tau_1^{-1} \mid V_f(\Phi)) \mid h(\Psi)), \]

where \( \Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi \) and \( \tau_1 \) is a fixed injection from \( V^* \) onto \( V \).

First, we shall prove that for each function \( \sigma : V_f(\Phi) \rightarrow V \setminus V_b(\Phi) \) and every formula \( \Phi \in F \) we have

\[(*) \quad h(S_f(\sigma)\Phi) = S_{\sigma^\Gamma} h(\Phi). \]

For \( \Phi \in F' \) the proof is by induction on the complexity of formula.

If \( \Phi = p(x) \), then \( h(S_f(\sigma)\Phi) = h(p(\sigma \circ x)) = S_{\sigma} p = S_{\sigma^\Gamma} h(\Phi) \).
If $\Phi = p(y)$, $y = \tau \circ x$, $\tau \in V^V$, we have

\[ h(S_f(\sigma)p(y)) = h(p(\sigma \circ \tau \circ x)) \quad \text{by definition } S_f \]

\[ = S_{\sigma \circ \tau}p \quad \text{by definition } h \]

\[ = S_{\sigma \circ V}S_{\tau}p \quad \text{by } (P_3) \text{ (see [1])} \]

\[ = S_{\sigma \circ V}h(p(y)) \quad \text{by definition } h. \]

If $\Phi = \neg \Psi$ or $\Phi = \Psi \lor \Theta$ it is easy to verify.

If $\Phi = (Py \geq r)\Psi$, then

\[ h(S_f(\sigma)\Phi) = h(S_f(\sigma)(Py \geq r)\Psi) \]

\[ = h((Py \geq r)S_f(\sigma_1)\Psi) \quad \text{where } \sigma_1 = \sigma \upharpoonright V_f(\Psi) \]

\[ = C_{(K)}(\sigma_1)V h(\Psi) \]

\[ = S_{\sigma \circ V}C_{\sigma_1}^{\upharpoonright(K)}h(\Psi) \]

\[ = S_{\sigma \circ V}C_{(K)}^{\upharpoonright}h(\Psi) \quad \text{since } \sigma \upharpoonright (K) = id \]

\[ = S_{\sigma \circ V}h((Py \geq r)\Psi). \]

To show that (*) holds for each formula from $F \setminus F'$, we shall prove, first, that

\[ (**\quad h(S(\sigma)\Phi) = S_{(\sigma \circ V_f(\Psi)))\setminus V} h(\Phi), \]

for each formula $\Phi \in F$ and each $\sigma : V(\Phi) \overset{1-1}{\longrightarrow} V$.

First, let be $\Phi \in F'$.

If $\Phi = p(x)$, then $h(S(\sigma)p(x)) = h(p(\sigma \circ x)) = S_{\sigma \circ V_f(\Psi)}h(\Phi)$.

If $\Phi = (\exists y)\Psi$, we have

\[ h(S(\sigma)\Phi) = h((\exists y)S(\sigma)\Psi) \]

\[ = C_{\sigma \circ (K)}h(S(\sigma)\Psi) \]

\[ = C_{\sigma \circ (K)}S_{(\sigma \circ V_f(\Psi)))\setminus V} h(\Psi) \quad \text{by induction hypothesis} \]

\[ = S_{(\sigma \circ V_f(\Psi)))\setminus V} C_{(K)}h(\Psi) \quad \text{by } (P_{12}) \]

\[ = S_{(\sigma \circ V_f(\Psi)))\setminus V} C_{(K)}h(\Psi) \quad \text{by } (P_{11}) \]

\[ = S_{(\sigma \circ V_f(\Psi)))\setminus V} h((\exists y)\Psi) = S_{(\sigma \circ V_f(\Psi)))\setminus V} h(\Psi). \]
If $\Phi \in F \setminus F'$ and $\sigma : V(\Phi) \overset{1-1}{\longrightarrow} V$, let be $\Theta = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)S(\sigma)\Phi$, $\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi$ and $\sigma_1 : V(\Psi) \longrightarrow V$ defined by $\sigma_1 \upharpoonright V_0(\Psi) = \tau_1 \circ \tau_0 \circ \tau_0^{-1} \circ \tau_1^{-1} \upharpoonright V_0(\Psi)$ and $\sigma_1 \upharpoonright V_f(\Psi) = \tau_1 \circ \sigma \circ \tau_1^{-1} \upharpoonright V_f(\Psi)$. Then

$$h(S(\sigma)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(\Theta) = S(\tau_1^{-1}V_j(\Theta))V h(S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)S(\sigma)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(S(\sigma_1)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(\Psi) = S(\tau_1^{-1}V_j(\Theta))V h(S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi) = h(S_f(\sigma)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(\Theta) = h(\Psi) = S(\sigma_1\Phi)\Phi.$$

By using (**) it is easy to verify that for each formula $\Phi \in F'$ holds

$$h(\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(\Psi),$$

where

$$\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi.$$

Now, we shall prove that (i) is valid for each formula $\Phi$ from $F \setminus F'$. Let be $\Theta = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)S(\sigma)\Phi$ and $\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi$. Then

$$h(S_f(\sigma)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(\Theta) = S(\tau_1^{-1}V_j(\Theta))V h(S_f(\sigma)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(S_f(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(S_f(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(S_f(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi) = S(\tau_1^{-1}V_j(\Theta))V h(S_f(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi) = S(\sigma_1\Phi)\Phi.$$

Now, we shall prove that $\Gamma \vdash L \cdot \Phi$ implies $h(\Phi) = 1$. It follows from $\Gamma \subseteq F$, $\Phi \in F$, that $\Gamma \vdash L \cdot \Phi$ if $\Gamma \vdash L \Phi$. So, it is enough to show that from $\Gamma \vdash L \Phi$ follows $h(\Phi) = 1$. At the beginning, we shall prove that each axiom maps to 1. First, let the axiom be from $F'$.

(A$_1$) It is sufficient to show that $h(\Phi) \neq 1$ implies $\Phi$ is not tautology. Let $I$ be maximal ideal of Boolean algebra $(B, +, \cdot, \neg, 0, 1)$ which contains $h(\Phi)$ and $\pi$ natural homomorphism from $B$ onto two-elements Boolean algebra $B/I$. From $h(\Phi) \neq 1$ it follows $\pi \circ h(\Phi) = 0$, i.e. $\Phi$ is not tautology.
(A2) \[ h((\forall y)(\Phi \rightarrow \Psi)) \rightarrow (\Phi \rightarrow (\forall y)\Psi) \]
\[ = h(-((\exists y)-((\Phi \rightarrow \Psi)) \rightarrow (\Phi \rightarrow -(\exists y)-\Psi))) \]
\[ = C_{(K)}(-h(-\Phi \lor \Psi)) + -h(\Phi) + -C_{(K)}(-h(\Psi)) \]
\[ = C_{(K)}(h(\Phi) \cdot -h(\Psi)) + -h(\Phi) + -C_{(K)}(-h(\Psi)) \]
\[ = C_{(K)}(h(\Phi) \cdot C_{(K)}(-h(\Psi))) + -h(\Phi) + -C_{(K)}(-h(\Psi)) \]
\[ \geq h(\Phi) \cdot C_{(K)}(-h(\Psi)) + -h(\Phi) + -C_{(K)}(-h(\Psi)) \]
\[ = -(h(\Phi) - C_{(K)}(-h(\Psi))) \cdot h(\Phi) \cdot C_{(K)}(-h(\Psi)) \]
\[ = -(0 + 0) = 1, \]

with condition \( V_f(\Phi) \cap \text{rang} \: y = \emptyset. \)

(A3) \[ h((\forall y)\Phi \rightarrow S_f(\tau)\Psi), \text{ where } \tau : \text{rang} \: y \longrightarrow V \setminus V_6(\Phi) \]
\[ = h((\exists y)-\Phi \lor S_f(\tau)\Phi) \]
\[ = C_{(K)}(-h(\Phi)) + S_{\tau \cdot} h(\Phi), \text{ where } \tau' = \tau \upharpoonright V \]
\[ = S_{\tau \cdot} C_{(K)}(-h(\Phi)) + S_{\tau \cdot} h(\Phi) \text{ by (P5) since } \]
\[ \tau' \upharpoonright (V \setminus \text{rang} \: y) = id \upharpoonright (V \setminus \text{rang} \: y) \]
\[ = S_{\tau \cdot} (C_{(K)}(-h(\Phi)) + h(\Phi)) \]
\[ \geq S_{\tau \cdot} (-h(\Phi) + h(\Phi)) \]
\[ \geq S_{\tau \cdot} 1 = 1. \]

(A4) \[ h((\forall y)\Phi \rightarrow (\forall \tau \circ y)\Phi) = C_{(K)} - h(\Phi) + -C_{(K)} - h(\Phi) = 1 \]

by (P13) supposing \( \tau : y \xrightarrow{1-1} \text{onto} \: y. \)

(A5) \[ h((Py \geq 0)\Phi) = C_{(K)}^0 h(\Phi) = 1 \text{ from (WPP2)}. \]

(A6) \[ h((Py \geq r)\Phi \rightarrow (Py \geq s)\Phi) = -C_{(K)}^r h(\Phi) + C_{(K)}^s h(\Phi) \geq -C_{(K)}^r h(\Phi) + C_{(K)}^r h(\Phi) = 1, \text{ for } r \geq s, \text{ by (WPP4)}. \]
\[ (A_7) \]
\[
 h \left( (Py \geq r) \Phi \land (Py \geq s) \Psi \land (Py \leq 0) (\Phi \land \Psi) \right) \\
\rightarrow (Py \geq \min\{1, r + s\}) (\Phi \lor \Psi) \\
= - \left( C_{(K)}^r h(\Phi) \cdot C_{(K)}^s h(\Psi) \cdot C_{(K)}^1 - (h(\Phi) \cdot h(\Psi)) \right) \\
+ C_{(K)}^{\min\{1, r+s\}} (h(\Phi) + h(\Psi)) \\
\geq - C_{(K)}^{\min\{1, r+s\}} (h(\Phi) + h(\Psi)) + C_{(K)}^{\min\{1, r+s\}} (h(\Phi) + h(\Psi)) = 1. 
\]

\[ (A_8) \]
\[
 h \left( (Py \leq r) \Phi \land (Py < s) \Psi \rightarrow (Py < \max\{0, r + s - 1\}) (\Phi \land \Psi) \right) \\
= - \left( C_{(K)}^{1-r} - h(\Phi) \cdot -C_{(K)}^s h(\Psi) \right) + C_{(K)}^{\max\{0, r+s-1\}} (h(\Phi) \cdot h(\Psi)) \\
\geq C_{(K)}^{\max\{0, r+s-1\}} (h(\Phi) \cdot h(\Psi)) + C_{(K)}^{\max\{0, r+s-1\}} (h(\Phi) \cdot h(\Psi)) = 1. 
\]

\[ (A_9) \]
\[
 h \left( (Py < s) \Phi \rightarrow (Py \leq s) \Phi \right) = -C_{(K)}^s h(\Phi) + C_{(K)}^{1-s} - h(\Phi) \\
\geq C_{(K)}^s h(\Phi) + -C_{(K)}^s h(\Phi) = 1, \text{ by } (WPP_k). 
\]

\[ (A_{10}) \]
\[
 h \left( (Py \geq s) \Phi \rightarrow (Py > r) \Phi \right) = -C_{(K)}^s h(\Phi) + C_{(K)}^r h(\Phi) \\
\geq - C_{(K)}^s h(\Phi) + C_{(K)}^s h(\Phi) = 1, \text{ by } (WPP_k), \text{ for } s > r. 
\]

\[ (A_{11}) \]
\[
 h \left( (Py > s) \Phi \rightarrow (Py \geq s^+) \Phi \right) = C_{(K)}^{1-s} - h(\Phi) + C_{(K)}^{s^+} h(\Phi) \\
\geq - C_{(K)}^{s^+} h(\Phi) + C_{(K)}^{s^+} h(\Phi) = 1, \text{ by } (WPP_k). 
\]

\[ (A_{12}) \]
\[
 h \left( (\forall y) \Phi \rightarrow (Py \geq 1) \Phi \right) = C_{(K)} h(\Phi) + C_{(K)}^1 h(\Phi) \\
\geq C_{(K)} h(\Phi) + -C_{(K)}^{0^+} h(\Phi) \text{ by } (WPP_2) \text{ (iii)} \\
\geq C_{(K)} h(\Phi) + -C_{(K)} - h(\Phi) \text{ from } (WPP_{10}) \text{ (ii)} \\
= 1. 
\]
Now, let $\Phi$ be axiom from $F \setminus F'$. Then $\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi$ is an axiom from $F'$. It follows $h(\Psi) = 1$. Then

$$h(\Phi) = S(\tau_1^{-1}\nu_f(\Psi))\nu h(\Psi) = S(\tau_1^{-1}\nu_f(\Psi))\nu 1 = 1.$$ 

It is easy to verify that all formulas from $\Gamma$ map to 1.

Now, we shall show that from formula mapped to 1, by any rule of inference, we get the formula which maps to 1, i.e. that all theorems map to 1. It is only nontrivial for ($R_3$) and ($R_4$).

($R_3$) Let be $\Phi \vdash_L S(\sigma)\Phi$, where $\sigma : V(\Phi) \xrightarrow{1-1} V$, $h(\Phi) = 1$. Then

$$h(S(\sigma)\Phi) = S(\sigma\nu_f(\Phi))\nu h(\Phi) = S(\sigma\nu_f(\Phi))\nu 1 = 1.$$ 

($R_4$) Let be $S_f(\sigma)\Phi \vdash_L \Phi$, where $\sigma : V_f(\Phi) \xrightarrow{1-1} V \setminus V_\nu(\Phi)$, $h(S_f(\sigma)\Phi) = 1$. If $\Phi \in F'$, we have $\Phi = S_f(\sigma^{-1})S_f(\sigma)\Phi$. It follows

$$h(\Phi) = S_{\sigma^{-1}}\nu h(S_f(\sigma)\Phi) = S_{\sigma^{-1}}\nu 1 = 1.$$ 

If $\Phi \in F \setminus F'$, let be $\Psi = S(\tau_1)S_f(\tau_0^{-1})S(\tau_0)\Phi$ and $\sigma_1 : V(S_f(\sigma)\Phi) \longrightarrow V$ defined by $\sigma_1 \upharpoonright V_\nu(S_f(\sigma)\Phi) = \tau_1 \circ \tau_0$ and $\sigma_1 \upharpoonright V_f(S_f(\sigma)\Phi) = \tau_1 \circ \sigma^{-1}$. It is clear that $\sigma_1$ is “1–1” function. Then

$$h(\Phi) = S(\tau_1^{-1}\nu_f(\Psi))\nu h(\Psi) = S(\tau_1^{-1}\nu_f(\Psi))\nu h(S(\sigma_1)S_f(\sigma)\Phi)$$
$$= S(\tau_1^{-1}\nu_f(\Psi))\nu S(\sigma_1\nu_f(S_f(\sigma)\Phi))\nu h(S_f(\sigma)\Psi)$$
$$= S(\tau_1^{-1}\nu_f(\Psi))\nu S(\sigma_1\nu_f(S_f(\sigma)\Phi))\nu 1 = 1.$$ 

Finally, we shall show that $g : B \rightarrow H_F$, $g(p) = (p(x))^F$, is “1–1”. If $g(p) = g(q)$, then $\Gamma \vdash_L p(x) \leftrightarrow q(x)$. It follows $h(p(x) \leftrightarrow q(x)) = 1$, and hence $p = q$. $\square$

**Theorem 3.** If $\mathbb{B}$ is a weak probability polyadic algebra with infinite dimension $\beta$, then there exists homomorphism of $\mathbb{B}$ to some weak probability polyadic set algebra.

**Proof.** By the previous theorem there exists the set of sentences $\Gamma$ in logic $L = L(\beta, \nu, \overline{\beta}^+, R)$ such that $H_F \equiv \mathbb{B}$. Since $\overline{B} > 1$, it follows that $\Gamma$ is consistent in $L$. By the completeness theorem, $\Gamma$ has a weak model $\mathfrak{A}$. By Lemma 4 $\mathfrak{A} = (A, \cup, \cap, \sim, A^0, \emptyset, c^{(K)}, c'^{(K)}, s_\tau)$ is a weak probability polyadic
set algebra and there exists homomorphism from $\mathbb{H}_F$ onto $A$. It means that there exists homomorphism from $\mathbb{B}$ onto $A$. \qed

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