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ON THE GUARANTEED CONVERGENCE OF A FAMILY OF SIMULTANEOUS ITERATIVE METHODS*

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Dedicated to Professor Radosav Ž. Đorđević on his 65th Birthday

Abstract. The construction of initial conditions which guarantee the safe convergence of iterative processes is one of the most important problems in solving nonlinear equations. In this paper we give initial conditions for a one parameter family of square-root iteration methods for the simultaneous approximation of all simple zeros of a polynomial. This family, based on the Hansen-Patrick third order method, has also the cubic convergence and generates some new methods. The presented initial conditions are of practical importance since depend only of available data: coefficients of a polynomial and initial approximations to the wanted zeros of a polynomial.

1. Introduction

The choice of initial points that guarantees the convergence of a given iterative method is of a great importance in solving nonlinear equations. Smale's point estimation theory, first introduced in [21] for Newton's method, deals with the domain of convergence and initial conditions in solving an equation $f(z) = 0$ using only the information of f at the initial point $\mathbf{z}^{(0)}$. Smale's result was improved by X. Wang and Han [23]. Their work was later extended by Curry [5] and Kim [11] to some higher-order iterative methods and generalized by Chen [4]. Point estimation theorems concerning simultaneous methods for solving polynomial equations were stated by M. Petković and his coauthors in several recent papers and collected in the recent book [17], and by Zhao and D. Wang [22], [24].

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In this paper we consider monic algebraic polynomials of the form

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \quad (a_i \in \mathbb{C})$$

which have only simple zeros. Initial conditions in this case should be a function of the polynomial coefficients $\mathbf{a} = (a_0, \dots, a_{n-1})$, its degree n and initial approximations $z_1^{(0)}, \dots, z_n^{(0)}$ to the zeros $\zeta_1, \dots, \zeta_n$ of P. Throughout this paper we will always assume that the polynomial degree n is ≥ 3.

For $m = 0, 1, \dots$ let

$$d^{(m)} = \min_{j \neq i} |z_i^{(m)} - z_j^{(m)}|$$

be the minimal distance between approximations obtained in the mth iteration, and let

$$\begin{aligned} N_i^{(m)} &= \frac{P(z_i^{(m)})}{P'(z_i^{(m)})} \quad \text{(Newton's correction)}, \\ W_i^{(m)} &= \frac{P(z_i^{(m)})}{\prod\limits_{j \neq i}(z_i^{(m)} - z_j^{(m)})} \quad \text{(Weierstrass' correction)}, \\ w^{(m)} &= \max_{1 \leq j \leq n} |W_j^{(m)}|, \end{aligned}$$

where $i \in I_n := \{1, \dots, n\}$. According to the results presented recently in [13]–[20], [22], [24], it turned out that suitable initial conditions, providing a safe convergence of iterative methods for the simultaneous determination of polynomial zeros, are of the form of the inequality

$$w^{(0)} < c(n) d^{(0)}, \tag{1}$$

where $c(n)$ is the quantity which depends only on the polynomial degree n. The motivation and discussion about initial conditions of the form (1) have been given by Petković, Herceg and Ilić in [15].

The convergence theorem which provides very simple verification of the safe convergence of a rather wide class of iterative methods for the simultaneous determination of polynomial zeros under a given initial condition of the form (1), is presented in Section 2. This theorem is applied in Section 4 to a new one parameter family of simultaneous methods for finding simple complex zeros of a polynomial, proposed recently in [10] and given briefly in Section 3. For this family of methods we state practically applicable initial conditions of the form (1) which provide a safe convergence of this method. The established initial conditions depend only on the vector $\mathbf{z}^{(0)} = (z_1^{(0)}, \dots, z_n^{(0)})$ of starting approximations and the values of P in the components of $\mathbf{z}^{(0)}$.

For simplicity, in our analysis we will sometimes omit the iteration index m and new entries in the later $(m+1)$-st iteration will be additionally stressed by the symbol $\widehat{}$ (*hat*). For example, instead of $z_i^{(m)}, z_i^{(m+1)}, W_i^{(m)}, W_i^{(m+1)}, d^{(m)}, d^{(m+1)}, N_i^{(m)}, N_i^{(m+1)}$, etc. we will write $z_i, \hat{z}_i, W_i, \widehat{W}_i, d, \hat{d}, N_i, \hat{N}_i$. According to this we denote

$$w = \max_i |W_i|, \quad \hat{w} = \max_i |\widehat{W}_i|.$$

To provide some estimates, in this paper we use circular complex arithmetic. For this reason, we give some basic operations and properties of this arithmetic. For more details see Alefeld and Herzberger [2, Ch. V]).

A disk Z with center $c = \operatorname{mid} Z$ and radius $r = \operatorname{rad} Z$ will be denoted with $Z = \{c; r\} = \{z : |z - c| \le r\}$. If $0 \notin Z$ (that is, $|c| > r$), then the exact inverse of Z is given by

$$Z^{-1} = \left\{ \frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2} \right\} = \left\{ \frac{1}{z} : z \in Z \right\},$$

and the inverse of Z in the centered form by

$$Z^I := \left\{ \frac{1}{c}; \frac{r}{|c|(|c| - r)} \right\} \supseteq \left\{ \frac{1}{z} : z \in Z \right\}. \tag{2}$$

Furthermore, if $Z_k = \{c_k; r_k\}$ $(k = 1, 2)$, then

$$Z_1 \pm Z_2 := \{c_1 \pm c_2; r_1 + r_2\} = \{z_1 \pm z_2 : z_1 \in Z_1, z_2 \in Z_2\}.$$

The product $Z_1 \cdot Z_2$ is defined as in Gargantini and Henrici [8]:

$$Z_1 \cdot Z_2 := \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\} \supseteq \{z_1 z_2 : z_1 \in Z_1, z_2 \in Z_2\}.$$

Then

$$\frac{Z_1}{Z_2} = Z_1 \cdot Z_2^{-1} \quad \text{or} \quad \frac{Z_1}{Z_2} = Z_1 \cdot Z_2^I \quad (0 \notin Z_2).$$

If $F(Z) \supseteq \{f(z):\ z \in Z\}$ is a *circular interval extension* of a given closed complex function f over a disk Z, then

$$|\text{mid } Z| - \text{rad } Z \le |f(z)| \le |\text{mid } F(Z)| + \text{rad } F(Z) \quad \text{for all } z \in Z \tag{3}$$

and

$$z \in Z \;\Rightarrow\; f(z) \in F(Z) \tag{4}$$

holds.

The square root of a disk $\{c; r\}$ in the centered form, where $c = |c|e^{i\theta}$ and $|c| > r$, is defined as the union of two disks (see Gargantini [7]):

$$\{c; r\}^{1/2} = \Big\{\sqrt{|c|}e^{i\frac{\theta}{2}}; \rho\Big\} \bigcup \Big\{-\sqrt{|c|}e^{i\frac{\theta}{2}}; \rho\Big\}, \quad \rho = \sqrt{|c|} - \sqrt{|c| - r}. \tag{5}$$

2. Point Estimation Theorem

Most of iterative methods for the simultaneous approximation of simple zeros of a polynomial can be expressed in the form

$$z_i^{(m+1)} = z_i^{(m)} - C_i(z_1^{(m)}, \dots, z_n^{(m)}) \quad (i \in I_n;\ m = 0, 1, \dots), \tag{6}$$

where $z_1^{(m)}, \dots, z_n^{(m)}$ are some distinct approximations to simple zeros ζ_1, $\dots$, ζ_n respectively, obtained in the mth iterative step. In what follows the term

$$C_i^{(m)} = C_i(z_1^{(m)}, \dots, z_n^{(m)})$$

will be called the *iterative correction term* or simply *correction.*

Let $\gamma \in (0, 1)$ be a contraction factor and let

$$g(\gamma) = \frac{1 + \gamma - \gamma^2}{1 - \gamma}.$$

Using the idea presented by Petković, Carstensen and Trajković in [14], the following convergence theorem was established in [16].

Theorem 1. *Let C_i be the iterative correction term of the form $C_i(z) = P(z)/F(z)$ with $F(z) \neq 0$ for $z = \zeta_i$ and $z = z_i^{(m)}$ $(i \in I_n;\ m = 0, 1, \dots)$. If for each $i, j \in I_n$ and $m = 0, 1, \dots$ the initial condition* (1) *implies*

(i) $|C_i^{(m+1)}| < \gamma |C_i^{(m)}|$ $(\gamma < 1)$;

(ii) $|z_i^{(0)} - z_j^{(0)}| > g(\gamma)(|C_i^{(0)}| + |C_j^{(0)}|)$ $(i \neq j)$,

then the iterative process (6) *is convergent.*

We emphasize that the class of iterative methods considered in Theorem 1 is rather wide and includes most frequently used methods for the simultaneous determination of polynomial zeros.

3. Family of Simultaneous Zero-finding Methods

Let f be a function of z and let α be a fixed parameter. Hansen and Patrick have derived in [9] one parameter family of iterative functions for finding simple zeros of f in the form

$$\hat{z} = z - \frac{(\alpha+1)f(z)}{f'(z)\left(\alpha + \sqrt{1-(\alpha+1)\dfrac{f(z)}{f'(z)}\dfrac{f''(z)}{f'(z)}}\right)}. \tag{7}$$

Here z is a current approximation and $\hat{z}$ is a new approximation to the wanted zero. This family includes the Ostrowski ($\alpha = 0$), Euler ($\alpha = 1$), Laguerre ($\alpha = 1/(\nu - 1)$) and Halley's method ($\alpha = -1$) and, as a limiting case ($\alpha \to +\infty$), Newton's method. All the methods of the family (7) have cubic convergence to a simple zero except Newton's method which is quadratically convergent.

Let

$$G_{1,i} = \sum_{j\neq i} \frac{W_j}{z_i - z_j}, \quad G_{2,i} = \sum_{j\neq i} \frac{W_j}{(z_i - z_j)^2}.$$

Starting from Hansen-Patrick's formula (7) the following one parameter family of iterative methods for the simultaneous approximation of all simple zeros of a polynomial P has been derived in [19]:

$$\hat{z}_i = z_i - \frac{(\alpha+1)W_i}{(1+G_{1,i})\left(\alpha + \sqrt{1+\dfrac{2(\alpha+1)W_iG_{2,i}}{(1+G_{1,i})^2}}\right)} \quad (i \in I_n). \tag{8}$$

It has been proved in [19] that the order of convergence of the iterative methods of the family (8) is equal to four for any fixed and finite parameter α. This family of methods provides 1) simultaneous determination of all zeros of a given polynomial and 2) the acceleration of the order of convergence from *three* to *four*. A number of numerical experiments showed that the proposed methods possess very good convergence properties.

An another one parameter family which is also based on Hansen-Patrick's formula (7) has been derived in the recent paper [10]. If $f \equiv P$ is an algebraic polynomial and approximations $z_1, \dots, z_n$ are close enough to the zeros $\zeta_1, \dots, \zeta_n$ of P, then substituting the approximation

$$\frac{P''(z_i)}{P'(z_i)} \cong 2\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{z_i - z_j}$$

in (7) we obtain the following family of iteration method:

$$\hat{z}_i = z_i - \frac{(\alpha+1)N_i}{\alpha + \sqrt{1 - 2(\alpha+1)N_i \sum_{j\neq i}(z_i - z_j)^{-1}}} \quad (i \in I_n). \tag{9}$$

The family (9) has a simple form compared to the fourth order family (8), but only a cubic convergence. Thus, the class of methods (8) is not faster than the basic Hansen-Patrick method (7).

We present some special cases of the iterative formula (9):

$\alpha = 0$, *Ostrowski-like method:*

$$\hat{z}_i = z_i - \frac{N_i}{\sqrt{1 - 2N_i \sum_{j\neq i}(z_i - z_j)^{-1}}} \quad (i \in I_n);$$

$\alpha = 1$, *Euler-like method:*

$$\hat{z}_i = z_i - \frac{2N_i}{1 + \sqrt{1 - 4N_i \sum_{j\neq i}(z_i - z_j)^{-1}}} \quad (i \in I_n);$$

$\alpha = 1/(n-1)$, *Laguerre-like method:*

$$\hat{z}_i = z_i - \frac{nN_i}{1 + \sqrt{(n-1)^2 - n(n-1)N_i \sum_{j\neq i}(z_i - z_j)^{-1}}} \quad (i \in I_n);$$

$\alpha = -1$, *Halley-like method:*

$$\hat{z}_i = z_i - \frac{N_i}{1 - N_i \sum_{j\neq i}(z_i - z_j)^{-1}} \quad (i \in I_n). \tag{10}$$

The last formula was considered for the first time by Maehly [12] and Börsch-Supan [3], but a practical application and an analysis were presented by Ehrlich [6] and Aberth [1] so that this method is most frequently called the Ehrlich-Aberth method.

For simplicity, we will introduce the abbreviation

$$t_i = 2(\alpha+1)N_i \sum_{j\neq i}(z_i - z_j)^{-1}$$

and consider the iteration formula (9) in the simpler form

$$\hat{z}_i = z_i - \frac{(\alpha+1)N_i}{\alpha + \sqrt{1 - t_i}} \quad (i \in I_n). \tag{11}$$

4. Initial Conditions and Safe Convergence

In this section we apply Theorem 1 and an initial condition of the form (1) to state the convergence theorem for the one parameter family (11) of simultaneous methods for finding polynomial zeros. Before establishing the main results, we give two necessary lemmas. In our consideration we will restrict the range of values of the parameter α appearing in the iterative formula (11) to the disk $\{-1; 5.5\}$, that is, we will always assume that

$$|\alpha + 1| < 5.5 \tag{12}$$

holds. This requirement will be explained later. We note that this range of α provides that all the above mentioned methods be defined. Moreover, in the similar way as in [19] it can be proved for large $|\alpha|$ the convergence of the square-root iteration methods (11) is slower and with growing $|\alpha|$ it becomes only quadratic. The convergence analysis and numerical examples show that the optimal behavior of the family (11) appears for α close to -1 (the Halley-like or Ehrlich-Aberth method (10), see Remark 2). For these reasons, the condition (12) can be regarded as a slight restriction only.

Lemma 1. *Assume that the following condition*

$$w < \frac{d}{13n} \tag{13}$$

is satisfied. Then each disk $\{z_i; \frac{13}{12}|W_i|\}$ $(i \in I_n)$ *contains one and only one zero of* P.

The above result follows according [17, Corollary 1.1].

Using the Lagrange interpolation of P at $z_1, \ldots, z_n$ we represent P in terms of W_j's in the form

$$P(t) = \left(\sum_{j=1}^{n} \frac{W_j}{t - z_j} + 1\right) \prod_{j=1}^{n} (t - z_j). \tag{14}$$

Hence, by applying the logarithmic derivative to (14) and putting $t = z_i$ in the obtained formula we find

$$\frac{P'(z_i)}{P(z_i)} = \sum_{j \neq i} \frac{1}{z_i - z_j} + \frac{1}{W_i}\Big(\sum_{j \neq i} \frac{W_j}{z_i - z_j} + 1\Big). \tag{15}$$

The relations (14) and (15) will be used in what follows.

Lemma 2. *Let $z_1, \dots, z_n$ be distinct approximations to the zeros $\zeta_1, \dots, \zeta_n$ of a polynomial P of degree $n \geq 3$, and let $\hat{z}_1, \dots, \hat{z}_n$ be new respective approximations obtained by the family* (11). *If the inequalities* (12) *and* (13) *hold, then for $i, j \in I_n$ we have*

(i) $\quad \dfrac{13}{15}|W_i| < |N_i| < \dfrac{13}{11}|W_i| < \dfrac{d}{11n}$;

(ii) $\quad |t_i| < \dfrac{2|\alpha+1|}{11}$;

(iii) $\quad \left|\dfrac{\alpha+1}{\alpha+\sqrt{1-t_i}}\right| < \dfrac{11}{9}$;

(iv) $\quad \left|\dfrac{\alpha+\sqrt{1-t_i}}{\alpha+1}\right| < \dfrac{13}{11}$.

Proof. Let

$$V_i = W_i \sum_{j \neq i} \frac{1}{z_i - z_j} + \sum_{j \neq i} \frac{W_j}{z_i - z_j} + 1.$$

Using (13) and the triangle inequality, we find

$$\begin{aligned} |V_i| &> 1 - |W_i| \sum_{j \neq i} \frac{1}{|z_i - z_j|} - \sum_{j \neq i} \frac{|W_j|}{|z_i - z_j|} \\ &> 1 - \frac{2(n-1)w}{d} > 1 - \frac{2(n-1)}{13n} > \frac{11}{13} \end{aligned} \tag{16}$$

and, in the similar way,

$$|V_i| < 1 + |W_i| \sum_{j \neq i} \frac{1}{|z_i - z_j|} + \sum_{j \neq i} \frac{|W_j|}{|z_i - z_j|} < \frac{15}{13}. \tag{17}$$

Now, using the identity (15) we find

$$|N_i| = \left|\frac{P(z_i)}{P'(z_i)}\right| = \frac{|W_i|}{|V_i|},$$

wherefrom, applying the inequalities (16) and (17), there follows

$$\frac{13}{15}|W_i| < |N_i| < \frac{13}{11}|W_i| < \frac{d}{11n} \tag{18}$$

and the assertion (i) is proved.

To prove (ii) we use (i) and the definition of d, and find

$$|t_i| \le 2|\alpha+1||N_i| \sum_{j\neq i} \frac{1}{|z_i - z_j|} < 2|\alpha+1| \frac{d}{11n} \cdot \frac{n-1}{d} < \frac{2|\alpha+1|}{11}.$$

According to (ii) we have

$$t_i \in \Big\{0; \frac{2|\alpha+1|}{11}\Big\} = \{0; t_0\},$$

where $t_0 = \dfrac{2|\alpha+1|}{11} < 1$ (in view of (12)).

As usual in an analysis of local convergence, we assume that approximations are reasonably close to the zeros, that is, $|t_i|$ in (11) is sufficiently small. We need only $|t_i| < 1$, which reduces to the necessary restriction (12). We emphasize that this restriction is not connected with the applied tools for the convergence analysis but the sequences of not so perfectly convergence properties of the family (11).

Using (4) and (5) we find

$$\begin{aligned} \sqrt{1-t_i} &\in (1 - \{0; t_0\})^{1/2} = (\{1; t_0\})^{1/2} = \{1; 1 - (1-t_0)^{1/2}\} \\ &= \Big\{1; \frac{t_0}{1 + (1-t_0)^{1/2}}\Big\} \subset \{1; t_0\} = \Big\{1; \frac{2|\alpha+1|}{11}\Big\} \end{aligned} \tag{19}$$

so that, by virtue of (2),

$$\frac{\alpha+1}{\alpha+\sqrt{1-t_i}} \in \frac{\alpha+1}{\alpha + \Big\{1; \dfrac{2|\alpha+1|}{11}\Big\}} = \frac{1}{\{1; \frac{2}{11}\}} = \Big\{1; \frac{2}{9}\Big\}. \tag{20}$$

Similarly, using (19) we obtain

$$\frac{\alpha+\sqrt{1-t_i}}{\alpha+1} \in \frac{\alpha + \Big\{1; \dfrac{2|\alpha+1|}{11}\Big\}}{\alpha+1} = \Big\{1; \frac{2}{11}\Big\}. \tag{21}$$

Now, applying (3), from (20) and (21) one estimates

$$\Big|\frac{\alpha+1}{\alpha+\sqrt{1-t_i}}\Big| < 1 + \frac{2}{9} = \frac{11}{9}, \tag{22}$$

$$\Big|\frac{\alpha+\sqrt{1-t_i}}{\alpha+1}\Big| < 1 + \frac{2}{11} = \frac{13}{11}. \tag{23}$$

Lemma 3. *Under the conditions* (12) *and* (13), *the following is valid:*

(*i*) $|\widehat{W}_i| < 0.47|W_i|$;

(*ii*) $\widehat{W} < \dfrac{\hat{d}}{13n}$.

Proof. From the iteration formula (11) and the inequalities (18) and (22), we obtain

$$|\hat{z}_i - z_i| = |N_i| \left| \frac{\alpha+1}{\alpha+\sqrt{1-t_i}} \right| < \frac{d}{11n} \cdot \frac{11}{9} = \frac{d}{9n}. \tag{24}$$

By (24) it follows

$$|\hat{z}_i - z_j| \geq |z_i - z_j| - |\hat{z}_i - z_i| > d - \frac{d}{9n} = \frac{9n-1}{9n} \tag{25}$$

and

$$|\hat{z}_i - \hat{z}_j| \geq |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| > d - 2 \cdot \frac{d}{9n} = \frac{9n-2}{9n} d. \tag{26}$$

From the last inequality we have

$$\hat{d} > \frac{9n-2}{9n} d, \quad \text{that is,} \quad d < \frac{9n}{9n-2} \hat{d}. \tag{27}$$

Putting $t = \hat{z}_i$ in (14), we obtain

$$P(\hat{z}_i) = (\hat{z}_i - z_i) u_i \prod_{j \neq i} (\hat{z}_i - z_j), \tag{28}$$

where

$$u_i = \frac{W_i}{\hat{z}_i - z_i} + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j} + 1.$$

From the iteration formula (11) and the identity (15) we obtain

$$\begin{aligned}
\frac{W_i}{\hat{z}_i - z_i} &= -W_i \cdot \frac{\alpha + \sqrt{1-t_i}}{(\alpha+1)N_i} \\
&= -W_i \cdot \frac{\alpha + \sqrt{1-t_i}}{\alpha+1} \left(\sum_{j \neq i} \frac{1}{z_i - z_j} + \frac{1}{W_i} \Big(\sum_{j \neq i} \frac{W_j}{z_i - z_j} + 1 \Big) \right) \\
&= -\frac{\alpha + \sqrt{1-t_i}}{\alpha+1} V_i,
\end{aligned}$$

where V_i is given at the beginning of the proof of Lemma 2. According to this and (21) there follows

$$\begin{aligned} u_i &= -\frac{\alpha+\sqrt{1-t_i}}{\alpha+1}V_i + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} + 1 \\ &\in -\Big\{1; \frac{2}{11}\Big\}V_i + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} + 1 = \{\eta; R\}, \end{aligned}$$

where

$$\begin{aligned} \eta &= -W_i\sum_{j\neq i}\frac{1}{z_i - z_j} - \sum_{j\neq i}\frac{W_j}{z_i - z_j} - 1 + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} + 1 \\ &= -W_i\sum_{j\neq i}\frac{1}{z_i - z_j} - (\hat{z}_i - z_i)\sum_{j\neq i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)}, \end{aligned}$$

and

$$R = \frac{2}{11}\left|W_i\sum_{j\neq i}\frac{1}{z_i - z_j} + \sum_{j\neq i}\frac{W_j}{z_i - z_j} + 1\right| = \frac{2}{11}|V_i|.$$

Now, using (13), (24) and (25), and the definition of d, we estimate

$$\begin{aligned} |\eta| &\le |W_i|\sum_{j\neq i}\frac{1}{|z_i - z_j|} + |\hat{z}_i - z_i|\sum_{j\neq i}\frac{|W_j|}{|\hat{z}_i - z_j||z_i - z_j|} \\ &< \frac{(n-1)w}{d} + \frac{d}{9n}\cdot\frac{(n-1)w}{\frac{9n-1}{9n}d\cdot d} = \frac{n-1}{13n} + \frac{n-1}{13n(9n-1)} < \frac{1}{13} \end{aligned}$$

and, by (17),

$$R < \frac{2}{11}|V_i| < \frac{2}{11}\cdot\frac{15}{13} = \frac{30}{143}.$$

Since $u_i \in \{\eta; R\}$, using the above bounds and (3), we find

$$|u_i| < |\eta| + R < \frac{1}{13} + \frac{30}{143} = \frac{41}{143}. \tag{29}$$

Dividing (28) with $\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)$ we obtain

$$\widehat{W}_i = \frac{P(\hat{z}_i)}{\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)} = (\hat{z}_i - z_i)u_i\prod_{j\neq i}\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}. \tag{30}$$

Using the bounds (24) and (26), we get

$$
\begin{aligned}
(31)\qquad \left|\prod_{j\neq i}\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| &\le \prod_{j\neq i}\left(1 + \frac{|\hat{z}_j - z_j|}{|\hat{z}_i - \hat{z}_j|}\right) < \prod_{j\neq i}\left(1 + \frac{\frac{1}{9n}d}{\frac{(9n-2)}{9n}d}\right)\\
&= \left(1 + \frac{1}{9n-2}\right)^{n-1} < \exp(1/9) \cong 1.1331.
\end{aligned}
$$

According to (18) and (22) we estimate

$$
(32)\qquad |\hat{z}_i - z_i| = |N_i|\left|\frac{\alpha+1}{\alpha+\sqrt{1-t_i}}\right| < \frac{13}{11}|W_i|\cdot\frac{11}{9} = \frac{13}{9}|W_i|.
$$

Using (29), (31) and (32), from (30) we find

$$
|\widehat{W}_i| = |\hat{z}_i - z_i||u_i|\left|\prod_{j\neq i}\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| < \frac{13}{9}|W_i|\cdot\frac{41}{143}\exp(1/9) \cong 0.4627|W_i|,
$$

so that

$$
(33)\qquad |\widehat{W}_i| < 0.47|W_i|,
$$

which proves (i) of Lemma 3.

With regard to the last inequality, (13) and (27) it follows

$$
\hat{w} < 0.47w < 0.47\cdot\frac{d}{13n} < \frac{0.47}{13n}\cdot\frac{9n}{9n-2}\hat{d} < \frac{\hat{d}}{13n}.
$$

Therefore, we prove the implication

$$
(34)\qquad w < \frac{d}{13n} \quad\Longrightarrow\quad \hat{w} < \frac{\hat{d}}{13n},
$$

which means that the assertion (ii) of Lemma 3 is valid.

This completes the proof of Lemma 3. □

Using Lemmas 2 and 3, and Theorem 1, we state the convergence theorem for the family of square-root iteration methods (11).

Theorem 2. *The family of iterative methods*

$$z_i^{(m+1)} = z_i^{(m)} - \frac{(\alpha+1)N_i^{(m)}}{\alpha + \sqrt{1 - t_i^{(m)}}} \quad (i \in I_n) \tag{35}$$

with the parameter $\alpha \in \{-1; 5.5\}$ *is convergent under the condition*

$$w^{(0)} < \frac{d^{(0)}}{13n}. \tag{36}$$

Proof. The correction $C_i^{(m)}$ appearing in (35) is given by

$$C_i = \frac{(\alpha+1)N_i^{(m)}}{\alpha + \sqrt{1 - t_i^{(m)}}} \quad (i \in I_n). \tag{37}$$

This correction has the required form $C_i = P(z_i)/F(z_i)$ (Theorem 1) with

$$F(z_i) = \frac{P'(z_i)\Big(\alpha + \sqrt{1 - t_i}\Big)}{\alpha + 1}.$$

We note that $F(z_i) \neq 0$. Indeed, in view of (21) we have

$$0 \notin \{1; 2/11\} \ni (\alpha + \sqrt{1 - t_i})/(\alpha + 1),$$

while by (i) of Lemma 2 it follows $|P'(z_i)| > 11n|P(z_i)|/d > 0$ for $z_i \neq \zeta_i$ and $|P'(\zeta_i)| \neq 0$ since ζ_i is a simple zero of P.

We will show now that the sequences $\big\{|C_i^{(m)}|\big\}$ $(i = 1, \dots, n)$ are monotonically decreasing. From (32) we immediate find

$$|C_i| = |\hat{z}_i - z_i| < \frac{13}{11}|W_i|. \tag{38}$$

In the proof of Lemma 3 we derived the implication (34).

$$w < \frac{w}{13n} \quad \Longrightarrow \quad \hat{w} < \frac{\hat{d}}{13n}.$$

Using the same argumentation and similar procedure, we prove by induction the following implication

$$w^{(0)} < \frac{d^{(0)}}{13n} \quad \Longrightarrow \quad w^{(m)} < \frac{d^{(m)}}{13n} \tag{39}$$

for each $m = 1, 2, \dots$. This means that all previously proved estimates hold for each index $m = 1, 2, \dots$. In particular, the assertion (i) of Lemma 3 is valid, that is

$$|W_i^{(m+1)}| < 0.47|W_i^{(m)}| \quad (i \in I_n;\ m = 0, 1, \dots). \tag{40}$$

According to this and the inequalities (33) and (38), we obtain

$$|\widehat{C}_i| < \frac{13}{11}|\widehat{W}_i| < \frac{13}{11} \cdot 0.47|W_i| < 0.56|W_i| = 0.56\Big|\frac{W_i}{\hat{z}_i - z_i}\Big||C_i| \tag{41}$$

since $|\hat{z}_i - z_i| = |C_i|$. By virtue of (i) of Lemma 2 we have $|W_i|/|N_i| < 15/13$, so that, by (23), we find

$$\Big|\frac{W_i}{\hat{z}_i - z_i}\Big| = \frac{|W_i|}{|N_i|}\Big|\frac{\alpha + \sqrt{1 - t_i}}{\alpha + 1}\Big| < \frac{15}{13} \cdot \frac{13}{11} = \frac{15}{11}.$$

Using the last bound, from (41) we obtain

$$|\widehat{C}_i| < 0.56 \cdot \frac{15}{11}|C_i| < 0.8|C_i|. \tag{42}$$

Hence, by induction and (39) we prove the inequality

$$|C_i^{(m+1)}| < 0.8|C_i^{(m)}| \ (i = 1, \dots, n;\ m = 0, 1, \dots),$$

which points to the monotonicity of the sequences $\{|C_i^{(m)}|\}$. Following Theorem 1 the contraction factor in (42) is $\gamma = 0.8$ and we calculate the constant $g(0.8) = 5.8$ appearing in (ii) of Theorem 1.

With the constant $g(0.8) = 5.8$ we should prove that the inclusion disks

$$S_1 = \{z_1^{(0)}; 5.8|C_1^{(0)}|\}, \dots, S_n = \{z_n^{(0)}; 5.8|C_n^{(0)}|\}$$

are disjoint (assertion (ii) of Theorem 1). In regard to (38) we have $|C_i^{(0)}| < 13w^{(0)}/11$, whence, by (36),

$$d^{(0)} > 13nw^{(0)} > 13n \cdot \frac{11}{13}|C_i^{(0)}| = 11n|C_i^{(0)}| \ \text{ for each } i \in I_n.$$

Using this bound we obtain for arbitrary pair of indices i, j $(i \neq j)$

$$|z_i^{(0)} - z_j^{(0)}| \geq d^{(0)} > 11n|C_i^{(0)}| \geq 5.5n\big(|C_i^{(0)}| + |C_j^{(0)}|\big)$$
$$> 5.8\big(|C_i^{(0)}| + |C_j^{(0)}|\big) = \text{rad } S_i + \text{rad } S_j.$$

Therefore, the inclusion disks $S_1, \ldots, S_n$ are disjoint, which completes the proof of Theorem 2. $\square$

Remark 1. As shown in [20], the fourth order family (8) converges under the weaker condition $w^{(0)} < d/(3n+3)$ in reference to (35). This means that the family (8) possesses better convergence properties than the family (11). The fast convergence of (8) is provided due to the additional term under the square root in (8) for which we have $2(\alpha+1)W_i G_{2,i} = O(|z_i - \zeta_i|^2)$, while for the corresponding term t_i in (11) we have only $t_i = O(|z_i - \zeta_i|)$.

Remark 2. It was proved in [18] that the Halley-like (Ehrlich-Aberth) method (10), appearing as a special case of the family (11), converges under considerably weaker condition $w^{(0)} < d^{(0)}/(2n+3)$ compared to (36). According to this and a number of numerical examples, it turns out that the parameter α in (11) should be chosen close to -1. For this reason, we may say that the condition (12) is not strong restriction for α.

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