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NUMERICAL SOLUTION OF IMPULSIVE DIFFERENTIAL EQUATIONS

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Dedicated to Prof. Radosav Ž. Đorđević for his 65th birthday

Abstract. In this paper an algorithm for solving impulsive differential equations is presented. The algorithm is based on well–known numerical methods and it gives numerical values of solution of impulsive differential equation. A new type of impulsive differential equations is presented, and numerical approach to their solving is given.

1. Introduction

In present literature regarded to impulsive differential equations solution was searched in form of analytical expression. Significantly results are presented by V. Lakshmikantham, D. Bainov, P. Simeonov, S. Kostadinov and N. van Minh ([1]-[10]). However, many impulsive differential equations can not be solved in this way or their solving is more complicated. From the other side, huge number of practical problems need not solution of impulsive differential equation in analytical form, but only need numerical values of solution. This is reason that impulsive differential equation can be solved numerically (see [12]). In this paper well–known numerical methods for solving ordinary differential equations are used (see [11]).

2. Impulsive Differential Equations

We denote set of real and set of integer numbers with $\mathbb{R}$ and $\mathbb{Z}$, respectively. Let it be $X = \mathbb{R}^n$ and $T = \{t_k \mid k \in \mathbb{Z}\} \subset \mathbb{R}$ where is $t_k < t_{k+1}$ for

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all $k \in \mathbb{Z}$, $t_k \to +\infty$ when $k \to +\infty$ and $t_k \to -\infty$ when $k \to -\infty$. Also, let $t_k^+ = t_k + 0$, and $t_k^- = t_k - 0$.

If $\Omega \subset \mathbb{R}$ is any real interval, we suppose that

$$\vec{x} \equiv \boldsymbol{x}(t) = [\, x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)\,]^T,$$

is vector of unknown functions, and

$$\boldsymbol{f}(t, \boldsymbol{x}) : \Omega \times X \mapsto X \ ,$$

$$\boldsymbol{f}(t, \boldsymbol{x}) = \begin{bmatrix} f_1(t, x_1(t), x_2(t), \ldots x_n(t)) \\ f_2(t, x_1(t), x_2(t), \ldots x_n(t)) \\ \vdots \\ f_n(t, x_1(t), x_2(t), \ldots x_n(t)) \end{bmatrix} ,$$

is continuous operator on every set $[t_k, t_{k+1}] \times X$.

Definition 2.1. A system of differential equations of the form

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t, \boldsymbol{x}) \qquad (t \neq t_k), \tag{2.1}$$

with conditions

$$\Delta \boldsymbol{x}|_{t=t_k} = \boldsymbol{x}(t_k^+) - \boldsymbol{x}(t_k^-) = I_k(\boldsymbol{x}(t_k)) \qquad (t = t_k), \tag{2.2}$$

where $I_k : X \mapsto X$ are continuous operators, $k = 0, \pm 1, \pm 2, \ldots$, is called impulsive differential equation (in further IDE).

A state of the process, simulated by IDE, at the moment $t = t_0$ is taken as a start condition

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0),$$

for solving differential equation (2.1).

Definition 2.2. Any set of functions $\varphi_i(t)$ $(i = 1, 2, \ldots, n)$ is said to be a solution of impulsive differential equation (2.1), (2.2) if for $t \in \mathbb{R} \setminus T$ satisfies the system of differential equations (2.1), and for $t \in T$ satisfies condition of the jump (2.2).

A problem of existence and uniqueness of the solutions of impulsive differential equations (2.1), (2.2) is reduced to that about corresponding ordinary differential equations (see [2]).

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t, \boldsymbol{x}).$$

Let $\boldsymbol{x}(t)$ be a solution of IDE (2.1), (2.2) which satisfies the start condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Also, let Ω^+ i Ω^- be maximal intervals on which the solution can be continued to the right, respectively to the left of the point $t = t_0$. Then next expression is valid (see [2]):

$$\boldsymbol{x}(t) = \begin{cases} \boldsymbol{x}_0 + \displaystyle\int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}(s))\, ds + \sum_{t_0 < t_k < t} I_k(\boldsymbol{x}(t_k)) & (t \in \Omega^+), \\ \boldsymbol{x}_0 + \displaystyle\int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}(s))\, ds - \sum_{t < t_k \le t_0} I_k(\boldsymbol{x}(t_k)) & (t \in \Omega^-). \end{cases} \tag{2.3}$$

3. Algorithm

We suppose that IDE (2.1), (2.2) with start condition $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ is given. Impulsive operators I_k, ($k \in \mathbb{Z}$) act at the moments t_k for all $k \in \mathbb{Z}$) and they can be described with the quadrat matrices of dimensions $n \times n$.

Now, we present an numerically algorithm for solving IDE, in further ASIDE (**A**lgorithm for **S**olving **I**mpulsive **D**ifferential **E**quations). If functions $x_1(t)$, $x_2(t)$, $\ldots$, $x_n(t)$ are solution of IDE (2.1), (2.2), it is possible to obtain values $x_1(t_z)$, $x_2(t_z)$, $\ldots$, $x_n(t_z)$ for fixed value t_z of parameter t by ASIDE. Without loss of generality, we suppose that $t_z > t_0$, i.e. $t_z \in \Omega^+$. Also, we denote index of iteration with j $(j = 0, 1, 2 \ldots)$.

ASIDE consists of following steps:

1° step: At the moment $t = t_0$, we set functions to the values that were given in start condition, i.e. $\boldsymbol{x} := \boldsymbol{x}_0$. We initialize the counter: $k := 0$.

2° step: Using numerical method (*NM*) we solve the functions of argument t taking it from halfsegment $(t_k, t_{k+1}]$, i.e. $\boldsymbol{x}^{[j+1]} := NM(\boldsymbol{x}^{[j]})$. We increase the counter: $k := k + 1$.

3° step: At the moment $t = t_k$ acts impulsive operator I_k, and brings rapidly changes (jumps) of functions $\boldsymbol{x}$ that amounts

$$\boldsymbol{J}(t_k) := I_k(\boldsymbol{x}\ (t_k)\ + \textstyle\sum_{t_0 < t_m < t_k} \boldsymbol{J}(t_m)).$$

4° step: We repeat steps 2°, and 3° while $t_{k+1} < t_z$.

5° step: Using numerical method (*NM*) we solve the functions of argument t taking it from half-segment $(t_k, t_z]$, i.e. $\boldsymbol{x}^{[j+1]} := NM(\boldsymbol{x}^{[j]})$.

6° step: We add to the functions $\boldsymbol{x}$ a sum of all jumps

$$\boldsymbol{x} := \boldsymbol{x} + \sum_{t_0 < t_k < t_z} \boldsymbol{J}(t_k)\,.$$

In 2° step we must choose m ($m \in \mathbb{N}$) nodes from the segment on which we solve, i.e. we must divide this segment to disjunctive subsegments

$$(t_k, t_{k+1}] = (t_k^{[0]}, t_k^{[1]}] \cup (t_k^{[1]}, t_k^{[2]}] \cup \ldots \cup (t_k^{[m-1]}, t_k^{[m]}]$$

where is $t_k^{[m]} = t_{k+1}$. We must do similarly in 5° step.

Using ASIDE algorithm we practically, do not reach analytical expressions for functions $x_1(t)$, $x_2(t)$, $\ldots$, $x_n(t)$, at all, but we solve approximative values of those functions for $t = t_z$.

For generating the sequence $\boldsymbol{x}^{[0]}, \boldsymbol{x}^{[1]}, \boldsymbol{x}^{[2]}, \ldots$ we choose some known numerical method (*NM*) for solving the system of differential equations (Euler's method, some multi–step method or some of the Runge-Kutta methods), which characteristics and accuracy are well known (see, for example [11]). We can choose the number of nodes and the sizes of subsegments on which we divide half-segments $(t_k, t_{k+1}]$. It is the most simple, from the aspect of programming, to choose equal number of the equidistant nodes on each half-segment. But, a way of choosing nodes can affect to accuracy of the result. A similar procedure can be applied in the case $t_z < t_0$, i.e. $t \in \Omega^-$. Only condition is that impulsive operators I_k for $k \le 0$ have their inverse operators I_k^{-1}. That is because the value of the argument t decreases, and at the moment $t = t_k$, when impulsive operator acts, an inverse mapping must be applied

$$\Delta \boldsymbol{x}|_{t=t_k} = I_k^{-1} \cdot \boldsymbol{x}(t_k).$$

4. Stroke–Impulsive Differential Equations

We considered another type of impulsive differential equations, where impulse does not act at the specific moment $t = t_k$, but at the moment when process, that is simulated by equation, comes to a specific state.

Definition 4.1. Stroke-impulsive differential equation (in further SIDE) is a system of differential equations of the form

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t, \boldsymbol{x}), \tag{4.1}$$

with jump-conditions

$$\Delta \boldsymbol{x}|_{u_k(\boldsymbol{x})=0} = I_k(\boldsymbol{x}_{u_k(\boldsymbol{x})=0}), \tag{4.2}$$

where $I_k : X \mapsto X$ are continuous operators, and $u_k(\boldsymbol{x}) = 0$, are some conditions ($k = 1, 2, \dots, m$).

A state of the process, simulated by SIDE, at the moment $t = t_0$ is taken as a start condition

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0),$$

for solving differential equation (4.1).

Definition 4.2. Any set of functions $\varphi_i(t)$ ($i = 1, 2, \dots, n$) is said to be a solution of stroke-impulsive differential equation (4.1), (4.2) if satisfies the system of differential equations (4.1), and satisfies condition of the jump (4.2) at the moments when conditions $u_k(\boldsymbol{x}) = 0$ ($k = 1, 2, \dots m$) are fulfilled.

Impulsive operators I_k, ($k \in \mathbb{Z}$), can be described with the quadrat matrices of dimensions $n \times n$.

We can use an numerically algorithm for solving SIDE, similarly with ASIDE, and we shall call it ASSIDE (**A**lgorithm for **S**olving **S**troke–**I**mpulsive **D**ifferential **E**quations). This way, we shall obtain values $x_1(t_z)$, $x_2(t_z)$, $\dots$, $x_n(t_z)$ for fixed value t_z of parameter t.

ASSIDE consists of following steps:

1° step: At the moment $t = t_0$, we set functions to the values that were given in start condition, i.e. $\boldsymbol{x} := \boldsymbol{x}_0$.

2° step: While $u_k(\boldsymbol{x}) \neq 0$ ($k = 1, 2, ..., m$) and $t < t_z$, using numerical method (NM) we solve the functions of argument $t = t_0, (h), t_z$, i.e. $\boldsymbol{x}^{[j+1]} := NM(\boldsymbol{x}^{[j]})$, where $\boldsymbol{x}^{[j]} = \boldsymbol{x}_{t=t_j}$, $t_j = t_0 + jh$.

3° step: If $t = t_z$ algorithm is finished.

4° step: Impulsive operator I_k acts and brings rapidly changes (jumps) of functions $\boldsymbol{x}$.

5° step: Increase $t := t + h$ and then go to step 2°.

In 2° step we must divide t-axis to disjunctive subsegments $(t_j, t_{j+1}]$. Using ASSIDE algorithm we do not reach analytical expressions for functions $x_1(t)$, $x_2(t)$, $\dots$, $x_n(t)$, but we solve approximative values of those functions for $t = t_z$.

We choose the numerical method (NM) on the same way, like in ASIDE.

5. Examples

Example 5.1. In this example, we solved an impulsive differential equation

$$\begin{aligned}
&\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t,\boldsymbol{x}) \qquad (t \neq t_k),\\
&\Delta\boldsymbol{x}|_{t=t_k} = I_k(\boldsymbol{x}(t_k)) \qquad (t = t_k),\\
&\boldsymbol{x}_0 = \boldsymbol{x}(t_0),
\end{aligned} \tag{5.1}$$

where $t_0 = 0.0$,

$$\boldsymbol{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \boldsymbol{x}_0 = \begin{bmatrix} -1.0 \\ 0.0 \end{bmatrix},$$

$$\boldsymbol{f}(t,\boldsymbol{x}) = \begin{bmatrix} 0.1666666x_1 + 0.1666666x_2 + 0.1666666 \\ -0.1666666x_1 - 0.1666666x_2 + 0.5833333 \end{bmatrix}$$

If impulsive operators

$$I_1 = \begin{bmatrix} 0.25 & 0.25 \\ 0.0 & -1.0 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 3.0 & 4.0 \\ 0.0 & -1.0 \end{bmatrix} \tag{5.2}$$

act at the moments $t_1 = 1.0$ and $t_2 = 2.0$, we found the states of the processes described by this IDE, at the moment $t_z = 2.5$.

We applied ASIDE algorithm using one of the simplest methods for numerical solving of differential equations, so called Euler's method,

$$\boldsymbol{x}^{[j+1]} = \boldsymbol{x}^{[j]} + h\boldsymbol{f}(t,\boldsymbol{x}^{[j]}), \tag{5.3}$$

where $j \in \mathbb{Z}$ is index of iteration, and h is a distance between a neighboring nodes. Then we applied ASIDE algorithm on the same IDE, but instead Euler's method (5.3) of the first order, we used Runge-Kutta method of the fourth order

$$\begin{aligned}
&\boldsymbol{x}^{[j+1]} - \boldsymbol{x}^{[j]} = \frac{h}{6}\cdot(\boldsymbol{K}_1 + 2\boldsymbol{K}_2 + 2\boldsymbol{K}_3 + \boldsymbol{K}_4)\\
&\boldsymbol{K}_1 = \boldsymbol{f}(t^{[j]},\boldsymbol{x}^{[j]})\\
&\boldsymbol{K}_2 = \boldsymbol{f}(t^{[j]} + \frac{h}{2},\boldsymbol{x}^{[j]} + \frac{h}{2}\boldsymbol{K}_1)\\
&\boldsymbol{K}_3 = \boldsymbol{f}(t^{[j]} + \frac{h}{2},\boldsymbol{x}^{[j]} + \frac{h}{2}\boldsymbol{K}_2)\\
&\boldsymbol{K}_4 = \boldsymbol{f}(t^{[j]} + h,\boldsymbol{x}^{[j]} + h\boldsymbol{K}_3),
\end{aligned} \tag{5.4}$$

Table 1

t_k	$x_1(t_k)$ (5.3)	$x_1(t_k)$ (5.4)	$x_1(t_k)$ (5.5)	$x_2(t_k)$ (5.3)	$x_2(t_k)$ (5.4)	$x_2(t_k)$ (5.5)
0.0	−1.0000000	−1.0000000	−1.0000000	0.0000000	0.0000000	0.0000000
0.1	−0.9993813	−0.9993738	−0.9993750	0.0743812	0.7445000	0.0743750
0.2	−0.9975126	−0.9974975	−0.9975000	0.1475125	0.1475749	0.1475000
0.3	−0.9943938	−0.9943712	−0.9943750	0.2193937	0.2194499	0.2193750
0.4	−0.9900250	−0.9899949	−0.9900000	0.2900250	0.2900748	0.2900000
0.5	−0.9844064	−0.9843749	−0.9843750	0.3594062	0.3593810	0.3593750
0.6	−0.9775375	−0.9774923	−0.9775000	0.4275375	0.4275746	0.4275000
0.7	−0.9694188	−0.9675436	−0.9693750	0.4944178	0.4978564	0.4942750
0.8	−0.9600500	−0.9599895	−0.9600000	0.5600500	0.5600746	0.5600000
0.9	−0.9494314	−0.9493632	−0.9493750	0.6244313	0.6244494	0.6243750
1.0^-	−0.9375375	−0.9374993	−0.9375000	0.6875625	0.6875117	0.6875000
1.0^+	−1.0000620	−0.9999962	−1.0000000	0.0000000	0.0000000	0.0000000
1.1	−0.9868438	−0.9868710	−0.9868750	0.0618812	0.0618760	0.0618750
1.2	−0.9725751	−0.9724958	−0.9725000	0.1225125	0.1225020	0.1225000
1.3	−0.9569563	−0.9568543	−0.9568750	0.1818938	0.1819367	0.1818750
1.4	−0.9467938	−0.9399778	−0.9448750	0.2388902	0.2400614	0.2467500
1.5	−0.9219688	−0.9218701	−0.9218750	0.2969062	0.2968798	0.2968750
1.6	−0.9026000	−0.9024948	−0.9025000	0.3525376	0.3525065	0.3525000
1.7	−0.8817688	−0.8818483	−0.8818750	0.4074562	0.4069353	0.4068750
1.8	−0.8601125	−0.8599717	−0.8600000	0.4600499	0.4600608	0.4600000
1.9	−0.8369938	−0.8368450	−0.8368750	0.5119312	0.5119346	0.5118750
2.0^-	−0.8126250	−0.8124933	−0.8125000	0.5625625	0.5625103	0.5625000
2.0^+	−1.0002500	−0.9999323	−1.0000000	0.0000000	0.0000000	0.0000000
2.1	−0.9999999	−0.9999073	−0.9997500	0.0005000	0.0004991	0.0004999
2.2	−0.9477624	−0.9474314	−0.9475000	0.0975126	0.0975018	0.0975000
2.3	−0.9474874	−0.9474039	−0.9472249	0.0979877	0.0975492	0.0979749
2.4	−0.8902748	−0.8899304	−0.8900000	0.1900249	0.1900034	0.1900000
2.5	−0.8596561	−0.8593048	−0.8593750	0.2344061	0.2343777	0.2343750
Δ	0.0002811	0.0000702	–	0.0000311	0.0000027	–

where $j \in \mathbb{Z}$ is index of iteration.

On the interval where we calculate, we take equidistant nodes with the step $h = 0.1$.

We compared obtained results with the results obtained using analytical expression that is solution of IDE (5.1)

$$
\boldsymbol{x} = \begin{cases} x_1(t) = 0.0625t^2 - 1.0, & \text{for } t \in (-\infty, 1), \\ x_2(t) = -0.0625t^2 + 0.75t, & \text{for } t \in (-\infty, 1), \\ \\ x_1(t) = 0.0625t^2 - 1.0625, & \text{for } t \in [1, 2), \\ x_2(t) = -0.0625t^2 + 0.75t - 0.6875, & \text{for } t \in [1, 2), \\ \\ x_1(t) = 0.0625t^2 - 1.25, & \text{for } t \in [2, \infty), \\ x_2(t) = -0.0625t^2 + 0.75t - 1.25. & \text{for } t \in [2, \infty). \end{cases} \tag{5.5}
$$

We implemented those methods on programming language FORTRAN-77 and the results that we obtained using ASIDE algorithm with numerical methods (5.3) and (5.4), as well as the results that we obtained using analytical expression (5.5), are given in the Table 1. Absolute errors of estimated results (Δ) are given too.

We can conclude that the accuracy of the results is better when we use a better numerical method. That fact shows us a possibility of increasing efficiency of ASIDE algorithm using more convenient numerical method (*NM*).

We can note that impulsive operators I_k, that act at the moments t_k, affect on error too. Namely, when operators I_k act, the values of the functions are approximative, and so we calculate approximative values of the jumps. ASIDE algorithm, constructed on this way, gives us possibilities for further improvement, in the sense of controlling influence of the impulsive operators to total error. That idea might be a basis for some further investigations.

Example 5.2. In this example, we consider a ball that is jumping on a flat horizontal surface (see Figure 1). Loss of energy, caused by the friction of surface, is characterized by constant μ.

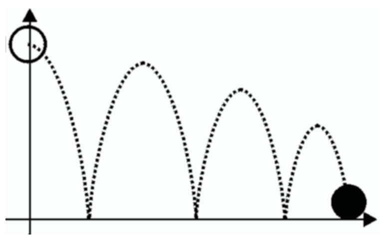

FIG. 1

This process is simulated by differential equation of second order

$$m\frac{d^2x}{dt^2} = F, \tag{5.6}$$

where m is a mass of the ball, $F = -mg$, is the force ($g \approx 9.81\frac{m}{s^2}$ is acceleration of Earths gravitation). Any time when ball touch the surface vertical component of the vector of velocity changes its sign. State of this process is described by vertical position, velocity and horizontal position of the ball (coordinates of the state). We shall compute approximative value of those tree coordinates at the moment $t_z = 0.5$. This process can be described with an SIDE

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -g, \quad \frac{dx_3}{dt} = v_0, \tag{5.7}$$

with start condition $\boldsymbol{x} = (x_1, x_2, x_3) = (h_0, 0, 0)$, $h_0 = 0.1\text{m}$, $v_0 = 1\text{m/s}$ and with condition of jump

$$I_k(x_1, x_2, x_3) = (x_1, -\mu \cdot x_2, x_3), \quad \text{for} \quad u(\boldsymbol{x}) = x_1 = 0. \tag{5.8}$$

Here is $\mu = 0.91$. In this model x_1 is vertical position, x_2 is velocity, x_3 is horizontal position of the ball.

Analytical expression of solution of SIDE (5.7), (5.8) is

$$\left.\begin{aligned} x_1(t) &= -4.905t^2 + 0.1, \\ x_2(t) &= -9.81t, \\ x_3(t) &= t, \end{aligned}\right\} \quad t \in [0.0000000, 0.1427843),$$

$$\left.\begin{aligned} x_1(t) &= -4.905t^2 + 2.075364t - 0.2819999\ , \\ x_2(t) &= -9.81t + 2.6753639, \\ x_3(t) &= t, \end{aligned}\right\} \quad t \in [0.1427843, 0.4026518),$$

$$\left.\begin{aligned} x_1(t) &= -4.905t^2 + 5.1099454t - 1.2622885, \\ x_2(t) &= -9.81t + 5.1099454, \\ x_3(t) &= t, \end{aligned}\right\} \quad t \in [0.4026518, 0.5000000).$$

We implemented ASSIDE algorithm (with Runge-Kutta method (5.4) with $h = 10^{-5}$) on programming language FORTRAN-77 and the results that we obtained, as well as the results that we obtained using analytical expression, are given in the Table 2. Absolute errors of estimated results (Δ) are given too.

Table 2

	$x_1(0.5)$	$\tilde{x}_1(0.5)$ ASSIDE	$x_2(0.5)$	$\tilde{x}_2(0.5)$ ASSIDE	$x_3(0.5)$	$\tilde{x}_3(0.5)$ ASSIDE
	0.0664342	0.0664035	0.2049454	0.2083093	0.5000000	0.4999907
Δ	-	0.000307	-	0.0033639	-	0.0000093

6. Further Possibilities

The examination of the problem in solving IDE and SIDE by numerical methods does not end by this algorithm. It is interesting, for example, to investigate the influence of nodes (i.e. the way of dividing the segments to the subsegments) to accuracy of results, because better accuracy can be obtained if nodes are more frequent on those parts of t-axe on which the state of the process change more rapidly. In connection with that, in presented algorithms we can use some numerical methods for solving differential equations which do not demand a set of equidistant nodes.

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