FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 15 (2000), 113–122

FINITE DIFFERENCE METHOD FOR THE HEAT EQUATION WITH COEFFICIENT FROM ANISOTROPIC SOBOLEV SPACE

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Dedicated to Professor R. Ž. Djordjević in the occasion of his 65th birthday

Abstract. In this paper we consider the first initial-boundary value problem for the heat equation with variable coeficient in a domain $(0,1)\times(0,T]$. We assume that the solution of the problem and the coefficient of equation belong to the corresponding anisotropic Sobolev spaces. Convergence rate estimate which is consistent with the smoothness of the data is obtained.

1. Introduction

For a class of finite difference schemes for parabolic initial-boundary value problem, the estimates of the convergence rates consistent with the smoothness of data, are of major interest, i.e.

$$\|u-v\|_{W_2^{r,r/2}(Q_{h\tau})} \le C(h+\sqrt{\tau})^{s-r}\|u\|_{W_2^{s,s/2}(Q)}\,, \quad s\ge r. \tag{1}$$

Here $u=u(x,t)$ denotes the solution of the original initial–boundary value problem, v denotes the solution of corresponding finite difference scheme, h and τ are discretisation parameters, $W_2^{s,s/2}(Q)$ denotes anisotropic Sobolev space, $W_2^{s,s/2}(Q_{h\tau})$ denotes discrete anisitropic Sobolev space, and C is a positive generic constant, independent of h,τ and u. For problems with variable coefficients constant C depends on the norms of coefficients.

Estimates of this type have been obtained for parabolic problems with coefficient which depends only on variable x [2]. In this paper we are deriving estimates for the parabolic problem with coefficient which depends on variables x and t. Our proof is based on Bramble–Hilbert lemma [3].

Received December 18, 1998.
1991 *Mathematics Subject Classification.* Primary 65M15; Secondary 46B70.

2. Initial–Boundary Value Problem and its Approximation

Let us define anisotropic Sobolev spaces $W_2^{s,s/2}(Q)$, $Q = \Omega \times I$, $I = (0,T)$, as follows [5]:

$$W_2^{s,s/2}(Q) = L_2(I, W_2^s(\Omega)) \cap W_2^{s/2}(I, L_2(\Omega))\,,$$

with the norm

$$\|f\|_{W_2^{s,s/2}(Q)} = \left(\int_0^T \|f(t)\|^2_{W_2^s(\Omega)} dt + \|f\|^2_{W_2^{s/2}(I,L_2(\Omega))} \right)^{1/2} .$$

We consider, as a model problem, the first initial–boundary value problem for parabolic equation with variable coefficient in the rectangular domain $Q = \Omega \times (0,T] = (0,1) \times (0,T]$:

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a\frac{\partial u}{\partial x}\Big) &= f\,, && (x,t) \in Q\,, \\ u &= 0\,, && (x,t) \in \partial\Omega \times [0,T]\,, \\ u(x,0) &= u_0(x)\,, && x \in \Omega\,. \end{aligned} \tag{2}$$

We assume that the generalized solution of the problem (2) belongs to the anisotropic Sobolev space $W_2^{s,s/2}(Q)$, $2 \le s \le 4$, with the right–hand side $f(x,t)$ which belongs to $W_2^{s-2,s/2-1}(Q)$. Consequently, coefficient $a = a(x,t)$ belongs to the space of multipliers $M\left(W_2^{s-1,(s-1)/2}(Q)\right)$, i.e. it is sufficient that

$$\begin{aligned} &a \in W_{3/(s-1)}^{s-1+\varepsilon,(s-1+\varepsilon)/2}(Q) \quad (\varepsilon > 0) && \text{for} \quad 2 \le s \le 5/2\,, \\ &a \in W_2^{s-1,(s-1)/2}(Q) && \text{for} \quad 5/2 < s \le 4\,. \end{aligned}$$

We assume that the coefficient $a(x,t)$ is decreasing function in variable t and $a(x,t) \ge c_0 > 0$.

Let ω be the uniform mesh in $\Omega = (0,1)$ with the step size h, $\bar\omega = \omega \cup \{0,1\} = \omega \cup \gamma$. Let θ_τ be the uniform mesh in $(0,T)$ with the step size τ, $\theta_\tau^+ = \theta_\tau \cup \{T\}$, $\bar\theta_\tau = \theta_\tau \cup \{0,T\}$. We define uniform mesh in Q: $Q_{h\tau} = \omega \times \theta_\tau$, $Q_{h\tau}^+ = \omega \times \theta_\tau^+$ and $\overline{Q}_{h\tau} = \bar\omega \times \bar\theta_\tau$, and we assume that the conditions

$$k_1 h^2 \le \tau \le k_2 h^2\,, \quad k_1, k_2 = \text{const} > 0\,,$$

are satisfied. Now, we define finite differences in the usual manner:

$$v_x = \frac{v^+ - v}{h} = v_{\bar{x}}^+ \,, \qquad v_{x\bar{x}} = \frac{v^+ - 2v + v^-}{h^2} \,,$$

where $v^{\pm}(x,t) = v(x \pm h, t)$, and

$$v_t(x,t) = \frac{v(x,t+\tau) - v(x,t)}{\tau} = v_{\bar{t}}(x, t+\tau) \,.$$

Also, we define the Steklov smoothing operators:

$$T_x^+ f(x,t) = \int_0^1 f(x + hx', t)\, dx' = T_x^- f(x+h, t) \,,$$

$$T_x^2 f(x,t) = T_x^+ T_x^- f(x,t) = \int_{-1}^1 (1 - |x'|) f(x + hx', t)\, dx' \,,$$

$$T_t^+ f(x,t) = \int_0^1 f(x, t + \tau t')\, dt' = T_t^- f(x, t+\tau) \,.$$

These operators commute and transform derivatives into differences:

$$T_x^2 \left(D_x^2 f(x,t) \right) = D_x^2 \left(T_x^2 f(x,t) \right) = f_{x\bar{x}}(x,t) \,,$$

$$T_t^- \left(D_t f(x,t) \right) = D_t \left(T_t^- f(x,t) \right) = f_{\bar{t}}(x,t) \,,$$

etc.

We approximate problem (2) with the following finite–difference scheme:

$$\begin{aligned} v_{\bar{t}} + L_h v &= T_x^2 T_t^- f && \text{in} \quad Q_{h\tau}^+ \,, \\ v &= 0 && \text{on} \quad \gamma \times \bar{\theta}_\tau \,, \\ v &= u_0 && \text{on} \quad \omega \times \{0\} \,, \end{aligned} \tag{3}$$

where

$$L_h v = -\frac{1}{2} \big((a v_x)_{\bar{x}} + (a v_{\bar{x}})_x \big) \,.$$

The finite–difference scheme (3) is the standard symmetric scheme with the averaged right–hand side. Note that for $s \leq 3.5$ the right–hand side may be discontinuous function, so scheme without averaging is not well defined.

3. Convergence of the Finite–Difference Scheme

Let u be the solution of initial-boundary value problem (2) and v - the solution of finite difference scheme (3). The error $z = u - v$ satisfies the conditions:

$$
\begin{aligned}
z_{\bar t} + L_h z &= \eta + \varphi \quad && \text{in} \quad Q^+_{h\tau}\,,\\
z &= 0 && \text{on} \quad \omega \times \{0\}\,,\\
z &= 0 && \text{on} \quad \gamma \times \bar\theta_\tau\,,
\end{aligned}
\tag{4}
$$

where

$$\eta = T_x^2 T_t^- \left(D_x(aD_x u)\right) - \frac{1}{2}\left((au_x)_{\bar x} + (au_{\bar x})_x\right)\,, \qquad \varphi = u_{\bar t} - T_x^2 u_{\bar t}\,.$$

We define discrete inner products

$$(v,w)_\omega = (v,w)_{L_2(\omega)} = h \sum_{x\in\omega} v(\cdot,t) w(\cdot,t)\,,$$

where $v(\cdot,t) = v(x,t)$, $(x,t) \in \omega \times \{t\}$, $t \in \theta^+_\tau$ is fixed,

$$(v,w)_{Q_{h\tau}} = (v,w)_{L_2(Q_{h\tau})} = h\tau \sum_{x\in\omega} \sum_{t\in\theta^+_\tau} v(x,t)w(x,t) = \tau \sum_{t\in\theta^+_\tau} (v,w)_\omega\,,$$

and discrete Sobolev norms

$$
\begin{aligned}
\|v\|^2_\omega &= (v,v)_\omega\,,\\
\|v\|^2_{Q_{h\tau}} &= (v,v)_{Q_{h\tau}}\,,\\
\|v\|^2_{W_2^{2,1}(Q_{h\tau})} &= \|v\|^2_{Q_{h\tau}} + \|v_x\|^2_{Q_{h\tau}} + \|v_{x\bar x}\|^2_{Q_{h\tau}} + \|v_{\bar t}\|^2_{Q_{h\tau}}\,.
\end{aligned}
$$

The following assertion holds true:

Lemma. *Finite–difference scheme* (4) *satisfies a priori estimate*

$$\|z\|_{W_2^{2,1}(Q_{h\tau})} \le \|\eta\|_{Q_{h\tau}} + \|\varphi\|_{Q_{h\tau}}\,. \tag{5}$$

Proof. Multiplying (4) by $L_h z = \frac{1}{2} L_h(z+\check z) + \frac{\tau}{2} L_h z_{\bar t}$, where $\check z = z(x, t-\tau)$ and summing through the nodes of ω we obtain:

$$
\begin{aligned}
\frac{1}{2\tau}\left(\|z\|^2_{L_h} - \|\check z\|^2_{L_h}\right) + \frac{\tau}{2}\|z_{\bar t}\|^2_{L_h} + \|L_h z\|^2_\omega &= (\eta+\varphi, L_h z)\\
&\le \frac{1}{2}\|\eta+\varphi\|^2_\omega + \frac{1}{2}\|L_h z\|^2_\omega\,,
\end{aligned}
$$

$$\|z\|^2_{L_h} - \|\check z\|^2_{L_h} + \tau^2\|z_{\bar t}\|^2_{L_h} + \tau\|L_h z\|^2_\omega \le \tau\|\eta+\varphi\|^2_\omega$$

and

$$\|z\|^2_{L_h} - \|\check{z}\|^2_{\check{L}_h} + \|\check{z}\|^2_{\check{L}_h} - \|\check{z}\|^2_{L_h} + \tau^2\|z_{\bar{t}}\|^2_{L_h} + \tau\|L_h z\|^2_\omega \le \tau\|\eta+\varphi\|^2_\omega\,,$$

where $\check{L}_h(t) = L_h(t-\tau)$. Recalling the condition that $a(x,t)$ is decreasing function in variable t we simply deduce that $\|\check{z}\|^2_{\check{L}_h} - \|\check{z}\|^2_{L_h} \ge 0$. We thus obtain

$$\|z\|^2_{L_h} - \|\check{z}\|^2_{\check{L}_h} + \tau\|L_h z\|^2_\omega \le \tau\|\eta+\varphi\|^2_\omega\,.$$

Summing through the nodes of θ^+_τ we obtain

$$\|z(T)\|^2_{L_h(T)} - \|z(0)\|^2_{L_h(0)} + \tau\sum_\tau^T \|L_h z\|^2_\omega \le \tau\sum_\tau^T \|\eta+\varphi\|^2_\omega\,.$$

Using the relations $\|z(T)\|^2_{L_h(T)} \ge 0$ and $\|z(0)\|^2_{L_h(0)} = 0$ we have

$$\tau\sum_\tau^T \|L_h z\|^2_\omega \le \tau\sum_\tau^T \|\eta+\varphi\|^2_\omega\,. \tag{6}$$

Using the relation $\|z_{\bar{t}}\| \le \|\eta+\varphi\| + \|L_h z\|$ we have

$$\tau\sum_\tau^T \|z_{\bar{t}}\|^2_\omega \le 4\tau\sum_\tau^T \|\eta+\varphi\|^2_\omega\,. \tag{7}$$

Finally, recalling well-known relations

$$\|L_h z\|_{Q_{h\tau}} \ge C\|z_{x\bar{x}}\|_{Q_{h\tau}}\,, \quad \|z\|_{Q_{h\tau}} \le C\|z_x\|_{Q_{h\tau}}\,, \quad \|z_x\|_{Q_{h\tau}} \le C\|z_{x\bar{x}}\|_{Q_{h\tau}}$$

and the relations (6) and (7) we simply obtain

$$\|z\|_{W_2^{2,1}(Q_{h\tau})} \le \|\eta\|_{Q_{h\tau}} + \|\varphi\|_{Q_{h\tau}}\,. \quad \square$$

In such a way, the problem of deriving the convergence rate estimate for finite–difference scheme (3) is now reduced to estimating the right–hand side terms in (5).

First of all, we decompose term η in the following way: $\eta = \sum_{k=1}^{7} \eta_k$, where

$$\begin{aligned}
\eta_1 &= T_x^2(aT_t^- D_x^2 u) - (T_x^2 T_t^- a)(T_x^2 T_t^- D_x^2 u)\,,\\
\eta_2 &= (T_x^2 T_t^- a - a)(T_x^2 T_t^- D_x^2 u)\,,\\
\eta_3 &= a(T_x^2 T_t^- D_x^2 u - u_{x\bar{x}})\,,\\
\eta_4 &= T_x^2 T_t^-(D_x a D_x u) - (T_x^2 T_t^- D_x a)(T_x^2 T_t^- D_x u)\,,\\
\eta_5 &= (T_x^2 T_t^- D_x a - 0.5(a_x + a_{\bar{x}}))(T_x^2 T_t^- D_x u)\,,\\
\eta_6 &= 0.5(a_x + a_{\bar{x}})(T_x^2 T_t^- D_x u - 0.5(u_x^- + u_{\bar{x}}^+))\,,\\
\eta_7 &= 0.25(a_x - a_{\bar{x}})(u_x^- - u_{\bar{x}}^+)\,.
\end{aligned}$$

Let us introduce the elementary rectangles

$$e = e(x,t) = \{(\xi,\nu):\ \xi \in (x-h, x+h)\,,\ \nu \in (t-\tau, t)\}.$$

The linear transformations $\xi = x + hx^*$, $\nu = t + \tau t^*$, defines a bijective mapping of the canonical rectangles $E = \{(x^*,t^*):\ |x^*| < 1\,,\ -1 < t^* < 0\}$ onto e. We define $u^*(x^*,t^*) = u(x + hx^*, t + \tau t^*)$ and so on.

The value of η_1 at a mesh point $(x,t) \in Q_{h\tau}^+$ can be expressed as

$$\begin{aligned}
\eta_1(x,t) = \frac{1}{h^2}\bigg\{ &\iint_E (1-|x^*|)a^*(x^*,t^*) D_x^2 u^*(x^*,t^*)\, dt^* dx^* \\
&- \iint_E (1-|x^*|)a^*(x^*,t^*)\, dt^* dx^* \\
&\times \iint_E (1-|x^*|) D_x^2 u^*(x^*,t^*)\, dt^* dx^* \bigg\}.
\end{aligned}$$

Then we deduce that η_1 is a bounded bilinear functional of the argument $(a^*, x^*) \in W_q^{\lambda,\lambda/2}(E) \times W_{2q/(q-2)}^{\mu,\mu/2}(E)$, where $\lambda \geq 0$, $\mu \geq 2$ and $q > 2$. Furthermore, $\eta_1 = 0$ whenever a^* is a constant function or u^* is a polynomial of degree two in x^* and degree one in t^*. Applying the bilinear version of the Bramble-Hilbert lemma we deduce that

$$|\eta_1(x,t)| \leq \frac{C}{h^2}|a^*|_{W_q^{\lambda,\lambda/2}(E)} |u^*|_{W_{2q/(q-2)}^{\mu,\mu/2}(E)}\,,$$
$$0 \leq \lambda \leq 1\,,\quad 2 \leq \mu \leq 3\,,\quad q > 2\,.$$

Returning from the canonical variables to the original variables we obtain:

$$|a^*|_{W_q^{\lambda,\lambda/2}(E)} \le Ch^{\lambda-\frac{3}{q}}|a|_{W_q^{\lambda,\lambda/2}(e)},$$

$$|u^*|_{W_{2q/(q-2)}^{\mu,\mu/2}(E)} \le Ch^{\mu-\frac{3(q-2)}{2q}}|u|_{W_{2q/(q-2)}^{\mu,\mu/2}(e)}.$$

Therefore,

$$|\eta_1(x,t)| \le Ch^{\lambda+\mu-\frac{7}{2}}|a|_{W_q^{\lambda,\lambda/2}(e)}|u|_{W_{2q/(q-2)}^{\mu,\mu/2}(e)},$$
$$0 \le \lambda \le 1\,, \quad 2 \le \mu \le 3\,, \quad q > 2\,.$$

Summing over the mesh $Q_{h\tau}^+$ we obtain

$$\|\eta_1\|_{Q_{h\tau}} \le Ch^{\lambda+\mu-2}\|a\|_{W_q^{\lambda,\lambda/2}(Q)}\|u\|_{W_{2q/(q-2)}^{\mu,\mu/2}(Q)},\tag{8}$$
$$0 \le \lambda \le 1\,, \quad 2 \le \mu \le 3\,.$$

Now suppose that $q > 2$ and $5/2 < s \le 4$. Then the following Sobolev imbeddings hold:

$$W_2^{\lambda+\mu,(\lambda+\mu)/2}(Q) \subset W_{2q/(q-2)}^{\mu,\mu/2}(Q) \quad \text{for} \quad \lambda \ge 3/q\,,$$
$$W_2^{\lambda+\mu-1,(\lambda+\mu-1)/2}(Q) \subset W_q^{\lambda,\lambda/2}(Q) \qquad \text{for} \quad \mu \ge 5/2 - 3/q\,.$$

Setting $\lambda + \mu = s$ in (8) yields:

$$\|\eta_1\|_{Q_{h\tau}} \le Ch^{s-2}\|a\|_{W_2^{s-1,(s-1)/2}(Q)}\|u\|_{W_2^{s,s/2}(Q)}, \quad 5/2 < s \le 4\,.\tag{9}$$

Similarly, for $2 \le s \le 5/2$ we have

$$|\eta_1(x,t)| \le Ch^{s-7/2}|a|_{W_{3/(s-1)}^{s-1+\varepsilon,(s-1+\varepsilon)/2}(e)}|u|_{W_2^{s,s/2}(e)}.$$

Summing over the mesh $Q_{h\tau}^+$ we obtain

$$\|\eta_1\|_{Q_{h\tau}} \le Ch^{s-2}\|a\|_{W_{3/(s-1)}^{s-1+\varepsilon,(s-1+\varepsilon)/2}(Q)}\|u\|_{W_2^{s,s/2}(Q)}, \quad 2 \le s \le 5/2\,.\tag{10}$$

By a similar argument, the term η_2 is a bounded bilinear functional of the argument $(a^*, x^*) \in W_q^{\lambda,\lambda/2}(E) \times W_{2q/(q-2)}^{\mu,\mu/2}(E)$, where $\lambda > 3/q$,$\mu >$

$3(q-2)/(2q)$ and $q>2$. Furthermore, $\eta_2=0$ whenever a^* is a polynomial of degree one in x^* or u^* is a polynomial of degree one in x^* . Applying the bilinear version of the Bramble-Hilbert lemma we deduce that

$$|\eta_2(x,t)| \le \frac{C}{h^2}|a^*|_{W_q^{\lambda,\lambda/2}(E)}|u^*|_{W_{2q/(q-2)}^{\mu,\mu/2}(E)}\,,$$

$$\frac{3}{q}<\lambda\le 2\,,\quad \frac{3(q-2)}{2q}<\mu\le 2\,.$$

Summing over the mesh $Q_{h\tau}^+$ we obtain

$$\|\eta_2\|_{Q_{h\tau}} \le Ch^{\lambda+\mu-2}\|a\|_{W_q^{\lambda,\lambda/2}(Q)}\|u\|_{W_{2q/(q-2)}^{\mu,\mu/2}(Q)}\,.$$

For $2\le s\le 5/2$, setting $q=3$, $\lambda=1+\varepsilon$, $\mu=s-(1+\varepsilon)$ and using the imbeddings

$$W_2^{\lambda+\mu,(\lambda+\mu)/2}(Q)\subset W_{2q/(q-2)}^{\mu,\mu/2}(Q) \quad \text{for} \quad \lambda\ge 3/q\,,$$

$$W_{3/(s-1)}^{\lambda+\mu-1+\varepsilon,(\lambda+\mu-1+\varepsilon)/2}(Q)\subset W_q^{\lambda,\lambda/2}(Q) \qquad \text{for} \quad \lambda\le 3/q+\varepsilon\,,$$

we obtain

(11) $$\|\eta_2\|_{Q_{h\tau}} \le Ch^{s-2}\|a\|_{W_{3/(s-1)}^{s-1+\varepsilon,(s-1+\varepsilon)/2}(Q)}\|u\|_{W_2^{s,s/2}(Q)}\,,\quad 2\le s\le 5/2\,.$$

For $5/2<s\le 4$, using the same technique as for η_1 we obtain

(12) $$\|\eta_2\|_{Q_{h\tau}} \le Ch^{s-2}\|a\|_{W_2^{s-1,(s-1)/2}(Q)}\|u\|_{W_2^{s,s/2}(Q)}\,,\quad 5/2<s\le 4\,.$$

The term η_3 is a bounded bilinear functional of the argument $(a^*,x^*)\in C(\overline{E})\times W_2^{s,s/2}(E)$, and $\eta_3=0$ whenever u^* is a polynomial of degree three in x^* and degree one in t^*. Recalling the imbeddings

$$W_{3/(s-1)}^{s-1+\varepsilon,(s-1+\varepsilon)/2}(Q)\subset C(\overline{Q}) \quad \text{for} \quad 2\le s\le 5/2\,,$$

$$W_2^{s-1,(s-1)/2}(Q)\subset C(\overline{Q}) \quad \text{for} \quad 5/2<s\le 4\,,$$

we obtain estimates of the form (9) and (10) for η_3 .

Using the same technique as before we obtain estimates of the form (9) and (10) for η_4 , η_5 , η_6 and η_7 .

Applying the linear version of the Bramble-Hilbert lemma we simply obtain estimates of the term φ :

(13) $$\|\varphi\|_{Q_{h\tau}} \le Ch^{s-2}\|u\|_{W_2^{s,s/2}(Q)}\,,\quad 2\le s\le 4\,.$$

Combining (5) with (9)–(13) we obtain the final result:

Theorem. *Finite–difference scheme* (3) *converges in the norm of the space* $W_2^{2,1}(Q_{h\tau})$ *and, with condition* $k_1 h^2 \le \tau \le k_2 h^2$*, the following estimates hold:*

$$\|u-v\|_{W_2^{2,1}(Q_{h\tau})} \le Ch^{s-2}\|a\|_{W_{3/(s-1)}^{s-1+\varepsilon,(s-1+\varepsilon)/2}(Q)}\|u\|_{W_2^{s,s/2}(Q)}, \quad 2 \le s \le 5/2\,,$$

$$\|u-v\|_{W_2^{2,1}(Q_{h\tau})} \le Ch^{s-2}\|a\|_{W_2^{s-1,(s-1)/2}(Q)}\|u\|_{W_2^{s,s/2}(Q)}, \qquad 5/2 < s \le 4\,.$$

These estimates are consistent with the smoothness of data.

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