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ON CONVERSE THEOREM OF APPROXIMATION IN VARIOUS METRICS FOR PERIODIC FUNCTIONS OF SEVERAL VARIABLES

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This paper is dedicated to Professor R. Ž. Djordjević for his 65th birthday

Abstract. The modulus of smoothness in the norm of space L_q of a 2π–periodic function of several variables is estimated by best approximations by trigonometric polynomials in the norm of L_p, $1 \leq p \leq q < +\infty$.

1. Introduction

The converse theorem of approximation in various metrics for 2π–periodic function of one variable was proved in [1]. In this paper we are proving one of the analogous theorems for functions of several variables. Actually we are improving and generalizing Theorem 6.3.5 in [3], and we are giving the implications of obtained result. In this way we are also getting one generalization of Theorem 1 in [1].

As usually, we say that $f(x_1, \dots , x_n) \in L_p([0, 2\pi])^n$ if f is measurable on Δ_n and is a 2π–periodic function with respect to every variable $x_1, \dots , x_n$, for which $\|f\|_p < +\infty$, where

$$\|f\|_p = \left(\int_{\Delta_n} \bigl| f(x_1, \dots , x_n) \bigr|^p \, dx_1 \dots dx_n \right)^{1/p} , \quad 1 \leq p < +\infty ,$$

$$\Delta_n = \bigl\{ \boldsymbol{x} = (x_1, \dots , x_n) : \ 0 \leq x_i \leq 2\pi , \ i = 1, \dots , n \bigr\} = [0, 2\pi]^n .$$

The notions of modulus of smoothness of a function and best approximation of a function by trigonometric polynomials are given in [3] and [2].

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Let

$$T_{\nu_1,\dots,\nu_n}(x_1,\dots,x_n)$$

be a trigonometric polynomial of order $\nu_1,\dots,\nu_n$ in the corresponding variables $x_1,\dots,x_n$. The best approximation $E_{\nu_1,\dots,\nu_n}(f)_p$ of a function $f\in L_p$ by trigonometric polynomials is the quantity (see [3], 2.2.6):

$$E_{\nu_1,\dots,\nu_n}(f)_p=\inf_T\big\|f-T_{\nu_1,\dots,\nu_n}\big\|_p\,. \tag{1.1}$$

The modulus of smoothness of order k of a function f with respect to x_i is the quantity (see [3], 3.3 and 3.4):

$$\omega_k(f;\delta_i)_p=\omega_k(f;0,\dots,0,\delta_i,0,\dots,0)_p=\sup_{|h_i|\le\delta_i}\big\|\Delta^k_{h_i}f\big\|_p\,, \tag{1.2}$$

where

$$\Delta^k_{h_i}f=\sum_{\nu=0}^{k}(-1)^{k-\nu}\binom{k}{\nu}f(x_1,\dots,x_{i-1},x_i+\nu h_i,x_{i+1},\dots,x_n)\,. \tag{1.3}$$

The mixed derivative of a function $f(x_1,\dots,x_n)$ of order r_j with respect to x_j we denote by

$$f^{(r_1,\dots,r_n)}=\frac{\partial^{r_1+\cdots+r_n}f}{\partial x_1^{r_1}\dots\partial x_n^{r_n}}\,.$$

By $a\ll b$, $a>0$, $b>0$, we will denote the inequality $a\le Cb$, where C is a positive constant.

2. The Main Result

In this section we are proving a theorem which is a generalization and improvement of Theorem 6.3.5 in [3].

Theorem 2.1. *Let $f(x_1,\dots,x_n)\in L_p\big([0,2\pi]^n\big)$ and let for given nonnegative integers r_j and natural numbers l_j, $j=1,\dots,n$, $l_i=1$, for some $i\in\{1,\dots,n\}$, $1\le p\le q<+\infty$ the following inequality holds*

$$\sum_{\nu=1}^{+\infty}\nu^{q\sigma-1}E_{\nu^{l_1},\dots,\nu,\dots,\nu^{l_n}}(f)_p<+\infty \tag{2.1}$$

where

$$\sigma=\sum_{j=1}^{n}l_j\Big(r_j+\frac1p-\frac1q\Big)\,. \tag{2.2}$$

Then the function f has a mixed derivative $f^{(r_1,\dots,r_n)}$ belonging to the space L_q and for any natural numbers k and m_i the following inequality holds

$$\omega_k\Big(f^{(r_1,\dots,r_n)};0,\dots,0,\frac{1}{m_i},0,\dots,0\Big)_q$$
$$\leq \frac{C}{m_i^k}\Big\{\|f\|_p^q+\sum_{\nu=1}^{m_i}\nu^{q(\sigma+k)-1}E_{\nu^{l_1},\dots,\nu,\dots,\nu^{l_n}}(f)_p\Big)^{1/q} \tag{2.3}$$
$$+\Big\{\sum_{\nu=m_i+1}^{+\infty}\nu^{q\sigma-1}E_{\nu_{l_1},\dots,\nu,\dots,\nu^{l_n}}(f)_p\Big\}^{1/q},$$

where constant C depends on k and σ only. The constant C does not depend on neither f nor $m_i=1,2,3\dots.$

Proof. Let

$$T_{\nu_1,\dots,\nu_{i-1},\nu,\nu_{i+1},\dots,\nu_n}=T_{\nu_1,\dots,\nu_{i-1},\nu,\nu_{i+1},\dots,\nu_n}(f;x_1,\dots,x_n) \tag{2.4}$$
$$\nu_j=\nu^{l_j}\,,\quad j=1,\dots,n\quad(\nu_i=\nu)\,,$$

be the trigonometric polynomials of the best approximation of function f in the space L_p. For trigonometric polynomials

$$S_m=T_{1,\dots,1}+\sum_{\nu=0}^{m}\Big[T_{2^{l_1(\nu+1)},\dots,2^{\nu+1},\dots,2^{l_n(\nu+1)}}-T_{2^{l_1\nu},\dots,2^{\nu},\dots,2^{l_n\nu}}\Big] \tag{2.5}$$

the following holds

$$f-S_m=f-T_{2^{l_1(m+1)},\dots,2^{m+1},\dots,2^{l_n(m+1)}}\,.$$

Since

$$\Big\|f-T_{2^{l_1(m+1)},\dots,2^{m+1},\dots,2^{l_n(m+1)}}\Big\|_p=E_{2^{l_1(m+1)},\dots,2^{m+1},\dots,2^{l_n(m+1)}}(f)_p \tag{2.6}$$

we conclude that

$$\|f-S_m\|\to 0\quad\text{as}\quad m\to+\infty\,. \tag{2.7}$$

This means that in the sense of L_p equality

$$f=T_{1,\dots,1}+\sum_{\nu=0}^{+\infty}\Big[T_{2^{l_1(\nu+1)},\dots,2^{\nu+1},\dots,2^{l_n(\nu+1)}}-T_{2^{l_1\nu},\dots,2^{\nu},\dots,2^{l_n\nu}}\Big] \tag{2.8}$$

holds.

In the following step we are proving that equality (2.8) also holds in the sense of L_q, $1 \le p \le q < +\infty$. To do this we will prove that the sequence S_m, $m = 0, 1, 2, \dots$, is a Cauchy sequence in L_q.

Applying the method by which the corresponding quantity in [1] was estimated (see estimation of quantity A for $q > 2$ in Lemma 1 in [1]), and taking into consideration the corresponding inequality of various metrics for trigonometric polynomials of several variables, we conclude that, for $t > m$,

$$\|S_t - S_m\|_q \ll \left\{ \sum_{\nu=m+1}^{t} 2^{\nu q\left(\frac{1}{p}-\frac{1}{q}\right)(l_1+\dots,l_n)} E^q_{2^{l_1\nu},\dots,2^{\nu},\dots,2^{l_n\nu}}(f)_p \right\}^{1/q}, \tag{2.9}$$

holds.

From (2.9) in view of the assumption (2.1) it follows that the sequence S_m is a Cauchy sequence in L_q. Since the space L_q is complete, there exists a function $h(x_1, \dots, x_n) \in L_q$ such that

$$\|h - S_m\|_q \to 0 \quad \text{as} \quad m \to +\infty\,. \tag{2.10}$$

Equality (2.8) and convergence (2.10) imply (see [3], 1.3.9) that equality (2.8) holds in L_q.

In the following step we are proving that in the sense of L_q equality

$$\begin{aligned} f^{(r_1,\dots,r_n)} &\overset{(q)}{=} T^{(r_1,\dots,r_n)}_{1,\dots,1} \\ &+ \sum_{\nu=0}^{+\infty}\Big[T^{(r_1,\dots,r_n)}_{2^{l_1(\nu+1)},\dots,2^{\nu+1},\dots,2^{l_n(\nu+1)}} - T^{(r_1,\dots,r_n)}_{2^{l_1\nu},\dots,2^{\nu},\dots,2^{l_n\nu}}\Big] \end{aligned} \tag{2.11}$$

holds.

Applying the same procedure which yielded inequality (2.9)), and using the Bernstein type inequality (see [3], 4.8.62(30); see also proof of Lemma 1 in [1]), we conclude that

$$\Big\|S^{(r_1,\dots,r_n)}_t - S^{(r_1,\dots,r_n)}_m\Big\|_q \ll \left\{ \sum_{\nu=m+1}^{t} 2^{\nu q\sigma} E^q_{2^{l_1\nu},\dots,2^{\nu},\dots,2^{l_n\nu}}(f)_p \right\}^{1/q} \tag{2.12}$$

holds.

In view of (2.12) and (2.1) we conclude that the sequence $S_m^{(r_1,\dots,r_n)}$ converges in L_q. Since equality (2.8) holds in L_q, it means that in the sense of L_q equality (2.11) holds (see [2], 4.4.7; [3], 6.3.31).

For modulus of smoothness of the function $f^{(r_1,\dots,r_n)}$ we have

$$\begin{aligned}(2.13)\qquad \omega_k\Big(f^{(r_1,\dots,r_n)};\frac{1}{m_i}\Big)_q &\le \omega_k\Big(f^{(r_1,\dots,r_n)} - S_m^{(r_1,\dots,r_n)};\frac{1}{m_i}\Big)_q \\ &\quad + \omega_k\Big(S_m^{(r_1,\dots,r_n)};\frac{1}{m_i}\Big)_q = I_1 + I_2\,.\end{aligned}$$

Now, we get

$$(2.14)\qquad I_1 \ll \Big\|f^{(r_1,\dots,r_n)} - S_m^{(r_1,\dots,r_n)}\Big\|_q\,.$$

Using the procedure which yielded inequality (2.12), and in view of equality (2.11) and (2.5), we obtain

$$(2.15)\qquad \Big\|f^{(r_1,\dots,r_n)} - S_m^{(r_1,\dots,r_n)}\Big\|_q \ll \Big\{\sum_{\nu=m+1}^{+\infty} 2^{\nu q\sigma} E^q_{2^{l_1\nu},\dots,2^{\nu},\dots,2^{l_n\nu}}(f)_p\Big\}^{\frac{1}{q}}\,.$$

In virtue of the properties of modulus of smoothness we have

$$(2.16)\qquad I_2 = \omega_k\Big(S_m^{(r_1,\dots,r_n)};\frac{1}{m_i}\Big)_q \ll \frac{1}{m_i^k}\big\|S_m^{(r_1,\dots,r_i+k,\dots,r_n)}\big\|_q\,.$$

The estimate for the norm $\|S_m^{(r_1,\dots,r_i+k,\dots,r_n)}\|_q$ follows from (2.5), in the same way as inequality (2.12). Only instead of number r_i we write r_i+k. Therefore, we get

$$\begin{aligned}(2.17)\qquad \Big\|S_m^{(r_1,\dots,r_i+k,\dots,r_n)}\Big\|_q &\ll \Big\|T_{1,\dots,1}\Big\|_p \\ &\quad + \Big\{\sum_{\nu=0}^{m} 2^{\nu q(\sigma+k)} E^q_{2^{l_1\nu},\dots,2^{\nu},\dots,2^{l_n\nu}}(f)_p\Big\}^{1/q}\,.\end{aligned}$$

To get (2.17) we use inequality

$$\big\|G_{2^{\nu+1}} - G_{2^\nu}\big\| \le \big\|G_{2^{\nu+1}} - f\big\| + \big\|f - G_{2^\nu}\big\|\,.$$

Hence

$$\left\| S_m^{(r_1,\dots,r_i+k,\dots,r_n)} \right\|_q \ll \left\{ \|f\|_p^q + \sum_{\nu=0}^{m} 2^{\nu q(\sigma+k)} E^q_{2^{l_1\nu},\dots,2^\nu,\dots,2^{l_n\nu}}(f)_p \right\}^{1/q}. \tag{2.18}$$

From (2.13), in view of (2.14), (2.15), (2.16) and (2.18), it follows

$$\begin{aligned} \omega_k\Big(f^{(r_1,\dots,r_n)}; \frac{1}{m_i}\Big)_q &\ll \frac{1}{m_i^k}\left\{ \|f\|_p^q + \sum_{\nu=0}^{m} 2^{\nu q(\sigma+k)} E^q_{2^{l_1\nu},\dots,2^\nu,\dots,2^{l_n\nu}}(f)_p \right\}^{1/q} \\ &+ \left\{ \sum_{\nu=m+1}^{+\infty} 2^{\nu q\sigma} E^q_{2^{l_1\nu},\dots,2^\nu,\dots,2^{l_n\nu}}(f)_p \right\}^{1/q}. \end{aligned} \tag{2.19}$$

Let us denote

$$F_{2^\nu} = E_{(2^\nu)^{l_1},\dots,2^\nu,\dots,(2^\nu)^{l_n}}\,. \tag{2.20}$$

Then

$$F_\mu = E_{\mu^{l_1},\dots,\mu,\dots,\mu^{l_n}} \tag{2.21}$$

and $F_\mu \downarrow 0$ as $\mu \to +\infty$.

Choosing m so that $2^m \le m_i < 2^{m+1}$, from (2.19) we get

$$\begin{aligned} \omega_k\Big(f^{(r_1,\dots,r_n)}; \frac{1}{m_i}\Big)_q \ll \frac{1}{m_i^k}&\left\{ \|f\|_p^q + \sum_{\nu=1}^{m_i} \nu^{q(\sigma+k)-1} F_\nu^q \right\}^{1/q} \\ &+ \left\{ \sum_{\nu=m_i+1}^{+\infty} \nu^{q\sigma-1} F_\nu^q \right\}^{1/q}. \end{aligned} \tag{2.22}$$

Inequality (2.3) follows from (2.22) in view of (2.21). $\square$

3. Some Consequences

We give now a few basic consequences of Theorem 2.1.

Corollary 3.1. *For $n = 1$ ($l_j = 1$, $r_j = r$, $\sigma = r + \frac{1}{p} - \frac{1}{q}$), Theorem 1 in* [1] *follows.*

Corollary 3.2. *If the condition*

$$\sum_{\nu=1}^{+\infty} \nu^{q\left[r+n(\frac{1}{p}-\frac{1}{q})\right]-1} E^q_{\nu,\dots,\nu,\dots,\nu}(f)_p < +\infty \tag{3.1}$$

holds, then the function f has a derivative $\frac{\partial^r f}{\partial x_i^r}$ with respect to any variable x_i. The derivative belongs to the space L_q. Also for the modulus of smoothness the corresponding inequality (2.3) *holds.*

Corollary 3.2 follows from the theorem for $l_1 = l_2 = \cdots = l_n = 1$, $r_j = 0$ for $j \neq i$, $r_i = r$.

Corollary 3.3. *If*

$$\sum_{\nu=1}^{+\infty} \nu^{q\left[r+(\frac{1}{p}-\frac{1}{q})(l_1+\cdots+l_n)\right]-1} E^q_{\nu^{l_1},\dots,\nu,\dots,\nu^{l_n}}(f)_p < +\infty, \tag{3.2}$$

then the function f has a derivative $\frac{\partial^r f}{\partial x_i^r} \in L_q$ and the corresponding inequality (2.3) *holds.*

This corollary follows from the theorem for $r_i = r$, $r_j = 0$ for $j \neq i$.

Corollary 3.4. *For $p = q$, Theorem* 2.1 *implies Theorem* 6.3.5 *in* [3]. *Indeed, from* (2.19) *using inequality $(\sum a_k)^s \leq \sum (a_k)^s$, $a_k \geq 0$, $0 < s \leq 1$, we get*

$$\begin{aligned} \omega_k\Big(f^{(r_1,\dots,r_n)};\frac{1}{m_i}\Big)_q \ll \frac{1}{m_i^k}\Big\{\|f\|_p + \sum_{\nu=0}^{m} 2^{\nu(\sigma+k)} E_{2^{l_1\nu},\dots,2^{\nu},\dots,2^{l_n\nu}}(f)_p\Big\} \\ + \sum_{\nu=m+1}^{+\infty} 2^{\nu\sigma} E_{2^{l_1\nu},\dots,2^{\nu},\dots,2^{l_n\nu}}(f)_p\,, \end{aligned} \tag{3.3}$$

and then

$$\omega_k\Big(f^{(r_1,\dots,r_n)};\frac{1}{m_i}\Big)_q \ll \frac{1}{m_i^k}\Big\{\|f\|_p+\sum_{\nu=1}^{m_i}\nu^{\sigma+k-1}E_{\nu^{l_1},\dots,\nu,\dots,\nu^{l_n}}(f)_p\Big\} + \sum_{\nu=m_i+1}^{+\infty}\nu^{\sigma-1}E_{\nu^{l_1},\dots,\nu,\dots,\nu^{l_n}}(f)_p\,. \tag{3.4}$$

For $q=p$ we have $\sigma=\sum_{j=1}^{n} l_j r_j$, and inequality 6.3.5 (24) in [3] follows from (3.4).

REFERENCES

1. N. A. Ilyssov: *The converse theorem of theory of approximation in various metrics.* Mat. Zametki **50** (1991), 57–65 (Russian).
2. S. M. Nikol'skiĭ: *Approximation of Functions of Several Variables and Imbedding Theorems.* (Second edition, revised and supplemented), Nauka, Moscow, 1977 (Russian).
3. A. F. Timan: *Theory of Approximation of Functions of a Real Variable.* Dover Publ., New York, 1994.

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