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THE ENVELOPE OF COMPOSITIONS CURVES

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This paper is dedicated to Professor R. Ž. Djordjević for his 65th birthday

Abstract. In this paper, different relationships and a functional dependence between the compositions $C(n,k)$ and parameters of compositions n and k have been analyzed. A combinatorial nature of this analysis enables a conclusion that families of compositions curves have an envelope. The envelope is a straight line with a slope which is proportional to $\log k$, i.e. the slope of the envelope is a function of the composition length k. The envelope theory is very important for consideration of temperature and chemical equilibrium of statistic thermodynamic systems and coding theory in telecommunication.

1. Introduction

The theory of probability, permutations and their operations, present one of the basic and most important objects in statistical physics and quantum mechanics. Permutations are defined like the rule of rearranging a finite collection of elements or like a bijective map of set $\mathbb{N}$ onto itself (see [2]). Also, applying some conditions onto forms of permutations we get many new combinatorial elements such as: combinations, variations, compositions, partitions, etc. Our analysis of statistical systems is based on partitions and compositions.

Definition 1. Let n $(\in \mathbb{N})$ be an integer $(n \geq 1)$. A partition of n is representation of n as sum of integers, not considering the order of terms of this sum

$$n = n_1 + n_2 + n_3 + \cdots + n_k \,, \quad n_i \geq 1 \quad (i = 1, 2, \ldots, k)\,, \tag{1}$$

where the numbers n_i are called parts of partition or summands.

An integer n can be presented like sum of k non-negative numbers, i.e. $n_i \geq 0$. This combinatorial object is known as composition.

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Definition 2. A composition of n is a particular arrangement of non-negative integers $[n_1, n_2, \dots, n_k]$, that sum up to n, and which total number is

$$C_k(n) = \binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!(k-1)!}\,. \tag{2}$$

These two related combinatorial objects, partitions and compositions, have a host of interesting applications and properties in our analysis of equilibrium in thermodynamic systems [1, 3].

Let us find all compositions of number n of lenght k. Total number of composition's set $[n_1, n_2, \dots, n_k]$ will be equal to (2).

Also, for each composition set the value m is determined as

$$m = n_1 + 2n_2 + \cdots + k\,n_k\,. \tag{3}$$

Also for each composition's set we can compute P_m, i.e. multinomial coefficient.

Definition 3. The multinomial coefficient $P_m[n_1, n_2, \dots, n_k, n]$,

$$P_m(n_1, n_2, \dots, n_k, n) = \frac{n!}{n_1!n_2!\cdots n_m!}\,, \tag{4}$$

gives the number of ways of partitioning N distinct objects into m sets of sizes n_i (with $n = \sum_{i=1}^{m} n_i$).

Therefore, each composition set can be presented or mapped by two values P_{mi} and m_i, i.e. by the pair (m_i, P_{mi}). Therefore, we get collection of ordered pairs (mi, P_{mi}) for $n = \text{const}$.

For a given value m_i and corresponding sets $[n_1, n_2, \cdots, n_k]$ (3) taken into account equation (4), the P_{ni} can be calculated too. By this calculation we can get a collections of ordered pairs (n_i, P_{ni}) for $m = \text{const}$. In effort to make our analysis easier and more convenient, in further analysis, the values $S_{mi} = \log P_{mi}$ instead of P_{mi} and $S_{ni} = \log P_{ni}$ instead of P_{ni} are used.

2. Software

All necessary computations of values m_i and P_i and appropriate composition sets is achieved by available software (see [4]). The software has been organized by modules. The modules are distributed in particular menus and subroutines. The values of multinomial coefficient are computed on the base of precise values of factorial function $F(N)$ (for $N < 171$) or $L(N) = \log F(N)$ for $N > 171$. This way of computation depends on the ability of system that has been used in this paper.

This analysis has been most easier for case $k = 2$. During the analysis the following values are computed:

$$P = e^S = \frac{n!}{n_1!n_2!}\ , \quad \text{i.e.} \quad S = \log P = \log \frac{n!}{n_1!n_2!}\ .$$

So, we mapped all composition sets (n_1, n_2) for $k = 2$ with pairs (m_i, S_{mi}) for $n = \text{const}$, and (n_i, S_i) for $m = \text{const}$.

The analysis for the cases $k \geq 3$ are more complex. For these cases the values of

$$P = \frac{n!}{n_1!n_2!\cdots n_k!}\ , \quad \text{i.e.} \quad S = \log P = \log \frac{n!}{n_1!n_2!\cdots n_k!}\ ,$$

are also computed. But in this case for $n = \text{const}$ we have more then one composition sets with a same value of m, defined in equation (3). Among all sets with identical values of m_i, the set with the greatest value of $S_{\max}$ is selected and presented by pair $(m_i, S_{\max})$ in the plane mOS (Fig. 1). For the case where $m = \text{const}$, the same procedure is also applied.

3. The Analysis

On the basis of equations (3), (4), every composition set of number n, of lenght k is presented in form of pairs $(m_i,\ S_{mi})$. In further analysis a functional dependence $S_{mi} = f(m_i)$ for $n = \text{const}$ and contrary $S_{ni} = f(n_i)$ for $m = \text{const}$ are investigated.

Refereing to Fig. 1, it is possible to notice the symmetrical distribution of pairs (m_i, S_m) in the plane mOS. These collections (Fig. 1) have a sharp maximum $S_{\max}$ only if $n \equiv 0 (\operatorname{mod} k)$. The value of $S_{\max}$ responds to $m_{\max}$, which is located in the middle of interval of available values of m_i. Now, it is very important to emphasize that this maximum responds to composition set where $n_1 = n_2 = \cdots = n_k$ i.e. for the case where all numbers n_i from composition set are equal. This fact will be used to determine and to calculate the equation of envelope.

In other situations, for the cases where $n \equiv 1 (\operatorname{mod} k)$, $n \equiv 2 (\operatorname{mod} k)$, $\ldots$, $n \equiv (k-1) (\operatorname{mod} k)$, a sharp maximum does not exists.

According to Fig. 2, the values of S_n for $m = \text{const}$, start from the zero value only for the case when $m \equiv 0 (\operatorname{mod} k)$. In all other cases the values for S_n start from the value that is greater than zero. The values of S_n are asymmetrically distributed in the interval of given values of n. The maximal values of S_n respond to composition sets where $n_1 = 2n_2 = \cdots = kn_k$.

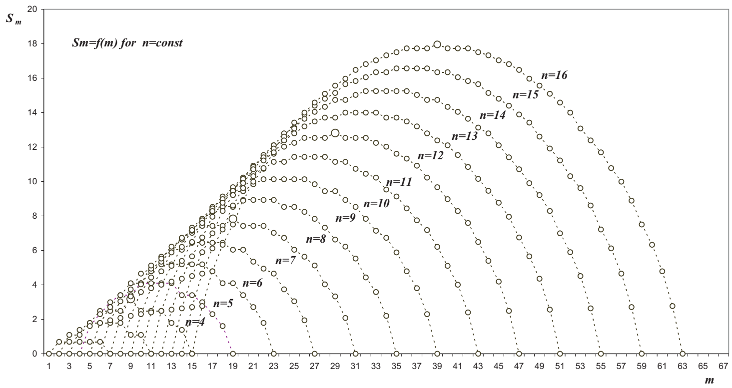

FIG. 1. Presentation of ordered pairs (m_i, P_{m_i}) in plane mOS

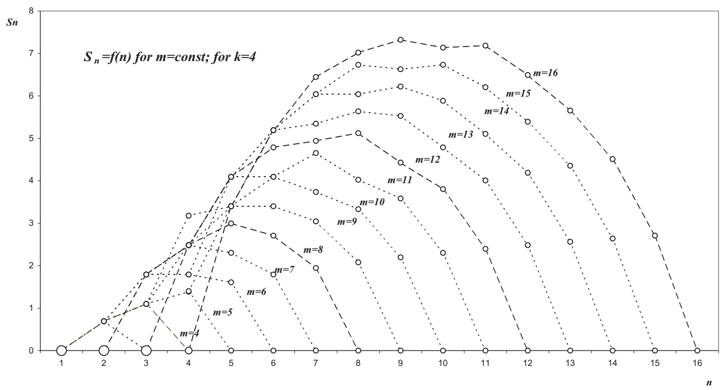

FIG. 2. Presentation of ordered pairs (n_i, S_{n_i}) in plane nOS

4. The Envelope of Sets of Compositions Curves

Analyzing graphical presentations of pairs (m_i, S_{mi}) for $n = \text{const}$ and (n_i, S_{ni}) for $m = \text{const}$, in the plane mOS and nOS, respectively, (Fig. 1

and Fig. 2), it can be seen that with an appropriate approximated function on the base of calculated dates, the family of curves with parameters n and m may be obtained (Fig. 3 and Fig. 4).

Every curve from the family has one and only one appropriate value of parameter n, for $N = \text{const}$. We are able to follow all curves of specific space by changing the parameter n in the specific space for $n = \text{const}$ (Fig. 3) or by changing a parameter m in the space where $m = \text{const}$ (Fig. 4). The previous conditions may be expressed as $F(m, C) = 0$ for $n = \text{const}$ and $F(n, C) = 0$ for $m = \text{const}$.

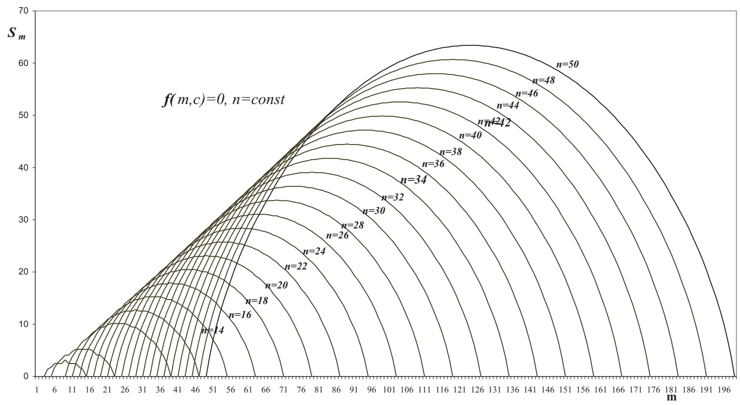

FIG. 3. The family of curves for $n = \text{const}$, $k = 4$

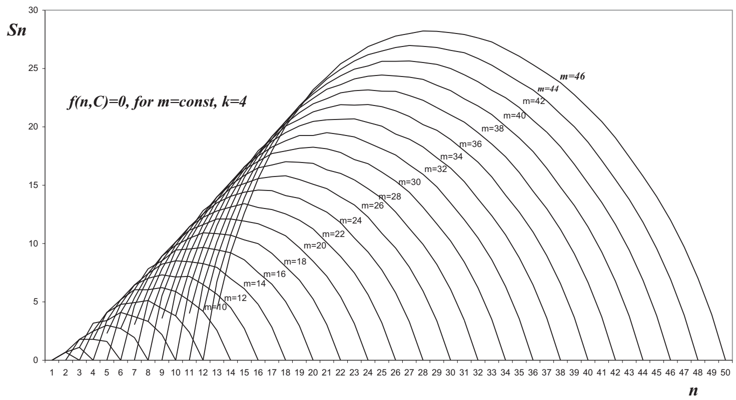

FIG. 4. The family of curves for $m = \text{const}$, $k = 4$

By choosing a maximum of S_{ni} for $m = \text{const}$ (Fig. 2), forming a pairs (m_i, S_{ni}) for each m $(m = 1, 2, \ldots, N)$ and presenting them in the plane mOS we can see that these pairs can be approximated with straight line which presents the envelope of family of curves $F(m, C) = 0$ for $n = \text{const}$ (Fig. 5). The pairs (n_i, S_{mi}) for case $n = \text{const}$ are located on the envelope of family of curves $F(n, C) = 0$ for $m = \text{const}$, in the plane nOS (Fig. 6).

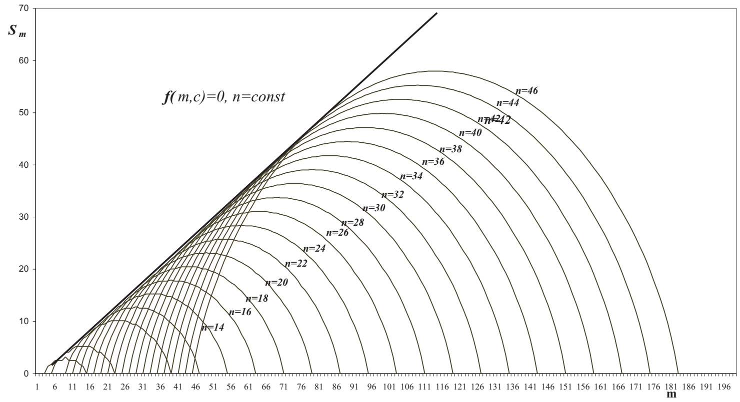

FIG. 5. The envelope of the family curves for $n = \text{const}$, $k = 4$

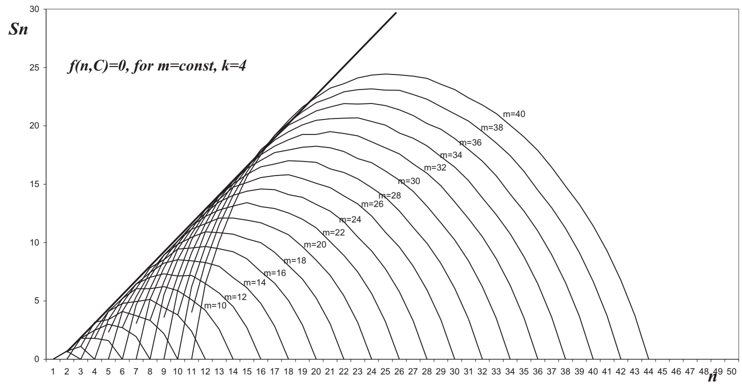

FIG. 6. The envelope of the family curves for $m = \text{const}$, $k = 4$

On the base of the fact that maximums of curves $S = f(m)$ for $n = \text{const}$ and $n \equiv 0 (\bmod k)$ respond to compositions presented by equal integers, it was possible to calculate the equation of the envelope of curves family for the case $n = \text{const}$:

$$S_{\max}^{k} = \log \frac{n!}{\left(\dfrac{n}{k}\right)!\left(\dfrac{n}{k}\right)!\cdots\left(\dfrac{n}{k}\right)!} \approx n \log k\,. \tag{5}$$

In order to get the previous expression, Stirling formula has been applied.

Hence, it can be concluded that, for large values of n ($m = n/k \gg k$), the envelope is almost straight line with asymptotical approach to the value of $\log k$ (see [1], [3]). The envelope divides the space onto two parts. The one with the compositions curves and second one, where the compositions curves do not exist. The envelope for the case $m = \text{const}$ will be presented in our future papers.

During the analysis of values S^k for the compositions of the length 2, 3, 4, 5, 6, ..., 10 on the basis of (5), it can be concluded that values of S^k are proportional to values $\log 2$, $\log 3$, ..., $\log 10$.

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