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COISOTROPIC SUBMANIFOLDS OF PSEUDO–FINSLER MANIFOLDS

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Dedicated to Prof. Radosav Ž. Đorđević for his 65th birthday

Abstract. We construct the transversal vector bundle of a coisotropic submanifolds of pseudo–Finsler manifold and obtain all structure equations of the degenerate immersion.

Introduction

The theory of Finsler manifolds is one of the most difficult theories in Finsler geometry. This is so because, in general, the induced Finsler connection does not inherit all the properties of Finsler connection on the ambient Finsler manifold. Despite of such difficulties, in case the metric of the enveloping Finsler manifold is positive definite, several important results have been obtained, some of them being brought together in separate chapters of monographs of Rund ([5]) and Bejancu ([1]).

Now, we consider a pseudo–Finsler manifold $F^{(m)} = (\tilde{M}, \tilde{F}, \tilde{g})$, where $\tilde{M}$ is a real m–dimensional manifold, and $\tilde{g}$ is a pseudo–Finsler metric on $\tilde{M}$ constructed in a usual way, by using the fundamental function $\tilde{F}$. Suppose M is an n-dimensional submanifold of $\tilde{M}$. Then the induced Finsler metric g on M might be non–degenerate or degenerate on M, or on subsets of M. In case g is non–degenerate on M we take the complementary orthogonal vector bundle $VTM^{\perp}$ of the vertical vector bundle VTM in $VT\tilde{M}_{|TM}$ and study the geometry og M based on the decomposition ([1], Ch.2)

$$VT\tilde{M}_{|TM} = VTM \ \perp \ VTM^{\perp}.$$

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In the present paper we suppose M is a coisotropic submanifold of $F^{(m)}$, that is, $VTM^{\perp}$ is a vector subbundle of VTM, and therefore the above theory of non–degenerate submanifolds does not apply anymore. We overcome this difficulty by considering a screen vector bundle that is a complementary vector bundle of $VTM^{\perp}$ in VTM. Then we construct the transversal vector bundle and obtain the induced geometric objects, the induced non–linear connection, second fundamental form, shape operator, the induced Finsler connection, etc. It is noteworthy that the local components of the second fundamental form do not depend on the screen vector bundle. Finally, we obtain all the structure equations of the degenerate immersion of M in $F(m)$.

1. The Transversal Vector Bundle of a Coisotropic Submanifold of a Psedo–Finsler Manifolds

Let $F^{(m)} = (\tilde{M}, \tilde{F})$ be a Finsled manifold, where $\tilde{M}$ is a real m-dimensional manifold and $\tilde{F}$ is the fundamental function of $F^{(m)}$ (see [4], p.101). In general, $\tilde{F}$ is not presumed to be smooth on the whole $T\tilde{M}$ but on an open subset of $T\tilde{M}$.

Throughout the paper we use the following range for indices:

$$i, j, k, \ldots \in \{1, \ldots, m\};\ \alpha, \beta, \gamma, \ldots \in \{1, \ldots, n\};\ a, b, c, \ldots \in \{n+1, \ldots, m\},$$

where $n < m$. Also we make use of Einstein convention, i.e. repeated idices with one upper index and one lower index denote summation over their range. By $F(S)$ and $\Gamma(E)$ we denote the algebra of smooth functions on a manifold S and the $F(S)$-module of smooth sections of a vector bundle E over S, respectively.

Denote by $(x^1, \ldots, x^m; y^1, \ldots, y^m) = (x, y)$ the local coordinates on $T\tilde{M}$ and define

$$\tilde{g}_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 \tilde{F}^2}{\partial y^i \partial y^j};\quad i, j \in \{1, \ldots, m\}.$$

Consider the vertical vector bundle $VT\tilde{M}$ of $\tilde{M}$ and define

$$\tilde{g} : \Gamma(VT\tilde{M}) \times \Gamma(VT\tilde{M}) \mapsto F(T\tilde{M})$$

by

$$\tilde{g}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \tilde{g}_{ij}(x, y),$$

where $\{\partial/\partial y^1, \ldots, \partial/\partial y^m\}$ is a local basis of $\Gamma(VT\tilde{M})$. Suppose $\tilde{g}$ is non-degenerate on $T\tilde{M}$, that is, $\operatorname{rank}[\tilde{g}_{ij}(x, y)] = m$ on any coordinate neighborhood of $T\tilde{M}$. Clearly, at any point (x, y) of $T\tilde{M}$, $\tilde{g}(x, y)$ is a pseudo-Euclidian metric on the fibre $VT\tilde{M}_{(x,y)}$. Denote by q the *index* of $\tilde{g}(x,y)$, i.e., q is the dimension of the largest subspace of $VT\tilde{M}_{(x,y)}$ on which $\tilde{g}(x,y)$ is negative definite. We further suppose $\tilde{g}$ is of constant index q on $T\tilde{M}$. In this case $\tilde{g}$ is said to be a *pseudo-Finsler metric* on $\tilde{M}$ and $F^{(m)} = (\tilde{M}, \tilde{F}, \tilde{g})$ is called a *pseudo-Finsler manifold.*

Let M be an n-dimensional submanifold of $\tilde{M}$ locally given by equations

$$x^i = x^i(u^1, \dots , u^n); \quad i \in \{1, \dots , m\}.$$

The local coordinates on TM are denoted by $(u^1, \dots , u^n; v^1, \dots , v^n) = (u, v)$. Denote by i the immersion of M in $\tilde{M}$ and by i_* the differential of i. Then a point (u^α, v^α) of TM is carried by i_* into $(x^i(u), y^i(u, v))$ where

$$y^i(u, v) = B^i_\alpha(u)v^\alpha;\ B^i_\alpha(u) = \frac{\partial x^i}{\partial u^\alpha}\ .$$

Hence the natural frame fields $\{\partial/\partial u^\alpha, \partial/\partial v^\alpha\}$ and $\{\partial/\partial x^i, \partial/\partial y^i\}$ on TM and $T\tilde{M}$ respectively, are related by

$$\frac{\partial}{\partial u^\alpha} = B^i_\alpha(u)\frac{\partial}{\partial x^i} + B^i_{\alpha\beta}(u)v^\beta\frac{\partial}{\partial y^i}, \tag{1.1}$$

and

$$\frac{\partial}{\partial v^\alpha} = B^i_\alpha(u)\frac{\partial}{\partial y^i}, \tag{1.2}$$

where we set

$$B^i_{\alpha\beta}(u) = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\ .$$

As a consequence of (1.2) we deduce that the vertical vector bundle VTM of M is a vector subbundle of $VT\tilde{M}_{|TM}$. Then $\tilde{g}$ induces a symmetric Finsler tensor field g on M, locally given by its components

$$g_{\alpha\beta}(u, v) = B^i_\alpha(u)B^j_\beta(u)\tilde{g}_{ij}(x(u), y(u, v))\ .$$

Next, consider the vector space

$$VTM^\perp_{(u,v)} \;=\; \{X \in VT\tilde{M}_{(u,v)};\ \tilde{g}(u, v)(X, Y) = 0;\quad Y \in VTM_{(u,v)}\},$$

and construct

$$VTM^\perp \;=\; \cup_{(u,v)\in TM} VTM^\perp_{(u,v)}\ .$$

Clearly, $VTM^{\perp}$ is a vector subbundle of $VT\tilde{M}_{|TM}$. In case $\tilde{g}$ is a Riemannian metric on $VT\tilde{M}$ the vector bundle $VTM^{\perp}$ is the complementary orthogonal vector bundle to VTM in $VT\tilde{M}_{|TM}$ and it is called the Finsler normal bundle of M. By using $VTM^{\perp}$ and the induced geometric objects on M it is developed a theory of Finsler submanifolds along with the theory of Riemannian manifolds.

In the present paper we suppose $\tilde{g}$ is neither positive definite nor negative definite, i.e., the index q og $\tilde{g}$ satisfies $0 < q < m$. Then M is said to be a *coisotropic submanifold* of $\tilde{M}$ if $VTM^{\perp}$ is a vector subbundle of VTM. It is easy to see that M is a coisotropic submanifold if and only if $m < 2n$ and the induced Finsler tensor field g is degenerate of rank $2n - m$, i.e., $\operatorname{rank}[g_{\alpha\beta}] = 2n - m$. In particular, any null hypersurface is a coisotropic submanifold (see [2]). The above name of coisotropic manifold is used by Libermann-Merle ([3]) and Vrănceanu-Rosca([6]) for submanifolds of symplectic manifolds and of pseudo-Riemannian manifolds, respectively.

As $VTM^{\perp}$ fails to be complementary to VTM in $VT\tilde{M}_{|TM}$ we shall construct a complementary (non-orthogonal) vector bundle to VTM in $VT\tilde{M}_{|TM}$ which will play the role of $VTM^{\perp}$ from the theory of non-degenerate Finsler submanifolds.

To this end, we consider a complementary vector bundle $S(VTM)$ of $VTM^{\perp}$ in VTM, that is we have

$$VTM = S(VTM) \perp VTM^{\perp} . \tag{1.3}$$

We use $\perp$ and $\oplus$ to denote an orthogonal and non-orthogonal direct sum, respectively. We call $S(VTM)$ a *screen vector bundle* of M. As $S(VTM)$ is a non-degenerate vector bundle we set

$$VT\tilde{M}_{|TM} = S(VTM) \perp S(VTM)^{\perp}, \tag{1.4}$$

where $S(VTM)^{\perp}$ is the orthogonal complementary vector bundle to $S(VTM)$ in $VT\tilde{M}_{|TM}$.

Theorem 1.1. *Let M be a coisotropic submanifold of $F^{(m)}$ and $S(VTM)$ be a screen vector bundle of M. Suppose $\mathcal{U}$ is a coordinate neighborhood of TM and $\{\xi_a\}$, $a \in \{n+1, \dots, m\}$ be a basis of $\Gamma(VTM^{\perp}_{|\mathcal{U}})$. Then there exist smooth sections $\{N_a\}$, $a \in \{n+1, \dots, m\}$ of $VT\tilde{M}_{|\mathcal{U}}$ such that*

$$\tilde{g}(N_a, \xi_b) = \delta_{ab}, \tag{1.5}$$

$$\tilde{g}(N_a, N_b) = 0, \tag{1.6}$$

$$\tilde{g}(N_a, X) = 0, \tag{1.7}$$

for any $a, b \in \{n+1, \ldots, m\}$ *and* $X \in \Gamma(S(VTM)_{|\mathcal{U}})$.

Proof. From (1.4) it follows that $S(VTM)^{\perp}$ is a non-degenerate vector bundle of rank $2(m-n)$. Moreover, $VTM^{\perp}$ is a vector subbundle of $S(VTM)^{\perp}$. Consider a complementary vector bundle F of $VTM^{\perp}$ in $S(VTM)^{\perp}$ and choose a basis $\{V_a\}$, $a \in \{n+1, \ldots, m\}$ of $\Gamma(F_{|\mathcal{U}})$. Due to (1.7) the sections we are looking for are locally expressed as follows

$$N_a = \sum_{b=n+1}^{m} \{A_{ab}\xi_b + B_{ab}V_b\}, \tag{1.8}$$

where A_{ab} and B_{ab} are smooth functions on $\mathcal{U}$. Then $\{N_a\}$ satisfies (1.5) if and only if

$$\sum_{c=n+1}^{m} B_{ac} g^*_{bc} = \delta_{ab},$$

where $g^*_{bc} = \tilde{g}(\xi_b, V_c)$, $b, c \in \{n+1, \ldots, m\}$. Clearly, $G = \det[g^*_{bc}]$ is everywhere non-zero on $\mathcal{U}$, otherwise $S(VTM)^{\perp}$ would be degenerate at least at a point of $\mathcal{U}$. Thus the above system has a unique solution

$$B_{ac} = \frac{(g^*_{ac})'}{G}, \tag{1.9}$$

where $(g^*_{bc})'$ is the cofactor of the element g^*_{ac} in the matrix $g^* = [g^*_{ac}]$. Finally, we define

$$A_{ab} = -\frac{1}{2} \sum_{c,d=n+1}^{m} \{B_{ac}B_{bd}\ \tilde{g}(V_c, V_d)\}, \tag{1.10}$$

and by direct calculations using (1.8)-(1.10) we obtain (1.6). This completes the proof of the theorem.

Theorem 1.2. *Let M be a coisotropic submanifold of $F^{(m)}$ and $S(VTM)$ be a screen vector bundle of M. Then there exists a complementary vector bundle tr(VTM) of $VTM^{\perp}$ in $S(VTM)^{\perp}$ such that $\{N_a\}$, $a \in \{n+1, \ldots, m\}$ from Theorem 1.1 be a basis of $\Gamma(tr(VTM)_{|\mathcal{U}})$.*

Proof. Consider $\{N_a, \xi_a, V_a\}$ and $\{N^*_a, \xi^*_a, V^*_a\}$, $a \in \{n+1, \ldots, m\}$ from Theorem 1.1, with respect to the coordinate neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$, such that $\mathcal{U}\cap\mathcal{U}^* \neq \emptyset$. As $\{\xi_a\}$ and $\{\xi_a^*\}$, $a \in \{n+1, \dots, m\}$ are bases of $\Gamma(VTM^{\perp}_{|\mathcal{U}})$ and $\Gamma(VTM^{\perp}_{|\mathcal{U}^*})$, respectively, we have

$$\xi_a^* = C_a^b \xi_b,$$

where C_a^b are smooth functions on $\mathcal{U}\cap\mathcal{U}^*$. Then by straightforward calculations using (1.8)-(1.10) for both coordinate neighborhoods we obtain

$$N_a^* = \frac{(C_a^b)'}{C} N_b,$$

where $C = \det[C_a^b]$. Thus there exists a vector bundle $\mathrm{tr}(VTM)$ of rank $m-n$ locally spanned on each $\mathcal{U}$ by $\{N_a\}$, given by (1.8) with coefficients from (1.9) and (1.10). Finally, we show that $\mathrm{tr}(VTM)$ is complementary to $VTM^{\perp}$. Suppose there exist a point $(u,v) \in TM$ and a vector $W_{(u,v)} \in VTM^{\perp}_{(u,v)} \cap \mathrm{tr}(VTM)_{(u,v)}$. Then

$$W_{(u,v)} = W^a(u,v)N_a(U,v) = W^a(u,v)\xi_a(u,v),$$

and using (1.5) we deduce

$$\tilde{g}_{(u,v)}(W_{(u,v)}, \xi_b(u,v)) = \tilde{g}_{(u,v)}(W^a(u,v)N_a(u,v), \xi_b(u,v)) = W^b(u,v),$$

and

$$\tilde{g}_{(u,v)}(W_{(u,v)}, \xi_b(u,v)) = \tilde{g}_{(u,v)}(W^a(u,v)\xi_a(u,v), \xi_b(u,v)) = 0.$$

Hence $W_{(u,v)} = 0_{(u,v)}$ and the proof is complete.

We call $\mathrm{tr}(VTM)$ constructed in Theorem 1.2 the *transversal vector bundle* of M. The above name is justified by the decompositions:

(1.11) $$VT\tilde{M}_{|TM} = S(VTM) \perp (VTM^{\perp} \oplus \mathrm{tr}(VTM)) = VTM \oplus \mathrm{tr}(VTM).$$

2. The Induced Geometric Objects on a Coisotropic Submanifold of a Pseudo–Finsler Manifolds

Let M be a coisotropic submanifold of a pseudo-Finsler manifold $F^{(m)} = (\tilde{M}, \tilde{F}, \tilde{g})$. Consider $F^{(m)}$ endowed with the Cartan connection $\tilde{FC} = (HT\tilde{M}, \tilde{\nabla})$, where $HT\tilde{M}$ is a complementary distribution to $VT\tilde{M}$

in $TT\tilde{M}$ and $\tilde{\nabla}$ is a special linear connection on $VT\tilde{M}$. Locally, $\tilde{FC}$ is represented by the tripple $(\tilde{N}^i_j, \tilde{F}^i_{j\,k}, \tilde{C}^i_{j\,k}$, where $\tilde{N}^i_j$ are the local coefficients for $HT\tilde{M}$, that is,

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \tilde{N}^j_i \frac{\partial}{\partial y^j}, \quad i \in \{1, \dots, m\}, \tag{2.1}$$

is a local basis of $\Gamma(HT\tilde{M})$, and $(\tilde{F}^i_{j\,k}, \tilde{C}^i_{j\,k})$ are the local coefficients of $\tilde{\nabla}$ with respect to the basis $\{\delta/\delta x^i, \partial/\partial y^i\}$, i.e., we have

$$\tilde{\nabla}_{\delta/\delta x^k} \frac{\partial}{\partial y^j} = \tilde{F}^i_{j\,k} \frac{\partial}{\partial y^i}, \quad \text{and} \quad \tilde{\nabla}_{\partial/\partial y^k} \frac{\partial}{\partial y^j} = \tilde{C}^i_{j\,k} \frac{\partial}{\partial y^i}.$$

Next, we consider the transversal vector bundle $\mathrm{tr}(VTM)$ with respect to the screen vector bundle $S(VTM)$. Then according to (1.11) we have

$$TT\tilde{M}_{|TM} = VT\tilde{M}_{|TM} \oplus HT\tilde{M}_{|TM} = VTM \oplus \mathrm{tr}(VTM) \oplus HT\tilde{M}_{|TM}.$$

Hence it is natural to look for an induced non-linear connection on TM which is a vector subbundle of $\mathrm{tr}(VTM) \oplus HT\tilde{M}_{|TM}$. With respect to this problem we may state the following result.

Theorem 2.1. *There exists a unique non-linear connection HTM on TM which is a vector subbundle of tr$(VTM) \oplus HT\tilde{M}_{|TM}$.*

Proof. Consider the local sections $\{N_a = N^i_a \partial/\partial y^i\}$, $a \in \{n+1, \dots, m\}$ constructed in Theorem 1.1. Then by (1.2) it follows that $[B^i_\alpha N^i_a]$ is the transition matrix from the natural field of frames $\{\partial/\partial y^1, \dots, \partial/\partial y^m\}$ in $VT\tilde{M}$ to the field of frames $\{\partial/\partial v^1, \dots, \partial/\partial v^n;\ N_{n+1}, \dots, N_m\}$ adapted to the last decomposition in (1.11). Thus we have

$$\begin{aligned} &\tilde{B}^\alpha_i B^i_\beta = \delta^\alpha_\beta;\ \tilde{B}^\alpha_i N^i_a = 0;\ \tilde{N}^a_i B^i_\alpha = 0, \\ &B^i_\alpha \tilde{B}^\alpha_j + N^i_a \tilde{N}^a_j = \delta^i_j;\ \tilde{N}^a_i N^i_b = \delta^a_b, \end{aligned} \tag{2.2}$$

where $[\tilde{B}^\alpha_i \tilde{N}^\alpha_i]$ is the inverse of the matrix $[B^i_\alpha N^i_a]$. We then define locally smooth functions N^β_α by

$$N^\beta_\alpha(u, v) = \tilde{B}^\beta_i (B^j_\alpha \tilde{N}^i_j) + B^i_{\alpha\gamma} v^\gamma. \tag{2.3}$$

Taking into account that $\tilde{N}^i_j$ are the local coefficients of $HT\tilde{M}$, one obtains that N^β_α are the local coefficients of a non-linear connection HTM on TM. Moreover, by using (1.1), (1.2), (2.1) and (2.2) we infer

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - N^\beta_\alpha \frac{\partial}{\partial v^\beta} \;=\; B^i_\alpha \frac{\delta}{\delta x^i} + H^a_\alpha N_a, \tag{2.4}$$

where we set

$$H^a_\alpha = \tilde{N}^a_i (B^j_\alpha \tilde{N}^i_j + B^i_{\alpha\gamma} v^\gamma). \tag{2.5}$$

Thus (2.3)-(2.5) prove the existence of the non-linear connection HTM as a vector subbundle of $\mathrm{tr}(VTM) \oplus HT\tilde{M}_{|TM}$. Next, suppose $HTM' = ({N'}^\beta_\alpha)$ is another non-linear connection on TM which is a vector subbundle of $\mathrm{tr}(VTM) \oplus HT\tilde{M}_{|TM}$. Then using (1.1), (1.2) and (2.1) we deduce

$$X_\alpha = \frac{\partial}{\partial u^\alpha} - {N'}^\beta_\alpha \frac{\partial}{\partial v^\beta} = B^i_\alpha \frac{\delta}{\delta x^i} + (B^j_\alpha \tilde{N}^i_j + B^i_{\alpha\gamma} v^\gamma - B^i_\beta {N'}^\beta_\alpha) \frac{\partial}{\partial y^i}\ ,$$

where $\{X_\alpha\}$ is a local basis of $\Gamma(HTM')$. On the other hand we have

$$X_\alpha = A^i_\alpha \frac{\delta}{\delta x^i} + P^a_\alpha N^i_a \frac{\partial}{\partial y^i}\ ,$$

where A^i_α and P^a_α are smooth functions locally defined on TM. Thus, we get

$$P^a_\alpha N^i_a = B^j_\alpha \tilde{N}^i_j + B^i_{\alpha\gamma} v^\gamma - B^i_\beta {N'}^\beta_\alpha. \tag{2.6}$$

Finally, contracting (2.6) with $\tilde{B}^\gamma_i$ and using (2.2) and (2.3) we obtain ${N'}^\beta_\alpha = N^\beta_\alpha$. Hence we have the uniqueness of the non-linear connection on TM which is a vector subbundle of $\mathrm{tr}(VTM) \oplus HT\tilde{M}_{|TM}$. The proof is complete.

Now, by using (1.11) we obtain

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \quad X \in \Gamma(TTM),\ Y \in \Gamma(VTM), \tag{2.7}$$

where $\nabla_X Y \in \Gamma(VTM)$ and $B(X,Y) \in \Gamma(\mathrm{tr}(VTM))$. It follows that ∇ is a linear connection on VTM and B is an $F(TM)$-bilinear mapping. We call B the *second fundamental form* of M and $FC = (HTM, \nabla)$ the *induced Finsler connection* on M by $\tilde{FC}$. Locally (2.7) becomes

$$\tilde{\nabla}_X Y = \nabla_X Y + B^a(X,Y) N_a, \tag{2.8}$$

where $B^a(X,Y) \in F(TM)$ and $\{N_a\}$ are the sections constructed in Theorem 1.1. We call (2.7) and (2.8) the *Gauss formulas* for the degenerate immersion of M in $\tilde{M}$.

By using (2.8) and (1.5) and taking into account that $\xi_a \in \Gamma(VTM^\perp)$, we deduce

$$B^a(X,Y) = \tilde{g}(\tilde{\nabla}_X Y, \xi_a), \quad X \in \Gamma(TTM),\ Y \in \Gamma(VTM). \tag{2.9}$$

Thus we may state:

Proposition 2.1. *The local components of the second fundamental form of a coisotropic submanifold M of $F^{(m)}$ do not depend on the screen vector bundle $S(VTM)$ of M.*

Moreover, we prove the following result.

Proposition 2.2. *The local components of the second fundamental form of a coisotropic submanifold M of $F^{(m)}$ are degenerate and identically vanish on $VTM^{\perp}$.*

Proof. Replace Y by ξ_b in (2.9) and taking into account that $\tilde{\nabla}$ is a metric connection on $VT\tilde{M}$ we obtain

$$B^a(X, \xi_b) \ + \ B^b(X, \xi_a) \ = \ 0, \quad a, b \in \{n+1, \dots, m\}. \tag{2.10}$$

Take $b = a$ in (2.10) and deduce

$$B^a(X, \xi_a) = 0, \ \text{(no summation)}, \quad X \in \Gamma(TM), \tag{2.11}$$

which implies that each B^a is degenerate. Also from (2.10) we derive

$$\begin{aligned} B^a(\xi_c, \xi_b) \ &+ \ B^b(\xi_c, \xi_a) \ = \ 0, \\ B^c(\xi_b, \xi_a) \ &+ \ B^a(\xi_b, \xi_c) \ = \ 0, \\ B^b(\xi_a, \xi_c) \ &+ \ B^c(\xi_a, \xi_b) \ = \ 0, \ a, b, c \in \{n+1, \dots, m\}. \end{aligned}$$

Finally, taking into account that B^a are symmetric bilinear forms on $VTM^{\perp}$, the above equations imply $B^a(\xi_b, \xi_c) = 0$, which completes the proof of the proposition.

Next, according to the same decomposition (1.11) we set

$$\tilde{\nabla}_X V = -A(V, X) + \nabla^t_X V, \quad X \in \Gamma(TTM), \ V \in \Gamma(tr(VTM)), \tag{2.12}$$

where $A(V, X) \in \Gamma(VTM)$ and $\nabla^t_X V \in \Gamma(\mathrm{tr}(VTM))$. It follows that ∇^t is a linear connection on the vector bundle $\mathrm{tr}(VTM)$ and

$$A_V : \Gamma(TTM) \mapsto \Gamma(VTM); \quad A_V(X) = A(V, X),$$

is an $F(TM)$ - linear operator. We call A_V the *shape operator* of M with respect to the transversal section V. As in case of non-degenerate Finsler submanifolds, (2.12) is called *Weingarten formula* for the degenerate immersion of M in $F^{(m)}$.

We now denote by P the projection morphism of VTM on $S(VTM)$ with respect to the decomposition (1.3). Then we set

$$\nabla_X PY = \nabla^*_X PY + B^*(X, PY) = \nabla^*_X PY + B^{*a}(X, PY)\xi_a, \tag{2.13}$$

for any $X \in \Gamma(TTM)$ and $Y \in \Gamma(VTM)$ where $\nabla^*_X PY \in \Gamma(S(VTM))$ and $B^*(X, PY) \in \Gamma(VTM^\perp)$. It is easy to check that ∇^* is a linear connection on $S(VTM)$ and B^* is a $(VTM^\perp)$-valued $F(TM)$-bilinear form on $\Gamma(TTM) \times \Gamma(S(VTM))$. Taking into account that $\tilde{\nabla}$ is a metric connection and by using (2.12) and (2.13) we obtain

$$B^{*a}(X, PY) = \tilde{g}(A_a(X), PY), \tag{2.14}$$

where $A_a = A_{N_a}$ for $a \in \{n+1, \ldots, m\}$.

It is important to note that, in general, the induced linear connection ∇ is not a metric connection. More precisely, we obtain

$$(\nabla_X g)(Y, Z) = B^a(X, Y)\eta_a(Z) + B^a(X, Z)\eta_a(Y), \tag{2.15}$$

for any $X \in \Gamma(TTM)$ and $Y, Z \in \Gamma(VTM)$, where η_a, $a \in \{n+1, \ldots, m\}$ are local q-forms defined by

$$\eta_a(Y) = g(Y, N_a), \quad Y \in \Gamma(VTM). \tag{2.16}$$

Theorem 2.2. *Let M be a coisotropic submanifold of $F^{(m)}$. Then ∇ is a metric connection on VTM if and only if the second fundamental form B satisfies*

$$B(X, PZ) = 0, \quad X \in \Gamma(TTM), \; Z \in \Gamma(VTM). \tag{2.17}$$

Proof. By using (2.14), (2.15), (1.7) and (2.11) we obtain

$$(\nabla_X g)(PY, PZ) = (\nabla_X g)(\xi_a, \xi_b) = 0,$$

and

$$(\nabla_X g)(\xi_a, PZ) = B^a(X, PZ),$$

which prove our assertion.

3. Equations of Gauss, Codazzi and Ricci for Coisotropic Submanifolds of Pseudo–Finsler Manifolds

Let M be a coisotropic submanifold of a pseudo-Finsler manifold $F^{(m)} = (\tilde{M}, \tilde{F}, \tilde{g})$. Consider the Cartian connection $\tilde{FC} = (HT\tilde{M}, \tilde{\nabla})$ and the induced Finsler connection $FC = (HTM, \nabla)$ on $F^{(m)}$ and M, respectively. Then the canonical almost product structure Q on TM is given by

$$Q\left(\frac{\delta}{\delta u^\alpha}\right) = \frac{\partial}{\partial v^\alpha} \ ; \ \ Q\left(\frac{\partial}{\partial v^\alpha}\right) = \frac{\delta}{\delta u^\alpha} \ .$$

By means of Q and ∇ we define on TM the linear connection ∇' as follows:

$$\nabla'_X Y = \nabla_X vY + Q(\nabla_X QhY), \quad X, Y \in \Gamma(TTM), \tag{3.1}$$

where v and h are the projection morphisms of TTM on VTM and HTM, respectively. It is easy to check that Q is parallel with respect to ∇'.

Next, denote by $\tilde{R}$, R and R^t the curvature tensor fields of linear connections $\tilde{\nabla}$, ∇ and ∇^t, respectively. Then by using (2.7) and (2.12) we obtain

$$\begin{aligned}\tilde{R}(X,Y)Z =& R(X,Y)Z + A(B(X,Z),Y) - A(B(Y,Z),X) \\ &+ (\nabla^t_X B)(Y,Z) - (\nabla^t_Y B)(X,Z) + B(T'(X,Y),Z),\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\tilde{R}(X,Y)V =& R^t(X,Y)V + B(Y, A(V,X)) - B(X, A(V,Y)) \\ &+ (\nabla_Y A)(V,X) - (\nabla_X A)(V,Y) + A(V, T'(X,Y)),\end{aligned} \tag{3.3}$$

for any $X, Y \in \Gamma(TTM)$, $Z \in \Gamma(VTM)$ and $V \in \Gamma(\mathrm{tr}(VTM))$, where T' is the torsion tensor field of ∇' and we set

$$(\nabla^t_X B)(Y,Z) = \nabla^t_X(B(Y,Z)) - B(\nabla'_X Y, Z) - B(Y, \nabla'_X Z),$$

and

$$(\nabla_X A)(V,Y) = \nabla_X(A(V,Y)) - A(\nabla^t_X V, Y) - A(V, \nabla'_X Y).$$

By using (1.5)–(1.7), (2.14) and (3.2) we deduce

$$\begin{aligned}&\tilde{g}(\tilde{R}(X,Y)Z, PU) = g(R(X,Y)Z, PU) \\ &\qquad + \sum_{a=n+1}^{m} \{B^a(X,Z)B^{*a}(Y,PU) - B^a(Y,Z)B^{*a}(X,PU)\},\end{aligned} \tag{3.4}$$

(3.5) $\tilde{g}(\tilde{R}(X,Y)Z,\xi) = \tilde{g}((\nabla^t_X B)(Y,Z) - (\nabla^t_Y B)(X,Z),\xi) + \tilde{g}(B(T'(X,Y),Z),\xi),$

(3.6) $$\tilde{g}(\tilde{R}(X,Y)Z,N) = \tilde{g}(R(X,Y)Z,N),$$

for any $U \in \Gamma(VTM)$, $\xi \in \Gamma(VTM^{\perp})$ and $N \in \Gamma(\mathrm{tr}(VTM))$. Similarly, by using properties of $\tilde{R}$ and (2.14), (2.16) and (3.6) we derive

(3.7) $$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)N_a,PU) &= g((\nabla_Y A)(N_a,X) - (\nabla_X A)(N_a,Y),PU)\\ &\quad - B^{*a}(T'(X,Y),PU) = -\tilde{g}(R(X,Y)PU,N_a),\end{aligned}$$

(3.8) $$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)N_a,\xi_b) &= g(R^t(X,Y)N_a,\xi_b) - B^b(Y,A_aX) - B^b(X,A_aY)\\ &= -\eta_a(R(X,Y)\xi_b),\end{aligned}$$

(3.9) $\tilde{g}(\tilde{R}(X,Y)N_a,N_b) = \eta_b(A_a(T'(X,Y)) + (\nabla_Y A)(N_a,X) - (\nabla_X A)(N_a,Y)).$

Hence we may state the following result.

Theorem 3.1. *Let M be a coisotropic submanifold of a pseudo-Finsler manifold $F^{(m)}$. Then the Gauss, Codazzi and Ricci equations* (*structure equations*) *for the degenerate immersion of M in $F^{(m)}$ are given by* (3.4)–(3.9).

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